# Section 7.8

Convolution

## Goals for Today

We need to consider some very specific inverse Laplace transform cases.

We also need to consider how to solve integral equations and integrodifferential equations.

## The Convolution Product

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$. The convolution of $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

Note that

$$(f * g)(t) = (g * f)(t) = \int_0^t g(t-\tau)f(\tau)d\tau$$

## Example 1

Compute the convolution product of $f(t) = t$ and $g(t) = t^2$.

## The Convolution Theorem

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$ and of exponential order $\alpha$. If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ both exist for $s > a \geq 0$, then

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

and, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

## Example 2

Use the Convolution Theorem to compute $\mathcal{L}\{(f * g)(t)\}$ given $f(t) = t$ and $g(t) = t^2$.

Then, use the result of Example 1 to verify your answer.

## Example 3

Find $\mathcal{L}^{-1}\{H(s)\}$ where $H(s) = \frac{k^2}{(s^2+k^2)^2}$

## Example 4

Use the Convolution Theorem to compute

$$\mathcal{L}\left\{\int_0^t \tau e^{t-\tau} d\tau\right\}$$

## Example 5

Solve the integral equation

$$y(t) + \int_0^t (t-\tau)y(\tau)d\tau = 1$$

## Example 6

Solve the integrodifferential equation

$$y'(t) + 3\int_0^t y(t-\tau)e^{-4\tau}d\tau = \sin t\,, \qquad y(0) = 0$$

## Example 7

Use the convolution theorem to obtain a formula for the solution to the initial value problem

$$y''' + 4y' = f(t), \qquad y(0) = 1, y'(0) = y''(0) = 0$$

Assume $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order.