Missouri University of Science and Technology — Founded 1870 | Rolla, Missouri

Section 3.2

Modeling with First-Order ODEs

Mathematical Models

A mathematical model is a mathematical description of a real-world system or phenomenon.

Goal for this section:
Use first-order ODEs to predict (or determine) what happens in a physical process.

Deterministic Models

There are three main types of deterministic models:

1. Law-based
   The model is constructed from physical laws
2. Empirical
   The model is constructed to match observations/expectations
3. A mixture of law-based and empirical observations

Mixture Problems

If $A(t)$ represents the amount of a compound dissolved in a tank at time t, we can determine the rate at which $A(t)$ changes over time as we pump a given mixture into the tank at a given rate and allow outflow from the tank at a given rate.

Specifically, we can use a DE of the form

$$\frac{dA}{dt} = R_{\text{in}} - R_{\text{out}}$$

where R represents the rate at which the dissolved compound is being pumped in or out.

Example 1

Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Another brine solution with concentration 2 pounds per gallon is pumped into the tank at a rate of 3 gallons per minute. When the solution is well-stirred, it is pumped out at the slower rate of 2 gallons per minute.

a) Set up an IVP for the amount of salt $A(t)$ in the tank at time t.

b) Solve the IVP to find an expression for $A(t)$.

Population Growth

Goal: Find a model for the population $y(t)$ of a given species at time $t > 0$ given an initial population $y(t_0) = y_0$.

Method: Set up an IVP based on empirical observations which will allow us to solve for $y(t)$.

A Simple Population Model

Assume the growth rate of the population is proportional to the current population.

This suggests an ODE:

$$y'(t) = ry(t), \text{ where } r > 0 \text{ is constant}$$

This ODE is both linear and separable!

Solution:

$$y(t) = Ce^{rt}$$

WARNING!

Your textbook *dramatically* overcomplicates this next portion.
Pay attention!

Modified Assumptions

Our previous ODE can be rewritten as

$$\frac{y'}{y} = r$$

which indicates the relative growth rate $\frac{y'}{y}$ is constant.

Let's assume the relative growth rate is linear and decreases as the population increases. This gives us the ODE

$$\frac{y'}{y} = r - ay$$

The Logistic Equation

$$\frac{y'}{y} = r - ay$$

can be rewritten as

$$y' = y(r - ay)$$

which is called the logistic equation.

The logistic equation is a nonlinear separable equation.

The Logistic Equation

$$y' = y(r - ay)$$

If $r - ay > 0$ (or $y < \frac{r}{a}$), then $y' > 0$ and the population will grow.

If $r - ay < 0$ (or $y > \frac{r}{a}$), then $y' < 0$ and the population will decay.

The carrying capacity of the population is $y = \frac{r}{a}$.

The Logistic Equation

$$y' = y(r - ay)$$

The carrying capacity of the population is $y = \frac{r}{a}$.

If we factor out a, we get

$$y' = ay\left(\frac{r}{a} - y\right)$$

or, equivalently,

$$y' = ay(M - y)$$

which is still called the logistic equation.

Example 2

Suppose that in 1885, the population of a certain country was 50 million and was growing at the rate of 750,000 people per year at that time. Suppose also that in 1940 its population was 100 million and was growing at a rate of 1 million per year. Assume that this population satisfies the logistic equation. Determine both the limiting population M and the predicted population for the year 2000.