Missouri University of Science and Technology — Founded 1870 | Rolla, Missouri

Section 6.1

Basic Theory of Higher-Order
Linear Differential Equations

Higher-Order Linear Differential Equations

Basic idea:

All the techniques we learned for second-order equations work essentially the same with higher-order equations.

Standard Form n^{th}-Order Linear Equations

The standard form of an n^{th}-order linear differential equation is

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t)$$

Theorem

The standard form initial value problem consisting of the differential equation

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t)$$

together with initial conditions

$$y(t_0) = k_0, y'(t_0) = k_1, \ldots, y^{(n-1)}(t_0) = k_{n-1}$$

is guaranteed to have a unique solution on an interval I containing t_0 provided that $a_{n-1}, \ldots, a_1, a_0$, and g are continuous on I.

Theorem

The most general solution of the linear homogeneous DE

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = 0$$

on an interval I is

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t)$$

where $y_1, y_2, \ldots$, and y_n are n linearly independent solutions of the equation on I.

We say that $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of the differential equation.

Recall: The Wronskian of Two Functions

The Wronskian of two differentiable functions f and g is the function

$$W[f,g](t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$$

If $W[f,g](t) \neq 0$, then the functions f and g are linearly independent.

The Wronskian of Three Functions

The Wronskian of three differentiable functions y_1, y_2, and y_3 is

$$W[y_1, y_2, y_3](t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{vmatrix}$$

If $W[y_1, y_2, y_3](t) \neq 0$, then the functions y_1, y_2, and y_3 are linearly independent.

The Determinant of a 3 × 3 Matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

The determinant of A is

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example 1

Compute the determinant.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix}$$

Determinants: An Alternate Approach

We can also calculate determinants using elementary row operations.

Key idea:

The determinant of a triangular matrix is the product of the entries on the main diagonal.

Elementary Row Operations

Replacement:

Replace one row by the sum of itself and a multiple of another row.

Replacement does not change the value of the determinant.

Interchange:

Interchange (swap) two rows.

Swapping two rows negates the determinant.

Scaling:

Multiply all entries in a row by a nonzero constant.

This one is usually not needed for determinants.

Example 1 (revisited)

Compute the determinant using row operations.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{vmatrix}$$

Example 2

Verify that $y_1 = 1$, $y_2 = t$, $y_3 = e^{-t}$, and $y_4 = te^{-t}$ form a fundamental set of solutions of the differential equation

$$y^{(4)} + 2y''' + y'' = 0$$

Higher-Order Nonhomogeneous Linear Equations

If we encounter a higher-order nonhomogeneous linear equation, we begin by finding the general solution y_h of the associated homogeneous equation and a particular solution y_p of the nonhomogeneous equation. Then, by superposition, the general solution is

$$y = y_h + y_p$$