Missouri University of Science and Technology — Founded 1870 | Rolla, Missouri

Section 4.10

Spring-Mass Systems
Forced Mechanical Vibrations

Recall: Spring-Mass Systems

A standard spring-mass system can be modeled using the differential equation

$$my'' + \gamma y' + ky = F(t)$$

where m is the mass of the object, γ is the damping constant, k is the spring constant, and $F(t)$ is the time-dependent external force acting on the system.

Goal for today: Consider what happens when $F(t) \neq 0$.

Forced Mechanical Vibrations

If $F(t) \neq 0$, the general solution of

$$my'' + \gamma y' + ky = F(t)$$

will be of the form

$$y(t) = y_h(t) + y_p(t)$$

$y_h(t)$ is the transient part of the solution.
$y_p(t)$ is the steady-state solution.

When $\gamma \neq 0$, note that $y_h(t) \to 0$ as $t \to \infty$.
Thus, $y(t) \to y_p(t)$ as $t \to \infty$.

Recall: Free Undamped Motion

With free undamped motion, we are solving the equation

$$my'' + ky = 0$$

which has general solution

$$y = C_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

Smaller values of $\sqrt{\frac{k}{m}}$ (typically from larger masses) yield slower vibrations.

Larger values of $\sqrt{\frac{k}{m}}$ (typically from stiffer springs) yield faster vibrations.

Forced Undamped Motion

With forced undamped motion, we are solving the equation

$$my'' + ky = F(t)$$

The transient part of the solution is

$$y_h = C_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

We often consider oscillating forces of the form

$$F(t) = F_0 \cos(\omega t)$$

where F_0 and ω are constants.

Forced Undamped Motion

$$my'' + ky = F_0 \cos(\omega t)$$

$$y_h = C_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

If $\omega = \sqrt{\frac{k}{m}}$, meaning the frequency of forcing matches the frequency of the unforced system, we get the steady-state solution

$$y_p = \frac{F_0}{2m\omega} t \sin\left(\sqrt{\frac{k}{m}}t\right)$$

which approaches ∞ as $t \to \infty$.

This is called resonance. Resonance is bad!

Example 1

An object that weighs 8 pounds stretches a spring 6 inches. The undamped system is acted upon by an external force of $8\cos(\omega t)$ pounds where ω is a positive constant. The object is released from static equilibrium. Let $y(t)$ be the displacement from static equilibrium at any positive time t seconds.

a) Set up an initial value problem.

b) Find the transient part $y_h(t)$ of the solution.

c) Find the steady-state solution $y_p(t)$ and solution $y(t)$ when $\omega \neq 8$ and graph $y(t)$ provided $\omega = 7.8$

Example 1

$$y = \frac{32}{64-\omega^2}[\cos\omega t - \cos 8t]$$

When $\omega = 7.8$, we get

$$y \approx 10.13[\cos 7.8t - \cos 8t]$$

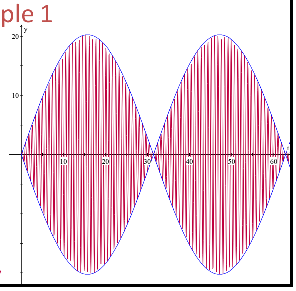
Recall from trig that

$$\cos A - \cos B = 2\sin\frac{B-A}{2}\sin\frac{B+A}{2}$$

Thus,

$$y \approx \underbrace{20.25\sin(0.1t)}_{\text{Longer period gives amplitude}}\underbrace{\sin(7.9t)}_{\text{Shorter period gives frequency}}$$

Longer period gives amplitude — Shorter period gives frequency



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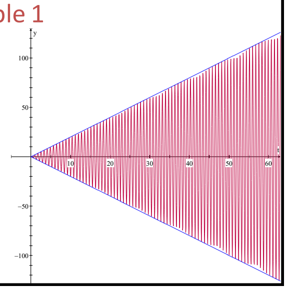
d) Find the steady-state solution $y_p(t)$ and solution $y(t)$ when $\omega = 8$ and graph $y(t)$.

Example 1

$y(t) = 2t\sin(8t)$

Amplitude ($2t$) Frequency ($\sin(8t)$)



Forced Damped Motion

With forced damped motion, we are solving the equation

$$my'' + \gamma y' + ky = F(t)$$

The transient part of the solution may or may not involve oscillation, depending on the roots of the auxiliary equation.

Resonance can still occur even with damping!