Missouri University of Science and Technology — Founded 1870 | Rolla, Missouri

Section 4.7

Variable Coefficient Equations

Variable Coefficient Equations

We have previously considered how to solve homogeneous constant coefficient linear ODEs of the form

$$ay'' + by' + cy = 0$$

where a, b, and c are constants.

We now wish to consider homogeneous linear ODEs with variable coefficients, which can be expressed in standard form as

$$y'' + p(t)y' + q(t)y = 0$$

Theorem

The standard form initial value problem

$$y'' + p(t)y' + q(t)y = g(t),\ y(t_0) = k_0,\ y'(t_0) = k_1$$

is guaranteed to have a unique solution on an interval I containing t_0 provided that p, q, and g are continuous on I.

Example 1

Determine the largest possible interval on which the IVP is guaranteed to have a unique solution.

$$y'' + 2y' + \frac{1}{t^2 - 4}y, \qquad y(0) = 5, y'(0) = 6$$

Cauchy-Euler Equations

A linear ODE of the form

$$at^2y'' + bty' + cy = 0$$

where a, b, and c are constants is a homogenous second-order Cauchy-Euler equation.

Solving Cauchy-Euler Equations

$$at^2y'' + bty' + cy = 0$$

Assume that at least one solution of the form

$$y = t^m \ (t > 0)$$

exists for some constant m. Then,

$$y' = mt^{m-1}$$
$$y'' = m(m-1)t^{m-2}$$

Substituting, we get

$$at^2m(m-1)t^{m-2} + btmt^{m-1} + ct^m = 0$$

Solving Cauchy-Euler Equations

$$at^2y'' + bty' + cy = 0$$

$$at^2m(m-1)t^{m-2} + btmt^{m-1} + ct^m = 0$$
$$t^m\,[am(m-1) + bm + c] = 0$$

Characteristic Equation:

$$am(m-1) + bm + c = 0$$

Solving Cauchy-Euler Equations

$$at^2y'' + bty' + cy = 0$$

Characteristic Equation:

$$am(m-1) + bm + c = 0$$

When solving the characteristic equation, we will encounter one of these three cases:

I. Two real, distinct roots
II. One real, repeated root
III. Two complex roots (which occur in a conjugate pair)

Cauchy-Euler: Distinct Real Roots

If we have two distinct real roots m_1 and m_2, we get two linearly independent solutions

$$y_1 = t^{m_1} \text{ and } y_2 = t^{m_2}$$

and a general solution of the form

$$y = C_1t^{m_1} + C_2t^{m_2}$$

Example 1

Find the general solution of the differential equation

$$t^2y'' - 4ty' + 4y = 0$$

Cauchy-Euler: Complex Roots

If we have two complex roots $m = \alpha \pm \beta i$, we get two complex solutions

$$z_1 = t^{\alpha+\beta i} \text{ and } z_2 = t^{\alpha-\beta i}$$

Unfortunately, we don't want complex solutions.
We want real solutions!

Real Solutions from Complex Solutions

$$\begin{aligned} z_1 &= t^{\alpha+\beta i} \\ &= t^{\alpha}t^{\beta i} \\ &= t^{\alpha}\left(e^{\ln t}\right)^{\beta i} \\ &= t^{\alpha}e^{\beta i \ln t} \\ &= t^{\alpha}[\cos(\beta \ln t) + i \sin(\beta \ln t)] \end{aligned}$$

Real Solutions:

$$y_1 = t^{\alpha}\cos(\beta \ln t) \text{ and } y_2 = t^{\alpha}\sin(\beta \ln t)$$

General Solution:

$$y = C_1t^{\alpha}\cos(\beta \ln t) + C_2t^{\alpha}\sin(\beta \ln t)$$

Example 2

Find the general solution of the differential equation

$$t^2y'' + 3ty' + 5y = 0$$

Reduction of Order

Note that two linearly independent solutions y_1 and y_2 cannot be constant multiples of each other on their interval of validity.

Thus, for any two linearly independent solutions y_1 and y_2, we have the relationship

$$y_2 = u(t)y_1$$

where $u(t)$ is a non-constant function.

Making this assumption about y_2 leads to a reduction of order.

Example 3

Find the general solution of the differential equation

$$x^2y'' + 5xy' + 4y = 0$$

for $x > 0$.

Cauchy-Euler: Repeated Real Roots

If we have a repeated real root m, we get the solution

$$y_1 = t^m$$

and use Reduction of Order to find a second linearly independent solution

$$y_2 = t^m \ln t$$

which then yields a general solution of the form

$$y = C_1 t^m + C_2 t^m \ln t$$

Example 4

Find the general solution of the differential equation

$$t^2 y'' - 2ty' + 2y = t^2$$

for $t > 0$.