Missouri University of Science and Technology — Founded 1870 | Rolla, Missouri

Section 4.6

Nonhomogeneous Equations:
Variation of Parameters

Nonhomogeneous Equations

Our goal in this section is to consider nonhomogeneous linear differential equations of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

assuming we can find (or are given) a set of linearly independent solutions $y_1(t)$ and $y_2(t)$ for the associated homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

Variation of Parameters

Consider the differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

If $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the associated homogeneous equation, then we begin by assuming that our particular solution of the nonhomogeneous equation has the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

Variation of Parameters – Derivation

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$
$$y_p{}'(t) = u_1{}'(t)y_1(t) + u_1(t)y_1{}'(t) + u_2{}'(t)y_2(t) + u_2(t)y_2{}'(t)$$
$$y_p{}'(t) = u_1(t)y_1{}'(t) + u_2(t)y_2{}'(t) + u_1{}'(t)y_1(t) + u_2{}'(t)y_2(t)$$

Assume

$$u_1{}'(t)y_1(t) + u_2{}'(t)y_2(t) = 0$$

Thus,

$$y_p{}'(t) = u_1(t)y_1{}'(t) + u_2(t)y_2{}'(t)$$

Variation of Parameters – Derivation

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$
$$y_p{}'(t) = u_1(t)y_1{}'(t) + u_2(t)y_2{}'(t)$$
$$y_p{}''(t) = u_1{}'(t)y_1{}'(t) + u_1(t)y_1{}''(t) + u_2{}'(t)y_2{}'(t) + u_2(t)y_2{}''(t)$$

Substituting into our nonhomogeneous equation, we get

$$(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = g$$

$$[u_1y_1'' + pu_1y_1' + qu_1y_1] + [u_2y_2'' + pu_2y_2' + qu_2y_2] + u_1'y_1' + u_2'y_2' = g$$

$$u_1[y_1'' + py_1' + qy_1] + u_2[y_2'' + py_2' + qy_2] + u_1'y_1' + u_2'y_2' = g$$

Variation of Parameters – Derivation

$$u_1[y_1'' + py_1' + qy_1] + u_2[y_2'' + py_2' + qy_2] + u_1'y_1' + u_2'y_2' = g$$

$$u_1[0] + u_2[0] + u_1'y_1' + u_2'y_2' = g$$

$$u_1{}'(t)y_1{}'(t) + u_2{}'(t)y_2{}'(t) = g(t)$$

Previously, we assumed

$$u_1{}'(t)y_1(t) + u_2{}'(t)y_2(t) = 0$$

Cramer's Rule

The system of linear equations

$$\begin{cases} ax + by = f \\ cx + dy = g \end{cases}$$

has solutions

$$x = \frac{\begin{vmatrix} f & b \\ g & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a & f \\ c & g \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

provided $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$

Variation of Parameters – Derivation

To solve the system

$$\begin{cases} u_1{}'y_1 + u_2{}'y_2 = 0 \\ u_1{}'y_1{}' + u_2{}'y_2{}' = g \end{cases}$$

use Cramer's Rule to get

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \text{ and } u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Variation of Parameters – Derivation

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \text{ and } u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Both denominators are simply the Wronskian W of y_1 and y_2.

Thus,

$$u_1' = \frac{-y_2 g}{W} \text{ and } u_2' = \frac{y_1 g}{W}$$

and we can integrate to find u_1 and u_2.

W_i

If W is the Wronskian of an appropriately sized set of linearly independent solutions, define W_i as the determinant which results from replacing the i^{th} column of W with the column

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Variation of Parameters

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If $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the associated homogeneous equation, then our particular solution of the nonhomogeneous equation has the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where

$$u_1 = \int \frac{g(t)W_1}{W} dt \text{ and } u_2 = \int \frac{g(t)W_2}{W} dt$$

Example 1

Find the general solution of

$$y'' + 4y = 3\csc 2t$$

on the interval $0 < t < \frac{\pi}{2}$

Example 2

Find the general solution of

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 3x^{3/2}\sin x$$

where $x > 0$ given the homogeneous solution

$$y_h(x) = C_1x^{-1/2}\sin x + C_2x^{-1/2}\cos x$$

Example 3 (if time)

Find the general solution of

$$x^2y'' - 3xy' + 4y = x^2\ln x$$

where $x > 0$ given the homogeneous solution

$$y_h(x) = C_1x^2 + C_2x^2\ln x$$