Section 2.3

Linear Equations

Recall: First-Order Equations

A first-order ODE is an ODE which can be written in the form

$$\frac{dy}{dt} = f(t, y)$$

A first-order linear ODE is an ODE which can be written in the form

$$a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

A first-order separable ODE is an ODE which can be written in the form

$$\frac{dy}{dt} = g(t)h(y)$$

First-Order Linear Equations

$$a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

If we divide both sides by $a_1(t)$, we obtain the standard form

$$\frac{dy}{dt} + p(t)y = q(t)$$

Goal: Rewrite the equation into a form where the left-hand side is the derivative of a single function. To do this, we multiply both sides of the equation by an integrating factor $\mu(t)$.

First-Order Linear Equations

$$\frac{dy}{dt} + p(t)y = q(t)$$

Multiplying by $\mu(t)$, we obtain

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)q(t)$$

It would be convenient if the left hand side was

$$\frac{d}{dt}[\mu(t)y] = \mu(t)\frac{dy}{dt} + \mu'(t)y$$

Thus, we need

$$\mu'(t) = \mu(t)p(t)$$

First-Order Linear Equations

$$\mu'(t) = \mu(t)p(t)$$

This is a separable DE, which we can solve to obtain the integrating factor

$$\mu(t) = e^{\int p(t)dt}$$

Solving First-Order Linear Equations

To solve any first-order linear ODE:

1. Rewrite the ODE in standard form
$$\frac{dy}{dt} + p(t)y = q(t)$$
2. Calculate the integrating factor
$$\mu(t) = e^{\int p(t)dt}$$
3. Multiply both sides of the standard form ODE by $\mu(t)$, obtaining
$$\frac{d}{dt}[\mu(t)y] = \mu(t)q(t)$$
4. Antidifferentiate both sides
5. Solve for y

Example 1

Find the general solution of the differential equation

$$y' + \left(\frac{1}{t}\right)y = 3\cos 2t$$

on the interval $(0, \infty)$.

Example 2

Find the solution of the initial value problem

$$(t+1)\frac{dy}{dt} + y = \ln t\ , \qquad y(1) = 10$$

Example 3

Find the general solution of

$$t\frac{dy}{dt} - 4y = t^6 e^t$$