Missouri University of Science and Technology

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Section 7.6

Transforms of Discontinuous Functions

Goals for Today

We have previously used Laplace transforms to solve IVPs with nice forcing functions.

Today, we want to consider discontinuous forcing functions.

The Unit Step Function

The unit step function $u(t)$ is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

The unit step function is often called the Heaviside function.

The unit step function can also be shifted horizontally, yielding

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

The Unit Step Function – Alternate Notation

In some texts, $u(t-a)$ is instead represented as $u_a(t)$.

Example 1

Sketch a graph of $u(t-a)$ where $a > 0$.

Example 2

Rewrite

$$g(t) = \begin{cases} 0 & t < a \\ 1 & a < t < b \\ 0 & t > b \end{cases}$$

using unit step functions.

The Rectangular Window Function

The rectangular window function $\Pi_{a,b}(t)$ is defined as

$$\Pi_{a,b}(t) = u(t-a) - u(t-b) = \begin{cases} 0 & t < a \\ 1 & a < t < b \\ 0 & t > b \end{cases}$$

Switches

To turn a function on at a and leave it on, use the "on switch"

$$u(t-a)$$

To turn a function on at a and off at b, use the "on-off switch"

$$\Pi_{a,b}(t) = u(t-a) - u(t-b)$$

To turn a function off at b (provided it was on since time began), use the "off switch"

$$1 - u(t-b)$$

If time begins at $t = 0$, the off switch can be written equivalently as

$$\Pi_{0,b}(t)$$

Example 3

Rewrite the function

$$g(t) = \begin{cases} \sin \pi t & t < 1 \\ \cos \pi t & t > 1 \end{cases}$$

in terms of window and step functions.

Example 4

Rewrite the function

$$f(t) = \begin{cases} 4 & t < 1 \\ t^2 & 1 < t < 2 \\ 3t & 2 < t < 4 \\ 2 & t > 4 \end{cases}$$

in terms of window and step functions.

Laplace Transforms of Unit Step Functions

If $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

provided $a > 0$.

Using this result, we see

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}, \qquad a > 0$$

and

$$\mathcal{L}\{\Pi_{a,b}(t)\} = \frac{1}{s}\left(e^{-as} - e^{-bs}\right), \qquad b > a > 0$$

Example 5

Solve the initial value problem

$$y'' + y = g(t)$$

provided $y(0) = 0$, $y'(0) = -1$, and

$$g(t) = \begin{cases} 1 & t < \pi \\ 0 & t > \pi \end{cases}$$

Example 6

Solve the initial value problem

$$y'' - 5y' + 6y = g(t)$$

provided $y(0) = 0, y'(0) = 0$, and

$$g(t) = \begin{cases} 4e^t & t < 2 \\ 0 & t > 2 \end{cases}$$

Express the final solution without using unit step functions or rectangular window functions.

Example 7

Compute the Laplace transform of

$$g(t) = \begin{cases} \sin t & t < \pi \\ 0 & t > \pi \end{cases}$$

Challenge Problem (on your own)

Solve the initial value problem

$$y'' + y' + \frac{5}{4}y = g(t)$$

provided $y(0) = 0, y'(0) = 0$, and

$$g(t) = \begin{cases} \sin t & t < \pi \\ 0 & t > \pi \end{cases}$$