Missouri University of Science and Technology — Founded 1870 | Rolla, Missouri

Section 4.9 with Section 4.1

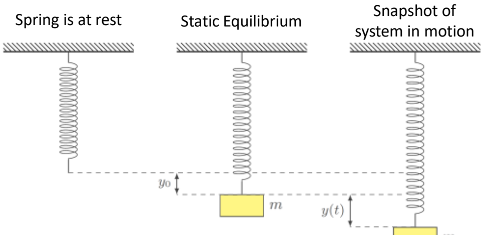
Spring-Mass Systems
Free Mechanical Vibrations

Spring-Mass Systems

Goal:
Use second-order linear differential equations to model spring-mass systems.

Basic Problem:
An object of mass m is attached to a vertical spring hanging from a rigid support. If we displace the object from its equilibrium position and/or push the object upwards or downwards, find the equation governing the vertical displacement $y(t)$ of the object from its equilibrium position at time t.

Spring-Mass Systems



Newton's Second Law

The sum of forces acting on an object equals the mass of the object times acceleration.

Symbolically,

$$\sum \mathbf{F} = m\mathbf{a}$$

Assumptions and Conventions

1. All motion of the object attached to the spring occurs along either a vertical or horizontal line. There is no swinging or twisting.
2. When working with vertical systems, the downward direction is positive.
3. We define zero as the position of the object in static equilibrium.

Assumptions and Conventions

4. The only forces acting on the object are:
   - The force due to gravity
   - The spring force
   - The damping force exerted by the surrounding medium as the object moves through it (e.g. air resistance or friction), unless we assume there is no damping
   - One or more time-dependent external forces, if such forces exist

Assumptions and Conventions

5. The force due to gravity is the positive force
$$F_{\text{grav}} = mg$$
6. The spring force is due to Hooke's Law:
The force exerted by the spring is proportional to its stretch (or compression) from its natural resting state, and it acts in the opposite direction from the stretch (or compression). Thus
$$F_{\text{spring}} = -k(y_0 + y)$$
where $k > 0$.

Assumptions and Conventions

7. The damping force is proportional to velocity and opposes the direction of motion. Thus,
$$F_{\text{damping}} = -\gamma y'$$
where $\gamma > 0$.
8. The external force, if any, is time dependent. Thus,
$$F_{\text{external}} = F(t)$$

Spring-Mass Systems

If we combine our assumptions and conventions with Newton's Second Law, we get
$$my'' = mg - k(y_0 + y) - \gamma y' + F(t)$$

Note that if the object is resting at static equilibrium, there is no velocity, acceleration, or external force. Thus,
$$m(0) = mg - k(y_0 + 0) - \gamma(0) + 0$$
$$0 = mg - ky_0$$
$$mg = ky_0$$

Spring-Mass Systems

If we combine our assumptions and conventions with Newton's Second Law, we get

$$my'' = mg - k(y_0 + y) - \gamma y' + F(t)$$

$$my'' = \underbrace{mg - ky_0}_{0} - ky - \gamma y' + F(t)$$

$$my'' = -ky - \gamma y' + F(t)$$

$$my'' + \gamma y' + ky = F(t)$$

Example 1

A 5 lb mass stretches a spring by $\frac{1}{3}$ ft. Find the spring constant k.

Units for the Spring Constant k

Standard: $\frac{\mathrm{lb}}{\mathrm{ft}}$

Metric: $\frac{\mathrm{N}}{\mathrm{m}}$ Recall: $1\ \mathrm{N} = 1\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}$

If we work with grams and centimeters, $1\ \mathrm{dyn} = 1\,\frac{\mathrm{g}\cdot\mathrm{cm}}{\mathrm{s}^2}$ and we would get a spring constant in $\frac{\mathrm{dyn}}{\mathrm{cm}}$.

Note: $1\ \mathrm{N} = 10^5\ \mathrm{dyn}$

Example 2

A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/s and there is no damping, determine the position $y(t)$ of the mass at any time t.

Free Undamped Motion

With free undamped motion, we are solving the equation

$$my'' + ky = 0$$

which has general solution

$$y = C_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

Smaller values of $\sqrt{\frac{k}{m}}$ (typically from larger masses) yield slower vibrations.

Larger values of $\sqrt{\frac{k}{m}}$ (typically from stiffer springs) yield faster vibrations.

Free Damped Motion

With free damped motion, we are solving the equation

$$my'' + \gamma y' + ky = 0, \qquad m, \gamma, k > 0$$

If we assume $y = e^{rt}$, we find

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Three cases:

I. Two distinct real roots

II. One real, repeated root

III. Two complex roots (which occur in a conjugate pair)

Free Damped Motion: Two Distinct Real Roots

If $\gamma^2 - 4mk > 0$, we get two real roots r_1 and r_2.

This yields the solution

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

and we note that $y(t)$ does not oscillate.

This is called overdamped motion.

Since r_1 and r_2 are both negative, $y(t) \to 0$ as $t \to \infty$.

Free Damped Motion: One Real Repeated Root

If $\gamma^2 - 4mk = 0$, we get the repeated real root $r_1 = \frac{-\gamma}{2m} < 0$.

This yields the solution

$$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$$

and we note that $y(t)$ does not oscillate.

This is called critically damped motion.

Since r_1 is negative, $y(t) \to 0$ as $t \to \infty$.

Free Damped Motion: Two Complex Roots

If $\gamma^2 - 4mk < 0$, we get the complex roots $r = \alpha \pm \beta i$.

This yields the solution

$$y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

and we note that $y(t)$ oscillates.

This is called underdamped motion.

If damping is very small ($\gamma \approx 0$), this solution is very similar to the solution of an undamped ($\gamma = 0$) system.

Free Damped Motion: Two Complex Roots

If $\gamma^2 - 4mk < 0$, we get the complex roots $r = \alpha \pm \beta i$.

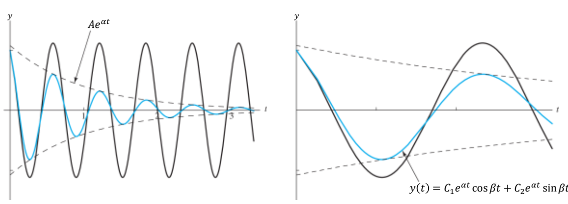
$$y(t) = C_1 e^{\alpha t} \cos \beta t + C_2 e^{\alpha t} \sin \beta t$$

The quasiperiod of the motion is

$$P = \frac{2\pi}{\beta}$$

and the quasifrequency of the motion is $\frac{1}{P}$

Quasiperiod and Quasifrequency



Example 3

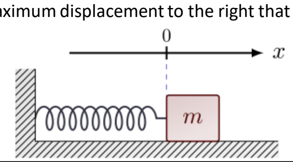
A mass weighing 16 pounds stretches a spring 3 inches. The mass is attached to a viscous damper with a damping constant of 2 pound-seconds per foot. If the mass is set in motion from its equilibrium position with a downward velocity of 3 inches per second, find and plot its position $y(t)$. Determine when the mass first returns to its equilibrium position. Finally, determine the quasiperiod.

Example 4

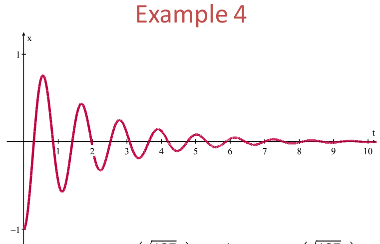
A $\frac{1}{4}$ kg mass is attached to a spring with stiffness 8 N/m.

The damping constant for the system is $\frac{1}{4}$ N-sec/m.

If the mass is moved 1 m to the left of equilibrium and released, what is the maximum displacement to the right that it will attain?



Example 4



$$x(t) = -e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{127}}{2}t\right) - \frac{1}{\sqrt{127}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{127}}{2}t\right)$$