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Section 7.9

Impulses and the Dirac Delta

Goals for Today

Now that we have considered IVPs with discontinuous forcing functions, we need to consider IVPs with impulsive forcing.

Impulsive Forces

An impulsive force is a relatively large force which acts over a relatively short time interval.

Note that we may not know exactly how large the force is or exactly how short the time interval is.

Impulse

The integral

$$I_g(\tau) = \int_{-\tau}^{\tau} g(t)dt$$

is called the impulse of the force g over the time interval $-\tau < t < \tau$.

Example 1

Consider the force defined by

$$g_n(t) = \begin{cases} \frac{n}{2} & -\frac{1}{n} < t < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

where n is a positive integer.

a) Sketch $g_n(t)$ for $n = 1, 2, 4$, and 8.

b) Calculate the impulse of $g_n(t)$ over the time interval $-\frac{1}{n} < t < \frac{1}{n}$.

The Dirac Delta

For each bounded continuous function f on $(-\infty, \infty)$, define

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)g_n(t)dt$$

δ is called the Dirac delta or the unit impulse.

Note that δ is not a function.

The Dirac Delta – Rewriting the Definition

Since $g_n(t)$ is zero outside $-\frac{1}{n} < t < \frac{1}{n}$, we can rewrite as follows:

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)g_n(t)dt$$

$$= \lim_{n\to\infty} \int_{-1/n}^{1/n} f(t)g_n(t)dt$$

$$= \lim_{n\to\infty} \int_{-1/n}^{1/n} f(t)\frac{n}{2}dt$$

$$= \lim_{n\to\infty} \frac{1}{\frac{2}{n}} \int_{-1/n}^{1/n} f(t)dt$$

This limit computes the average value of $f(t)$ over the interval $-\frac{1}{n} < t < \frac{1}{n}$ as that interval becomes increasingly small.

Thus, this limit converges to $f(0)$.

The Sifting Property of the Dirac Delta

For each bounded continuous function f on $(-\infty, \infty)$, define

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)g_n(t)dt = \lim_{n\to\infty} \frac{1}{\frac{2}{n}} \int_{-1/n}^{1/n} f(t)dt = f(0)$$

This is the sifting property of the Dirac delta.

If we translate the Dirac delta horizontally by a, we obtain the more generalized version of the sifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

The Laplace Transform of the Dirac Delta

Recall that the Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)dt$$

Thus, the Laplace transform of the Dirac delta is

$$\mathcal{L}\{\delta(t-a)\} = \int_{0}^{\infty} e^{-st}\delta(t-a)dt = e^{-as}$$

Similarly,

$$\mathcal{L}\{f(t)\delta(t-a)\} = \int_{0}^{\infty} e^{-st} f(t)\delta(t-a)dt = f(a)e^{-as}$$

Example 2

Find the solution to the initial value problem

$$2y'' + 4y' + 10y = \delta(t-5), \qquad y(0) = y'(0) = 0$$

Example 3

Find the solution to the initial value problem

$$y'' + 4y = \delta\left(t - \frac{\pi}{4}\right)\sin t\,, \qquad y(0) = 1, y'(0) = 0$$

Example 4 (if time)

Find the solution to the initial value problem

$$y'' + y = \delta\left(t - \frac{\pi}{2}\right) + \delta(t - 2\pi), \qquad y(0) = 1, y'(0) = 0$$