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Section 4.2

Homogeneous Linear Equations

Recall: First-Order Linear Equations

A first-order linear ODE is an ODE which can be written in the form

$$a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

Second-Order Linear Equations

A second-order linear ODE is an ODE which can be written in the form

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

If $f(t) = 0$, the equation is called homogeneous.

Example 1

Solve the DE.

$$y'' - 4y = 0$$

Homogeneous Equations with Constant Coefficients

To solve an equation of the form

$$ay'' + by' + cy = 0$$

where a, b, and c are constants, begin by assuming that a solution of the form $y = e^{rt}$ exists.

Differentiating, we get

$$y' = re^{rt} \text{ and } y'' = r^2 e^{rt}$$

Substituting into the original DE, we obtain

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

Homogeneous Equations with Constant Coefficients

$ay'' + by' + cy = 0$ Assume $y = e^{rt}$

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

Factoring yields

$$e^{rt}(ar^2 + br + c) = 0$$

and since the exponential factor is always nonzero, we obtain

$$ar^2 + br + c = 0$$

which is called the auxiliary (or characteristic) equation of the DE

The Auxiliary Equation

$ay'' + by' + cy = 0$ Assume $y = e^{rt}$

$$ar^2 + br + c = 0$$

When solving the auxiliary equation, we will encounter one of these three cases:

I. Two real, distinct roots
II. One real, repeated root
III. Two complex roots (which occur in a conjugate pair)

Example 2

Find the general solution of the differential equation

$$y'' + 3y' + 2y = 0$$

The Superposition Principle

If y_1 and y_2 are two solutions of a linear homogeneous differential equation, then the linear combination

$$y = C_1 y_1 + C_2 y_2$$

is also a solution for any values of the constants C_1 and C_2.

Example 3

Find the solution of the initial value problem

$$y'' + 8y' - 9y = 0, y(1)=1, y'(1)=0$$

Then, describe the behavior of the solution as $t \to \infty$.

The Determinant of a 2 × 2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A is

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 4

Compute the determinant.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Theorem

The system of equations

$$ax + by = f$$
$$cx + dy = g$$

will have a unique solution when

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

This is equivalent to saying the coefficient matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible.

Theorem

The initial value problem

$$ay'' + by' + cy = 0, y(t_0) = k_0, y'(t_0) = k_1$$

is guaranteed to have a unique solution for all t in $(-\infty, \infty)$.

The Wronskian

The Wronskian of two differentiable functions f and g is the function

$$W[f, g](t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$$

If $W[f, g](t) \neq 0$, then the functions f and g are linearly independent.

Example 3 (revisited)

When solving the initial value problem

$$y'' + 8y' - 9y = 0, y(1) = 1, y'(1) = 0$$

we obtained two solutions of the differential equation:

$$y_1 = e^{-9t}, y_2 = e^{t}$$

Calculate the Wronskian of these two solutions.

Are they linearly independent?

Theorem

The most general solution of the linear homogeneous DE

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$$

on an interval I is

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

where y_1 and y_2 are two linearly independent solutions of the equation on I.

Repeated Real Roots

Consider the DE $ay'' + by' + cy = 0$.

When solving the auxiliary equation $ar^2 + br + c = 0$ using the quadratic formula $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, if $b^2 - 4ac = 0$ we obtain only one (repeated) real root $r_1 = -\frac{b}{2a}$.

Thus, $y_1 = e^{r_1 t}$ is one solution of the DE.

It can be shown that $y_2 = te^{r_1 t}$ is also a solution.

Example 5

Find the general solution of the differential equation

$$4y'' - 12y' + 9y = 0$$

Example 6

Consider the DE $ay'' + by' + cy = 0$ where $b^2 - 4ac = 0$.

Thus, we obtain the repeated real root $r_1 = -\frac{b}{2a}$ and we know $y_1 = e^{r_1 t}$ is one solution of the DE.

Verify that $y_2 = te^{r_1 t}$ is also a solution.