S	MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY	Founded 1870 Rolla, Mesouri
	Section 4.7	
	Variable Coefficient Equations	

Variable Coefficient Equations

We have previously considered how to solve homogeneous constant coefficient linear ODEs of the form ay''+by'+cy=0

where a, b, and c are constants.

We now wish to consider homogeneous linear ODEs with variable coefficients, which can be expressed in standard form as y''+p(t)y'+q(t)y=0

Theorem

The standard form initial value problem $y''+p(t)y'+q(t)y=g(t), \ y(t_0)=k_0, \ y'(t_0)=k_1$ is guaranteed to have a unique solution on an interval I containing t_0 provided that p,q, and g are continuous on I.

Example 1

Determine the largest possible interval on which the IVP is guaranteed to have a unique solution.

$$y'' + 2y' + \frac{1}{t^2 - 4}y$$
, $y(0) = 5$, $y'(0) = 6$

Cauchy-Euler Equations

A linear ODE of the form

$$at^2y'' + bty' + cy = 0$$

where $a,\,b,\,{\rm and}\,\,c$ are constants is a homogenous second-order Cauchy-Euler equation.

Solving Cauchy-Euler Equations

$$at^2y'' + bty' + cy = 0$$

Assume that at least one solution of the form

$$y = t^m \, (t > 0)$$

exists for some constant m. Then,

$$y' = mt^{m-1}$$

 $y'' = m(m-1)t^{m-2}$

Substituting, we get

$$at^2m(m-1)t^{m-2} + btmt^{m-1} + ct^m = 0$$

Solving Cauchy-Euler Equations

$$at^2y'' + bty' + cy = 0$$

$$at^{2}m(m-1)t^{m-2} + btmt^{m-1} + ct^{m} = 0$$

 $t^{m} [am(m-1) + bm + c] = 0$

Characteristic Equation:

$$am(m-1) + bm + c = 0$$

Solving Cauchy-Euler Equations

$$at^2y'' + bty' + cy = 0$$

Characteristic Equation:

$$am(m-1) + bm + c = 0$$

When solving the characteristic equation, we will encounter one of these three cases:

- I. Two real, distinct roots
- II. One real, repeated root
- III. Two complex roots (which occur in a conjugate pair)

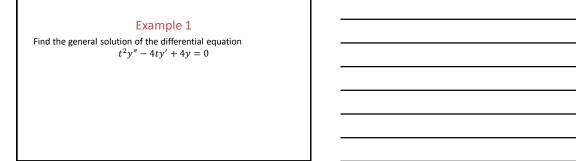
Cauchy-Euler: Distinct Real Roots

If we have two distinct real roots $m_{\rm 1}$ and $m_{\rm 2}$, we get two linearly independent solutions

$$y_1 = t^{m_1}$$
 and $y_2 = t^{m_2}$

and a general solution of the form

$$y=C_1t^{m_1}+C_2t^{m_2}$$



Cauchy-Euler: Complex Roots

If we have two complex roots $m=\alpha\pm\beta i$, we get two complex solutions

$$z_1 = t^{\alpha + \beta i}$$
 and $z_2 = t^{\alpha - \beta i}$

Unfortunately, we don't want complex solutions. We want real solutions!

Real Solutions from Complex Solutions

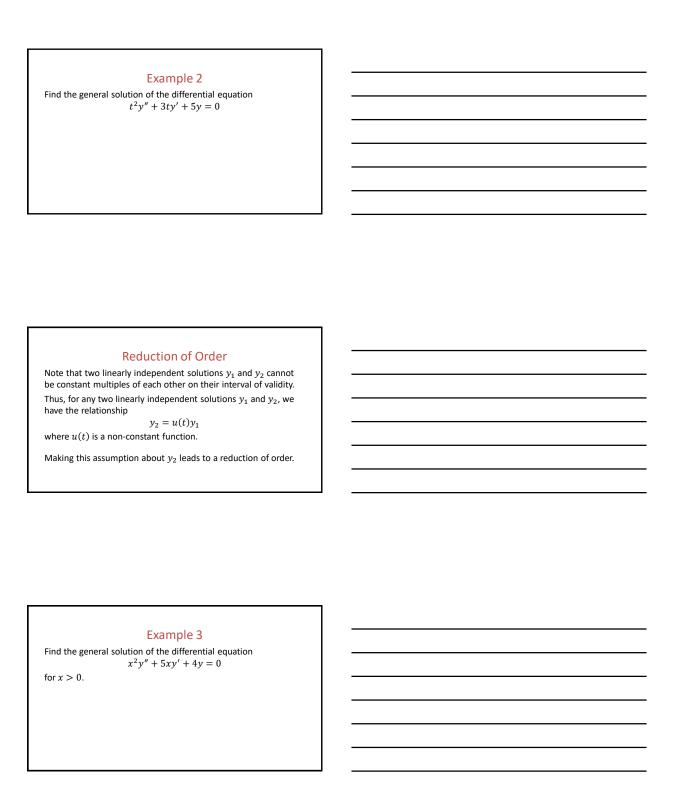
$$\begin{split} z_1 &= t^{\alpha+\beta i} \\ &= t^{\alpha} t^{\beta i} \\ &= t^{\alpha} \left(e^{\ln t}\right)^{\beta i} \\ &= t^{\alpha} e^{\beta i \ln t} \\ &= t^{\alpha} [\cos(\beta \ln t) + i \sin(\beta \ln t)] \end{split}$$

Real Solutions:

$$y_1 = t^{\alpha} \cos(\beta \ln t)$$
 and $y_2 = t^{\alpha} \sin(\beta \ln t)$

General Solution:

$$y = C_1 t^{\alpha} \cos(\beta \ln t) + C_2 t^{\alpha} \sin(\beta \ln t)$$



Cauchy-Euler: Repeated Real Roots

If we have a repeated real root m, we get the solution $y_1 = t^m \label{eq:y1}$

$$y_1 = t^n$$

and use Reduction of Order to find a second linearly independent solution

$$y_2 = t^m \ln t$$

which then yields a general solution of the form $y = C_1 t^m + C_2 t^m \ln t$

$$y = C_1 t^m + C_2 t^m \ln t$$

Example 4

Find the general solution of the differential equation $t^2y''-2ty'+2y=t^2 \label{eq:continuous}$

$$t^2 y'' - 2ty' + 2y = t^2$$

for t > 0.