Exam 1 Review Sheet

Math 3304

Section 1.1 - Background

Problem 1. For each of the following differential equations, give the order, if it is linear or nonlinear, and if it is homogeneous or nonhomogeneous, or say that the term in not applicable.

$$4y'' - 3\sin(t)y' + 8t^2 = 0$$

$$yy' + y^{(4)}\sec(t) + 12t = 0$$

$$8 + \frac{t}{t+1}y' + y\sqrt{t} = 0$$

$$t\frac{d^2}{dt^2}\left[\frac{dy}{dt} + ty\right] = 4y$$

Problem 2. For the following ordinary differential equations, give the order, if it is linear or nonlinear, and if it is homogeneous.

$$tu'' + t^2 - \sin(t)u' = 0$$

$$y' + y(1-y) = 0$$

$$\sqrt{\tan(x^4)}y' + x - y = 0$$

$$u\frac{d^3u}{dx^3} = e^x\frac{du}{dx}$$

Problem 3. Determine the order of each differential equation and state whether it is linear or nonlinear.

$$(1-x)y'' - 4xy' + 5y = \cos(x)$$

$$\ln(x)\frac{d^3y}{dx^3} - (\frac{dy}{dx})^4 + y = 0$$

$$\sin(t)y'' - cos(t)y' = y$$

$$\frac{d^2y}{dx^2} = \sqrt{\frac{dy}{dx}}$$

$$y''' + ty' + y\cos(y) = 0$$

Problem 4. For the following Differential Equations find the order, whether it is linear or nonlinear, and if it is homogeneous.

1. $\frac{d^2y}{dt^2} + \sin(t + y) = e^t$
2. $t^2y^{(5)} + \cos(t)y - t^3 = 0$
3. $\frac{d^3y}{dt^3} + \frac{1}{y} = 0$
4. $e^u u''' + e^{-t}u = 1 + t^2$

Problem 5. For the following Differential Equations, identify if it is ordinary or partial, give the order, whether it is linear or nonlinear, and if it is homogeneous or nonhomogeneous. If the term homogeneous does not apply to the differential equation, write NA.

1. $y''' + ty' + (\cos^2(t))y = 0$
2. $\theta'' + \sin(\theta) = 0$
3. $sx''' + x' + sx = s^3$
4. $xy'' + 2y' + x(y^2 - 1)^{3/2} = 0$

Problem 6. Classify each differential equation by completing the columns in the following table. For each nonlinear differential equation identify the term which makes it nonlinear.

Table for identification of characteristics of Differential Equations

Differential Equation	Order	Linear (Y/N)	Nonlinear Term
$x^{(5)} = 2y^3x' - 6x$			
$v'' = \sqrt{uv^2 - 1} - (1 + u)v$			
$y(p')^2 + ln(y)p = 0$			
$(rcos(r))w' - w = 3$			

Problem 7. Classify each differential equation by completing the columns in the following table.

Differential Equation	Order	Linear (Y/N)
$aa'' + a^3 = 1$		
$v''\text{sin}t + v\text{cos}t + 1 = 0$		
$y(x) = x\frac{dy}{dx} + \frac{dy}{dx}$		
$y^{(2025)} + y = 0$		

Differential Equation	Order	Linear (Y/N)
$z(y)z'(y) = z''(y)$		

Problem 8. Classify each differential equation by completing the columns in the following table.

Differential Equation	Order?	Linear? (Y or N)
$x^2 - 7xy' + 15y = 0$		
$\left(\frac{dy}{dx}\right)^2 - xy = 1$		
$\sin(t^4)y'(t) + \cos(t^2)y''(t) = \sin(2t)$		
$(1+2y)\frac{d^3y}{dt^3} + t^5\frac{dy}{dt} = 0$		
$\left(\frac{dz}{dx}\right)^5 + \frac{d^4z}{dx^4} + z\cos^2(x) = x^{1/2}$		

Problem 9. Classify each differential equation by completing the columns in the following table.

Differential Equation	Order?	Linear? (Y or N)
$y''' + y'' + e^x = 0$		
$y'' + y' = \sin(t+y)$		
$t^2y'' + 2ty' + 4y = t^3$		
$\frac{d^3z}{dx^3} + z\cos^2(x) = \sqrt{x}$		
$(1+g^2)\frac{d^2g}{dt^2} = g + t^2$		

Section 1.2 - Solutions and Initial Value Problems

Problem 10. Determine the longest interval in which the given initial value problem is certain to have a unique solution

$$(x-3)y'' + (lnx)y' - tan(x)y = sin(x), y(2) = 1, y'(2) = 3$$

Problem 11. Find the largest open interval I in which the following IVP has a unique solution

$$ty'' + \frac{2}{t-5}y' + 3y = t$$

Problem 12. Determine for which values of m the function $f(x) = x^m$ is a solution to $x^2y'' + 4xy' - 18y = 0$ for $x > 0$.

Section 1.3 - Direction Fields

Problem 13. Draw the direction field for the ODE

$$y' = 2y - 1$$

Section 2.2 - Separable Equations

Problem 14. Consider the initial value problem

$$ty' - y = t + \frac{1}{t}, y(1) = 0$$

a) Solve the initial value problem.
b) Determine the largest interval on which the solution is defined.
c) Determine the behavior of the solution as t becomes very large.

Problem 15. Find the explicit solution to the initial value problem

$$y' - ty^2 = t, y(0) = 1$$

Problem 16. Solve the initial value problem

$$e^t y' - \left(\frac{1 + e^t}{y + 1}\right) y = 0, y(0) = 1$$

Problem 17. Find the explicit solution of the initial value problem

$$y' = \frac{x^2 + 5x + 3}{2(y + 1)}$$

Problem 18. Find an explicit solution to the initial value problem

$$x\frac{dy}{dx} = \frac{1}{y^3}, y(1) = 1$$

Problem 19. Find the explicit solution of the differential equation

$$y' = \sqrt{1 - y}$$

Problem 20. Solve the initial value problem

$$y' = te^{t-y}, \quad y(0) = 0$$

Problem 21. Find the explicit solution of the initial value problem

$$y' = \frac{2t}{y + t^2 y}, \qquad y(0) = -2$$

Problem 22. Find the explicit solution of the initial value problem

$$y' = \frac{1}{y-2} + y - 2$$

Problem 23. Find the explicit solution of

$$y' = (t + y - 1)^2$$

hint: you can change variables by letting $v = t + y - 1$

Problem 24. Find the explicit solution of the initial value problem:

$$e^t + yy' = 0, \quad y(0) = 2$$

Problem 25. Find the explicit solution of the initial value problem:

$$e^t + yy' = 0, \quad y(0) = 2$$

Problem 26. Find the explicit solution of the initial value problem:

$$\frac{dy}{dt} = 12t^5 e^{-y}, \quad y(1) = 0$$

Problem 27. Find the explicit solution of the initial value problem:

$$yy' = e^t, \quad y(0) = 1$$

Problem 28. Find the explicit solution of the initial value problem:

$$y' = \frac{t}{y + t^2 y}, \quad y(0) = -2$$

Problem 29. Find the explicit solution of the initial value problem:

$$(1 + t^2)y' - 2ty^2 = 0, \quad y(0) = -2$$

Section 2.3 - Linear Equations

Problem 30. Solve the initial value problem.

$$ty' = y + 2t^2, y(2) = 10$$

Problem 31. Find the general solution on the given domain.

$$(20 + t)y' + 2y = \frac{3}{2}(20 + t), t > 0$$

Problem 32. Find the general solution to the differential equation

$$y'' - 3y' = 0$$

Problem 33. Find the general solution to the differential equation

$$y'' - 2y' + 5y = 0$$

Problem 34. Find the general solution to the differential equation

$$y'' + 6y' + 9y = 0$$

Problem 35. Solve the initial value problem

$$4t^2 - ty' = 2y, y(1) = 2$$

Problem 36. Find the general solution of the differential equation

$$y' = \frac{1 + xy}{x^2}, x > 0$$

Problem 37. Find the general solution of the differential equation.

$$t\frac{dy}{dt} + 2y = t^{-3}, t > 0$$

Problem 38. Solve the initial value problem $ty' = y + t^3 \sin(t), \quad y(\pi) = 0$

Problem 39. Solve the initial value problem $ty' = y + 2t^2 \sin(2t), t > 0$

Problem 40. Find the value of y_0 for which the solution of the initial value problem

$$ty' - y = t^2 e^{-t}, \quad y(1) = y_0$$

approaches zero as t goes to infinity.

Problem 41. Find the general solution to the differential equation: $ty' - 2y = t^4 cos(t)$

Problem 42 (10). Find the general solution of

$$y'' + 4y' + 3y = 0$$

Problem 43. Find the explicit general solution of the differential equation:

$$ty' - 2y = t^3 e^{-t} - 4$$

Problem 44. Find the explicit general solution of the differential equation:

$$ty' - 2y = t^3 e^{-t} - 4$$

Problem 45. Find the explicit general solution of the differential equation:

$$y' + 4x^{-1}y = x^{-8}$$

Problem 46. Find the explicit general solution of the differential equation:

$$y' - 2t^{-1}y = t^2e^{-t}$$

Problem 47. Solve the differential equation:

$$y' - \frac{1}{t}y = te^{-t}$$

on the interval $(0, \infty)$.

Problem 48. Find the general solution to the equation:

$$x\frac{dy}{dx} + 3(y + x^2) = \frac{\sin x}{x}, \quad x > 0$$

Section 3.2 - Compartmental Analysis

Problem 49. A population of insects in a certain region has a daily birth rate that equals the square of the current population. Assume that the population's daily death rate is triple the current insect population. On any given day, there is a net migration into the region of 2 million insects. If there are half a million insects initially, write but DO NOT SOLVE an initial value problem which models the population of insects in the region at any time t > 0.

Problem 50. A 1000-gallon tank originally holds 300 gallons of water solution containing 100 pounds of salt. Then water containing 2 pounds of salt per gallon is poured into the tank at a rate of 5 gallons per minute, and the well-stirred mixture is allowed to leave the tank at a rate of 3 gallons per minute.
(a) How long will it take before the tank begins to overflow?
(b) Set up, BUT DO NOT SOLVE, an initial value problem that models the amount of salt in the tank at all times prior to the moment when the tank overflows.

Problem 51. Let P(t) denote the population of fish in a certain lake at time t. Suppose the birth rate of the fish is twice the current fish population and the death rate of the fish equals the square of the current fish population. Also suppose that the fish are harvested at a constant rate h. Write down, but DO NOT SOLVE, a differential equation that models the fish population P(t).

Problem 52. A tank initially contains 200 liters of pure water. A mixture of salt and water containing 10 grams per liter of salt enters the tank at a rate of 3 liters per minute, and the well-stirred mixture leaves the tank at a rate of 4 liters per minute. Set up, but DO NOT SOLVE, an initial value problem that models the amount A(t) of salt in the tank at any time t.

Problem 53. A 1000 gallon tank originally holds 200 gallons of water solution containing 50 pounds of salt. A water solution containing 2 pounds of salt per gallon is poured into the tank at a rate of 4 gallons per minule. The well-stirred mixture is allowed to drain from the tank at the rate of 2 gallons per minute.
a) How long will it take for the tank to begin to overflow?

b) Set up but DO NOT SOLVE an initial value problem which models the amount A(t) of salt in the tank at time t for the all values of t prior to the moment when the tank overflows.

Problem 54. A 500 gallon tank originally contains 200 gallons of pure water. Then water containing two pounds of salt per gallon is poured into the tank at a rate of four gallons per minute, and the well-stirred mixture leaves at a rate of five gallons per minute. Write but do NOT solve, an initial value problem for the amount Q(t) of salt in the tank at time t.

Problem 55. A tank initially contains 10 gallons of water in which 2 pounds of salt is dissolved. A mixture containing 3 pounds of salt per gallon of water is pumped into the tank at a rate of 5 gallons per minute. The well-mixed solution is pumped out at a rate of 3 gallons per minute.
(a) Write but DO NOT SOLVE an initial value problem for the amount of salt, Q(t), in the tank at time t.
(b) If the volume of the tank is 200 gallons, find the time t when the tank becomes full.

Problem 56. A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical. Water containing 0.01g/gal of chemical flows into the pond at a rate of 300gal/h, and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond. Let Q(t) be the amount of the chemical in the pond at time t. Write but do not solve an Initial Value Problem for Q(t).

Problem 57. One theory of epidemic spread postulates that the time rate of change in the infected population is proportional to the product of the number of individuals who have the disease with the number of disease free individuals. Assuming that the population of mice in a certain meadow has a stable value of one thousand. Use the theory of epidemic spread to write, but do not solve, an initial value problem that models the number $N(t)$ of infected mice at time $t > 0$ if ten mice were initially infected

Problem 58. A brine solution of salt flows at a constant rate of 10 gallons per minute into a large tank that initially held 1,000 gallons of pure water. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 gallons per minute. The concentration entering the tank is 0.5 pounds per gallon. Write, BUT DO NOT SOLVE, an initial value problem to model the amount $A(t)$ of salt (in pounds) in the tank at time t (in minutes).

Problem 59. The initial mass of a certain species of fish is 10 million tons. The mass of fish, if left alone, would increase by 4 times the current mass; however, commercial fishing removes fish mass at a constant rate of 13 million tons per year. Write, BUT DO NOT SOLVE, an initial value problem to model the mass $M(t)$ of fish (in million tons) at time t (in years).

Section 4.2 - Homogeneous Linear Equations: The General Solution

Problem 60. Find the general solution of the differential equation.

$$y'' - 5\pi^2 y = 0$$

Problem 61. Find the general solution of the differential equation.

$$y'' + y' + y = 0$$

Problem 62. Solve the initial value problem

$$y'' - 4y' + 4y = 0, y(0) = 5, y\left(\frac{\ln(5)}{2}\right) = 0$$

Problem 63. Find the general solution of the differential equation.

$$y'' + 2y' - y = 0$$

Problem 64. Find the general solution of the differential equation.

$$y'' + 2y' + 4y = 0$$

Problem 65. Find the general solution of

$$y'' + 4y' + y = 0$$

Problem 66. Find the general solution of

$$4y'' + 12y' + 9y = 0$$

Problem 67. Find the General solution of

$$y'' + 2y' + y = 0$$

and describe the behavior as

$$t \to \infty$$

Problem 68. Find the General solution of

$$y'' + 4y' + 4y = 0$$

and describe the behavior as

$$t \to \infty$$

Problem 69. Obtain a general solution of the following differential equation:

$$\frac{1}{t}y' + y = e^{t^2/2}$$

Problem 70. Find the general solution of the differential equation:

$$y'' + 3y' + y = 0$$

Problem 71. Find the explicit general solution of the differential equation:

$$ty' - 2y = t^3 e^{-t} - 4$$

Problem 72. Find the general solution of the differential equation:

$$y'' - 4y' - 5y = 0$$

Problem 73. Find the general solution of the differential equation:

$$y'' - 5y' + 6y = 0$$

Problem 74. Find the general solution of the differential equation:

$$y'' + 6y' + 9y = 0$$

Problem 75. Find the general solution of the differential equation:

$$y'' - 2y' = 0$$

Problem 76. Find the general solution of the differential equation:

$$y'' - 6y' + 9y = 0$$

Problem 77. Find the exact solution to the initial value problem:

$$y'' + 2y' - 8y = 0, \quad y(0) = 3, \quad y'(0) = -6$$

Section 4.3 - Auxiliary Equations with Complex Roots

Problem 78. Find the general solution of

$$y'' + 2y' + 4y = 0$$

Problem 79. Find the General solution of

$$y'' + 4y' + 5y = 0$$

and describe the behavior as

$$t \to \infty$$

Problem 80. Find the general solution of the following differential equation:

$$y'' + 2y' + 3y = 0$$

Problem 81. Find the general solution of

$$2y'' + 2y' + y = 0$$

Problem 82. Find the general solution of the differential equation:

$$y'' + 36y = 0$$

Problem 83. Find the general solution of the differential equation:

$$y'' + 3y' + y = 0$$

Problem 84 (10). Solve the initial value problem

$$y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Problem 85. Find the general solution of the differential equation:

$$y'' + 36y = 0$$

Problem 86. Find the general solution of the differential equation:

$$y'' + 36y = 0$$

Problem 87. Find the general solution of the differential equation:

$$y'' + 2y' + 5y = 0$$

Problem 88. Find the general solution of the differential equation:

$$y'' + 25y = 0$$

Problem 89. Find the general solution to the given differential equation:

$$y'' + 4y' + 5y = 0$$

Section 4.4 - Nonhomogeneous Equations: The Method of Undetermined Coefficients

Problem 90. Find the general solution of the differential equation

$$y''' - 3y' + 2y = 2e^{3t}$$

The homogeneous solution is given at the end of the solution, if you prefer. It may not be given on an exam.

Problem 91. Find the solution of the initial value problem:

$$y'' - 2y' + y = te^{t} + 4, \quad y(0) = 1, \quad y'(0) = 1$$

Problem 92. Find the general solution to the differential equation:

$$3y'' + y' - 2y = 2\cos(x)$$

Section 4.5 - The Superposition Principle and Undetermined Coefficients Revisited

Problem 93. Given that $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions for

$$y'' - 5y' + 6y = 0$$

Use the method of undetermined coefficients to solve the differential equation

$$y'' - 5y + 6y = 3e^{2t}$$

Problem 94. Use the method of undetermined coefficients to solve the IVP

$$y'' - 6y' + 9y = 4e^{t}, \quad y(0) = 0, \quad y'(0) = 2$$

Exam 1 Review Sheet

Math 3304

Section 1.1 - Background

Problem 1. For each of the following differential equations, give the order, if it is linear or nonlinear, and if it is homogeneous or nonhomogeneous, or say that the term in not applicable.

$$4y'' - 3\sin(t)y' + 8t^2 = 0$$

$$yy' + y^{(4)}\sec(t) + 12t = 0$$

$$8 + \frac{t}{t+1}y' + y\sqrt{t} = 0$$

$$t\frac{d^2}{dt^2}\left[\frac{dy}{dt} + ty\right] = 4y$$

Solution 1.

1. second order, linear, nonhomogeneous
2. 4th order, nonlinear, homogeneity does not apply
3. first order, linear, nonhomogeneous
4. For this one, just distribute the second derivative across the sum. Then it is clear that the equation is third order, linear, and homogeneous.

Problem 2. For the following ordinary differential equations, give the order, if it is linear or nonlinear, and if it is homogeneous.

$$tu'' + t^2 - \sin(t)u' = 0$$

$$y' + y(1-y) = 0$$

$$\sqrt{\tan(x^4)}y' + x - y = 0$$

$$u\frac{d^3u}{dx^3} = e^x\frac{du}{dx}$$

Solution 2.

1. 2nd order, linear, nonhomogeneous

2. 1st order, nonlinear, homogeneity only applies to linear DEs
3. 1st order, linear, nonhomogeneous
4. 3rd order, nonlinear, homogeneity only applies to linear DEs

Problem 3. Determine the order of each differential equation and state whether it is linear or nonlinear.

$$(1-x)y'' - 4xy' + 5y = \cos(x)$$

$$\ln(x)\frac{d^3y}{dx^3} - (\frac{dy}{dx})^4 + y = 0$$

$$\sin(t)y'' - cos(t)y' = y$$

$$\frac{d^2y}{dx^2} = \sqrt{\frac{dy}{dx}}$$

$$y''' + ty' + y\cos(y) = 0$$

Solution 3.

1. 2nd order, linear
2. 3rd order, nonlinear
3. 2nd order, linear
4. 2nd order, nonlinear
5. 3rd order, nonlinear

Problem 4. For the following Differential Equations find the order, whether it is linear or nonlinear, and if it is homogeneous.

1. $\frac{d^2y}{dt^2} + \sin(t+y) = e^t$
2. $t^2y^{(5)} + \cos(t)y - t^3 = 0$
3. $\frac{d^3y}{dt^3} + \frac{1}{y} = 0$
4. $e^u u''' + e^{-t}u = 1 + t^2$

Solution 4.

1. 2nd order, Nonlinear, N/A
2. 5th order, Linear, Nonhomogeneous

1. 3rd order, Nonlinear, N/A
2. 3rd order, Nonlinear, N/A

Problem 5. For the following Differential Equations, identify if it is ordinary or partial, give the order, whether it is linear or nonlinear, and if it is homogeneous or nonhomogeneous. If the term homogeneous does not apply to the differential equation, write NA.

1. $y''' + ty' + (\cos^2(t))y = 0$
2. $\theta'' + \sin(\theta) = 0$
3. $sx''' + x' + sx = s^3$
4. $xy'' + 2y' + x(y^2 - 1)^{3/2} = 0$

Solution 5.

1. Ordinary, 3rd order, linear, homogeneous
2. Ordinary, 2nd order, nonlinear, NA
3. Ordinary, 3rd order, linear, nonhomogeneous
4. Ordinary, 2nd order, nonlinear, NA

Problem 6. Classify each differential equation by completing the columns in the following table. For each nonlinear differential equation identify the term which makes it nonlinear.

Table for identification of characteristics of Differential Equations

Differential Equation	Order	Linear (Y/N)	Nonlinear Term
$x^{(}5) = 2y^3x' - 6x$			
$v'' = \sqrt{uv^2 - 1} - (1 + u)v$			
$y(p')^2 + ln(y)p = 0$			
$(rcos(r))w' - w = 3$			

Solution 6.

Table for identification of characteristics of Differential Equations

Differential Equation	Order	Linear (Y/N)	Nonlinear Term
$x^{(}5) = 2y^3x' - 6x$	5	Y	-
$v'' = \sqrt{uv^2 - 1} - (1 + u)v$	2	N	$\sqrt{uv^2 - 1}$
$y(p')^2 + ln(y)p = 0$	1	N	$(p')^2$
$(rcos(r))w' - w = 3$	1	Y	-

Problem 7. Classify each differential equation by completing the columns in the following table.

Differential Equation	Order	Linear (Y/N)
$aa'' + a^3 = 1$		
$v''\sin t + v\cos t + 1 = 0$		
$y(x) = x\dfrac{dy}{dx} + \dfrac{dy}{dx}$		
$y^{(2025)} + y = 0$		
$z(y)z'(y) = z''(y)$		

Solution 7. Below is the classification for each differential equation:

Differential Equation	Order	Linear (Y/N)
$aa'' + a^3 = 1$	2	N
$v''\sin t + v\cos t + 1 = 0$	2	Y
$y(x) = x\dfrac{dy}{dx} + \dfrac{dy}{dx}$	1	Y
$y^{(2025)} + y = 0$	2025	Y
$z(y)z'(y) = z''(y)$	2	N

- **Row 1:** Nonlinear because of the term aa'' (product of dependent variable and its derivative) and a^3.
- **Row 2:** Linear; coefficients are functions of the independent variable t only.
- **Row 3:** Linear; the equation can be rewritten as $y = (x + 1)y'$.
- **Row 4:** Linear; the 2025^{th} derivative is just a high order.
- **Row 5:** Nonlinear because of the product $z(y)z'(y)$.

Problem 8. Classify each differential equation by completing the columns in the following table.

Differential Equation	Order?	Linear? (Y or N)
$x^2 - 7xy' + 15y = 0$		
$\left(\dfrac{dy}{dx}\right)^2 - xy = 1$		
$\sin(t^4)y'(t) + \cos(t^2)y''(t) = \sin(2t)$		
$(1 + 2y)\dfrac{d^3y}{dt^3} + t^5\dfrac{dy}{dt} = 0$		

Differential Equation	Order?	Linear? (Y or N)
$\left(\frac{dz}{dx}\right)^5 + \frac{d^4z}{dx^4} + z\cos^2(x) = x^{1/2}$		

Solution 8. Below is the classification for each differential equation:

Differential Equation	Order?	Linear? (Y or N)
$x^2 - 7xy' + 15y = 0$	1st	Yes
$\left(\frac{dy}{dx}\right)^2 - xy = 1$	1st	No
$\sin(t^4)y'(t) + \cos(t^2)y''(t) = \sin(2t)$	2nd	Yes
$(1+2y)\frac{d^3y}{dt^3} + t^5\frac{dy}{dt} = 0$	3rd	No
$\left(\frac{dz}{dx}\right)^5 + \frac{d^4z}{dx^4} + z\cos^2(x) = x^{1/2}$	4th	No

- **Row 1:** 1st order because the highest derivative is y'. It is linear because y and y' appear only to the first power and are not multiplied together.
- **Row 2:** 1st order. It is nonlinear because the derivative $\frac{dy}{dx}$ is squared.
- **Row 3:** 2nd order because of $y''(t)$. It is linear; even though the coefficients are complicated functions of t, the dependent variable y and its derivatives are linear.
- **Row 4:** 3rd order. It is nonlinear because of the term $(1+2y)\frac{d^3y}{dt^3}$, which involves a product of the dependent variable y and its derivative.
- **Row 5:** 4th order. It is nonlinear because the first derivative $\frac{dz}{dx}$ is raised to the 5th power.

Problem 9. Classify each differential equation by completing the columns in the following table.

Differential Equation	Order?	Linear? (Y or N)
$y''' + y'' + e^x = 0$		
$y'' + y' = \sin(t+y)$		
$t^2y'' + 2ty' + 4y = t^3$		
$\frac{d^3z}{dx^3} + z\cos^2(x) = \sqrt{x}$		
$(1+g^2)\frac{d^2g}{dt^2} = g + t^2$		

Solution 9. Below is the classification for each differential equation:

Differential Equation	Order?	Linear? (Y or N)
$y''' + y'' + e^x = 0$	3rd	Y
$y'' + y' = \sin(t + y)$	2nd	N
$t^2y'' + 2ty' + 4y = t^3$	2nd	Y
$\frac{d^3z}{dx^3} + z\cos^2(x) = \sqrt{x}$	3rd	Y
$(1 + g^2)\frac{d^2g}{dt^2} = g + t^2$	2nd	N

- **Row 1:** 3rd order because the highest derivative is y'''. It is linear because y and its derivatives are not raised to a power or multiplied together.
- **Row 2:** 2nd order. It is nonlinear because the dependent variable y is inside a sine function ($\sin(t + y)$).
- **Row 3:** 2nd order. It is linear; the coefficients depend only on the independent variable t.
- **Row 4:** 3rd order. It is linear because the dependent variable z and its derivatives are all to the first power.
- **Row 5:** 2nd order. It is nonlinear because the term $(1 + g^2)$ multiplies a derivative, meaning the dependent variable g is multiplied by its own derivative.

Section 1.2 - Solutions and Initial Value Problems

Problem 10. Determine the longest interval in which the given initial value problem is certain to have a unique solution

$$(x - 3)y'' + (lnx)y' - tan(x)y = sin(x), y(2) = 1, y'(2) = 3$$

Solution 10. We begin by writing the equation in standard form.

$$y'' + \frac{lnx}{x - 3}y' - \frac{tanx}{x - 3}y = \frac{sinx}{x - 3}$$

We check for the what values make the coefficient functions are undefined.

$$\frac{lnx}{x - 3}: x \leq 0, x = 3$$
$$\frac{tanx}{x - 3}: x = \pm\frac{(2n + 1)\pi}{2}, x = 3$$
$$\frac{sinx}{x - 3}: x = 3$$

These values CANNOT be in our interval. This leads to the possible intervals:

$$(0, \frac{pi}{2}), (\frac{\pi}{2}, 3), (3, \frac{3\pi}{2}), (\frac{3\pi}{2}, \frac{5\pi}{2}), \cdots$$

We have our initial conditions at $x = 2$. Thus, the longest interval is

$$(\frac{\pi}{2}, 3)$$

Problem 11. Find the largest open interval I in which the following IVP has a unique solution

$$ty'' + \frac{2}{t-5}y' + 3y = t$$

Solution 11. Getting the ODE in standard form yields:

$$y'' + \frac{2}{t(t-5)}y' + \frac{3}{t}y = 1$$

By looking at the coefficient functions, they are undefined when: $t = 0$, and $t = 5$. As we have the initial condition at $t = 1$, then the largest interval is:

$$I = (0,5)$$

Problem 12. Determine for which values of m the function $f(x) = x^m$ is a solution to $x^2y'' + 4xy' - 18y = 0$ for $x > 0$.

Solution 12. To find the values of m, we substitute the function $y = x^m$ and its derivatives into the given differential equation.

1. Find the Derivatives:

$$y = x^m$$
$$y' = mx^{m-1}$$
$$y'' = m(m-1)x^{m-2}$$

2. Substitute into the Differential Equation:

$$x^2(m(m-1)x^{m-2}) + 4x(mx^{m-1}) - 18(x^m) = 0$$

Simplify each term using exponent rules ($x^a \cdot x^b = x^{a+b}$):

$$m(m-1)x^m + 4mx^m - 18x^m = 0$$

3. Solve the Characteristic Equation: Since $x > 0$, we can factor out x^m and divide both sides by it:

$$x^m[m(m-1) + 4m - 18] = 0$$
$$m^2 - m + 4m - 18 = 0$$
$$m^2 + 3m - 18 = 0$$

Now, we factor the quadratic equation:

$$(m+6)(m-3)=0$$

This gives us two possible values for m:

$$m=-6 \quad \text{and} \quad m=3$$

Final Answer: The values of m are $m=3,-6$.

Section 1.3 - Direction Fields

Problem 13. Draw the direction field for the ODE

$$y'=2y-1$$

Solution 13. Please use software to compare your solutions. You can use maple, python, Matlab, Bluffton slope and direction fields to solve this. Remember many times we will use software to solve these types of problems.

Section 2.2 - Separable Equations

Problem 14. Consider the initial value problem

$$ty'-y=t+\frac{1}{t}, y(1)=0$$

a) Solve the initial value problem.
b) Determine the largest interval on which the solution is defined.
c) Determine the behavior of the solution as t becomes very large.

Solution 14. a) The equation is 1st order, linear, so we want to find an integrating factor. We can rearrange the equation into standard form.

$$y'+(-\frac{1}{t})y=1+t^{-2}$$

Then to find an integrating factor, we write

$$p(t)=-\frac{1}{t}$$

$$\int p(t)dt=-\int \frac{1}{t}dt=-\ln|t|=\ln(|t|^{-1})$$

$$\mu(t)=e^{\ln(|t|^{-1})}=|t|^{-1}$$

And we must be careful to avoid the case t = 0. Thus, the intervals possible intervals on which the solution is defined are

$$(-\infty, 0), (0, \infty)$$

Since the interval must contain 1 due to the initial condition, we choose the second interval, where t is positive. Then we can further simplify the integrating factor as

$$\mu(t) = t^{-1}$$

Multiply both sides by the integrating factor and proceed.

$$t^{-1}y' + (-t^{-2})y = t^{-1} + t^{-3}$$
$$\frac{d}{dt}[t^{-1}y] = t^{-1} + t^{-3}$$
$$t^{-1}y = \int (t^{-1} + t^{-3})dt$$
$$t^{-1}y = \ln(t) - \frac{1}{2}t^{-2} + C$$
$$y = t\ln(t) - \frac{1}{2}t^{-1} + Ct$$

To find the constant, apply the initial condition.

$$0 = 1\ln(1) - \frac{1}{2}(1)^{-1} + C$$
$$\Rightarrow \frac{1}{2} = C$$

Then the solution to the initial value problem is

$$y(t) = t\ln(t) - \frac{1}{2}t^{-1} + \frac{1}{2}t$$

b) In our work in part a, we have seen that the largest interval where the solution is defined is

$$(0, \infty)$$

c) To find the behavior as t becomes large, we take limits.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(t\ln(t) - \frac{1}{2}t^{-1} + \frac{1}{2}t\right)$$
$$\lim_{t\to\infty}(t\ln(t)) = \infty$$
$$\lim_{t\to\infty}\left(\frac{1}{2}t^{-1}\right) = 0$$
$$\lim_{t\to\infty}\left(\frac{1}{2}t\right) = \infty$$
$$\Rightarrow \lim_{t\to\infty} y(t) = \infty$$

So y grows without bound as t increases.

Problem 15. Find the explicit solution to the initial value problem

$$y' - ty^2 = t, y(0) = 1$$

Solution 15. The differential equation is first order and nonlinear, so we must use separation of variables. In order to get the equation in the correct form, write

$$\begin{aligned} y' &= t + ty^2 \\ y' &= t(1 + y^2) \\ \frac{1}{1 + y^2} y' &= t \\ \Rightarrow \int \frac{1}{1 + y^2} dy &= \int t dt \end{aligned}$$

And now we need to evaluate these integrals. The integral on the right is immediate.

$$\int t dt = \frac{1}{2} t^2 + C$$

The integral on the left uses trigonometric substitution, specifically

$$y = \tan(\theta)$$

. Then we use these facts:

$$dy = \sec^2(\theta) d\theta$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

And write

$$\int \frac{1}{1 + y^2} dy = \int \frac{\sec^2(\theta)}{\sec^2(\theta)} d\theta = \int d\theta = \theta + C$$

We then just note from our substitution that

$$\theta = \tan^{-1}(y)$$

Then we may finally write

$$\tan^{-1}(y) = \frac{1}{2} t^2 + C$$

To find the constant, we may apply the initial condition.

$$\begin{aligned} \tan^{-1}(1) &= \frac{1}{2}(0^2) + C \\ \frac{\pi}{4} &= C \end{aligned}$$

Then the implicit solution is

$$\tan^{-1}(y) = \frac{1}{2} t^2 + \frac{\pi}{4}$$

Since we are asked to find the explicit solution, take the tangent of both sides, yielding

$$y = \tan\left(\frac{1}{2}t^2 + \frac{\pi}{4}\right)$$

And this is the general solution.

Problem 16. Solve the initial value problem

$$e^t y' - \left(\frac{1+e^t}{y+1}\right) y = 0, y(0) = 1$$

Solution 16. Note that the equation is first order, nonlinear, so it must be that we have to use separation of variables. To get the equation into the correct form, write

$$e^t y' = \frac{y(1+e^t)}{y+1}$$
$$\frac{y+1}{y} dy = \frac{1+e^t}{e^t} dt$$

Then we integrate both sides.

$$\int \left(1 + \frac{1}{y}\right) dy = \int (e^{-t} + 1) dt$$
$$y + \ln|y| = t - e^{-t} + C$$

Note that we have

$$y(0) = 1$$

as an initial condition and thus we can drop the absolute value in the above equation. Rewriting, we have

$$y + \ln(y) = t - e^{-t} + C$$

Then plug in the initial condition to find the constant.

$$1 + \ln 1 = 0 - e^0 + C 1 + 0 = -1 + C C = 2$$

Then the implicit solution to the initial value problem is

$$y + \ln(y) = t - e^{-t} + 2$$

Problem 17. Find the explicit solution of the initial value problem

$$y' = \frac{x^2 + 5x + 3}{2(y+1)}$$

Solution 17. Note the equation is separable and multiply both sides by

$$2(y+1)$$

We have

$$2(y+1)dy = (x^2+5x+3)dx$$
$$\int (2y+2)dy = \int (x^2+5x+3)dx$$
$$y^2+2y = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 3x + C$$

Plug in the initial condition. You can find the explicit formula first, but it is worse. Using

$$y(0) = -2$$
$$(-2)^2 + 2(-2) = \frac{1}{3}(0)^3 + \frac{5}{2}(0)^2 + 3(0) + C$$
$$0 = C$$

Then the implicit solution is

$$y^2+2y = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 3x$$

To find the explicit formula, we can complete the square.

$$(y+1)^2 - 1 = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 3x$$
$$y+1 = \pm\sqrt{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 3x + 1}$$
$$y = -\sqrt{\frac{1}{3}x^3 + \frac{5}{2}x^2 + 3x + 1} - 1$$

Note that above we have used the negative version after taking square roots to satisfy the initial condition.

Problem 18. Find an explicit solution to the initial value problem

$$x\frac{dy}{dx} = \frac{1}{y^3}, y(1) = 1$$

Solution 18.

$$y^3 dy = \frac{1}{x} dx$$
$$\int y^3 dy = \int \frac{1}{x} dx$$
$$\frac{1}{4} y^4 = \ln|x| + C$$
$$y^4 = 4\ln|x| + C$$
$$y^4 = ln|x|^4 + C$$
$$y^4 = ln(x^4) + C$$
$$y = \sqrt[4]{\ln(x^4) + C}$$
$$y(1) = 1 = \sqrt[4]{\ln(1^4) + C} \Longrightarrow C = 1$$
$$y = \sqrt[4]{\ln(x^4) + 1}$$

Problem 19. Find the explicit solution of the differential equation

$$y' = \sqrt{1 - y}$$

Solution 19.

$$\frac{dy}{dx} = \sqrt{1 - y}$$
$$\frac{dy}{\sqrt{1 - y}} = dx$$
$$\int \frac{dy}{\sqrt{1 - y}} = \int dx$$

By performing a u-substitution on the left hand side we have:

$$u = 1 - y \quad \Rightarrow \quad du = -dy \quad \Rightarrow \quad dy = -du$$

$$\int \frac{-du}{\sqrt{u}} = \int dx$$
$$-2\sqrt{u} = x + C$$
$$2\sqrt{1 - y} = -x + C$$
$$\sqrt{1 - y} = \frac{C - x}{2}$$
$$1 - y = \left(\frac{C - x}{2}\right)^2$$
$$y(x) = 1 - \left(\frac{C - x}{2}\right)^2$$

Also, note that $y(x) = 1$ is also a solution to the differential equation. This is called a singular solution to the differential equation.

Problem 20. Solve the initial value problem

$$y' = te^{t-y}, \quad y(0) = 0$$

Solution 20. This is a first order, nonlinear ODE.

$$\frac{dy}{dt} = te^{t}e^{-y}$$
$$e^{y}dy = te^{t}dt$$
$$\int e^{y}dy = \int te^{t}dt$$
$$e^{y} = te^{t} - e^{t} + c$$

Using the initial conditions:

$$e^{0} = 0 - 1 + c \Rightarrow c = 1 + 1 = 2$$

Thus, the solution is:

$$e^{y} = te^{t} - e^{t} + 2$$
$$y = \ln(te^{t} - e^{t} + 2)$$

Note: Either of these solutions is acceptable.

Problem 21. Find the explicit solution of the initial value problem

$$y' = \frac{2t}{y + t^{2}y}, \qquad y(0) = -2$$

Solution 21.

$$y' = \frac{2t}{y(1 + t^{2})}$$
$$\frac{dy}{dt} = \frac{2t}{y(1 + t^{2})}$$
$$ydy = \frac{2t}{1 + t^{2}}dt$$
$$\int ydt = \int \frac{2t}{1 + t^{2}}, dt$$
$$\frac{1}{2}y^{2} = \ln(1 + t^{2}) + C$$
$$y^{2} = 2\ln(1 + t^{2}) + C_1$$
$$y = \pm\sqrt{2\ln(1 + t^{2}) + C_1}$$

Using our initial conditions, we find:

$$y(0) = -2 \Rightarrow -2 = \pm\sqrt{2\ln(1) + C_1}$$
$$\Rightarrow -2 = \pm\sqrt{C_1}$$

(Since the square root is always positive we need the negative square root.)

$$\Rightarrow -2 = -\sqrt{C_1}$$
$$\Rightarrow C_1 = 4$$

Leading to:

$$y(t) = -\sqrt{2\ln(1 + t^2) + 4}$$

Problem 22. Find the explicit solution of the initial value problem

$$y' = \frac{1}{y-2} + y - 2$$

Solution 22. The differential equation is nonlinear and we will use separation of variables. First we rewrite the equation:

$$\frac{dy}{dt} = \frac{1}{y-2} + y - 2$$
$$= \frac{1}{y-2} + \frac{(y-2)^2}{y-2}$$
$$= \frac{1 + (y-2)^2}{y-2}$$

Now, we separate the variables so we have the y's on one side and the t's on the other.

$$\frac{y-2}{1 + (y-2)^2} dy = dt$$
$$\int \frac{y-2}{1 + (y-2)^2} dy = \int dt$$

On the left hand side we need to make a u-sub.

$$u = 1 + (y-2)^2$$
$$du = 2(y-2)dy$$
$$\Rightarrow \frac{1}{2} du = (y-2)dy$$

Integrating yields:

$$
\begin{aligned}
\frac{1}{2}\int \frac{1}{u}du &= t + C \\
\frac{1}{2}ln|u| &= t + C \\
\frac{1}{2}ln|1 + (y - 2)^2| &= t + C \\
ln|1 + (y - 2)^2| &= 2t + C \\
|1 + (y - 2)^2| &= e^{2t+C} \\
&= e^{C}e^{2t} \\
\Rightarrow 1 + (y - 2)^2 &= \pm e^{C}e^{2t} \\
1 + (y - 2)^2 &= C_0e^{2t} \qquad C_0 \text{is an arbitrary constant}
\end{aligned}
$$

Using the initial conditions, $y(0) = -1$, we have:

$$1 + (-1 - 2)^2 = C_0e^{2(0)} \Rightarrow C_0 = 10$$

So, we have:

$$
\begin{aligned}
1 + (y - 2)^2 &= 10e^{2t} \\
y - 2 &= \pm\sqrt{10e^{2t} - 1} \\
y &= 2 \pm \sqrt{10e^{2t} - 1}
\end{aligned}
$$

We need to take the negative case to satisfy the initial condition. Thus,

$$y = 2 - \sqrt{10e^{2t} - 1}$$

Problem 23. Find the explicit solution of

$$y' = (t + y - 1)^2$$

hint: you can change variables by letting $v = t + y - 1$

Solution 23. We use the substitution $v = t + y - 1$, taking the derivative of this gives $v' = 1 + y'$. Then, the original differential equation is equivalent to:

$$v'(t) - 1 = (v(t))^2$$

Then, we obtain:

$$
\begin{aligned}
v' &= v^2 + 1 \\
\frac{dv}{dt} &= v^2 + 1 \\
\frac{1}{v^2 + 1}dv &= dt
\end{aligned}
$$

Integrating yields:

$$arctan(v) = t + C$$
$$v = tan(t + C)$$

Substituting back in terms of y yields:

$$t + y - 1 = tan(t + C) \Rightarrow y(t) = 1 - t + tan(t + C)$$

where C is an arbitrary constant.

Problem 24. Find the explicit solution of the initial value problem:

$$e^t + yy' = 0, \quad y(0) = 2$$

Solution 24. We use **Separation of Variables** to solve this equation:

$$yy' = -e^t$$
$$y\frac{dy}{dt} = -e^t$$
$$y\,dy = -e^t\,dt$$

Now, integrate both sides:

$$\int y\,dy = -\int e^t\,dt$$
$$\frac{y^2}{2} = -e^t + C$$

Multiply by 2 and solve for $y(t)$ to get the general implicit form:

$$y^2 = -2e^t + C \quad \text{(where } 2C \text{ is renamed } C\text{)}$$
$$y(t) = \pm\sqrt{-2e^t + C}$$

Next, we apply the initial condition $y(0) = 2$:

- Since $y(0) = 2$ is positive, we choose the positive root: $y(t) = \sqrt{-2e^t + C}$
- Substitute $t = 0$ and $y = 2$:

$$2 = \sqrt{-2e^0 + C}$$
$$2 = \sqrt{-2(1) + C}$$
$$2 = \sqrt{-2 + C}$$

- Square both sides to solve for C:

$$4 = -2 + C$$
$$C = 6$$

Explicit Solution:

$$y(t) = \sqrt{-2e^t + 6}$$

Problem 25. Find the explicit solution of the initial value problem:

$$e^t + yy' = 0, \quad y(0) = 2$$

Solution 25. We use **Separation of Variables** to solve this equation:

$$yy' = -e^t$$
$$y\frac{dy}{dt} = -e^t$$
$$y\,dy = -e^t\,dt$$

Now, integrate both sides:

$$\int y\,dy = -\int e^t\,dt$$
$$\frac{y^2}{2} = -e^t + C$$

Multiply by 2 and solve for $y(t)$ to get the general implicit form:

$$y^2 = -2e^t + C \quad \text{(where } 2C \text{ is renamed } C\text{)}$$
$$y(t) = \pm\sqrt{-2e^t + C}$$

Next, we apply the initial condition $y(0) = 2$:

- Since $y(0) = 2$ is positive, we choose the positive root: $y(t) = \sqrt{-2e^t + C}$
- Substitute $t = 0$ and $y = 2$:

$$2 = \sqrt{-2e^0 + C}$$
$$2 = \sqrt{-2(1) + C}$$
$$2 = \sqrt{-2 + C}$$

- Square both sides to solve for C:

$$4 = -2 + C$$
$$C = 6$$

Explicit Solution:

$$y(t) = \sqrt{-2e^t + 6}$$

Problem 26. Find the explicit solution of the initial value problem:

$$\frac{dy}{dt} = 12t^5 e^{-y}, \quad y(1) = 0$$

Solution 26. This is a first-order differential equation. We can solve it using the method of **Separation of Variables**.

Separate the variables: Multiply both sides by e^y and dt to group the y terms and t terms:

$$e^y\,dy = 12t^5\,dt$$

Integrate both sides:

$$\int e^y\,dy = \int 12t^5\,dt$$

$$e^y = 2t^6 + C$$

Solve for y explicitly: Take the natural logarithm of both sides to isolate y:

$$y(t) = \ln(2t^6 + C)$$

Apply the initial condition $y(1) = 0$: Substitute $t = 1$ and $y = 0$ into the equation to solve for C:

$$0 = \ln(2(1)^6 + C)$$
$$0 = \ln(2 + C)$$

Recall that $\ln(x) = 0$ when $x = 1$. Therefore:

$$2 + C = 1$$
$$C = -1$$

Explicit Solution: Substitute $C = -1$ back into the explicit form:

$$y(t) = \ln(2t^6 - 1)$$

Problem 27. Find the explicit solution of the initial value problem:

$$yy' = e^t, \quad y(0) = 1$$

Solution 27. We solve this differential equation using the method of **Separation of Variables**.

Separate the variables: Rewrite y' as $\frac{dy}{dt}$ and move all y terms to one side and t terms to the other:

$$y\frac{dy}{dt} = e^t$$

$$y\,dy = e^t\,dt$$

Integrate both sides:

$$\int y\,dy = \int e^t\,dt$$

$$\frac{y^2}{2} = e^t + C$$

Apply the initial condition $y(0) = 1$: Substitute $t = 0$ and $y = 1$ to solve for C:

$$\frac{(1)^2}{2} = e^0 + C$$
$$\frac{1}{2} = 1 + C$$
$$C = \frac{1}{2} - 1 = -\frac{1}{2}$$

Solve for y explicitly: Substitute $C = -\frac{1}{2}$ back into the equation:

$$\frac{y^2}{2} = e^t - \frac{1}{2}$$

Multiply the entire equation by 2 to isolate y^2:

$$y^2 = 2e^t - 1$$

$$y(t) = \pm\sqrt{2e^t - 1}$$

Since the initial condition $y(0) = 1$ is positive, we must choose the positive root.

Explicit Solution:

$$y(t) = \sqrt{2e^t - 1}$$

Problem 28. Find the explicit solution of the initial value problem:

$$y' = \frac{t}{y + t^2 y}, \quad y(0) = -2$$

Solution 28. This is a first-order differential equation. We can solve it by using the method of **Separation of Variables**.

1. Separate the Variables: First, we factor the denominator on the right side:

$$\frac{dy}{dt} = \frac{t}{y(1 + t^2)}$$

Now, we move all y terms to the left and all t terms to the right:

$$y\,dy = \frac{t}{1 + t^2}\,dt$$

2. Integrate Both Sides:

$$\int y\,dy = \int \frac{t}{1 + t^2}\,dt$$

For the left side, the integral is simple power rule. For the right side, we use a u-substitution where $u = 1 + t^2$ and $du = 2t\,dt$ (or $\frac{1}{2}du = t\,dt$):

$$\frac{y^2}{2} = \frac{1}{2}\int \frac{1}{u}\,du$$
$$\frac{y^2}{2} = \frac{1}{2}\ln|1+t^2| + C_1$$

Multiply the entire equation by 2 to simplify:

$$y^2 = \ln(1+t^2) + C$$

(Note: $1+t^2$ is always positive, so we can drop the absolute value bars. $C = 2C_1$ is our arbitrary constant.)

3. Solve for y: Take the square root of both sides:

$$y(t) = \pm\sqrt{\ln(1+t^2) + C}$$

4. Apply the Initial Condition $y(0) = -2$: Since the initial value $y(0) = -2$ is negative, we must choose the **negative root**:

$$-2 = -\sqrt{\ln(1+0^2) + C}$$

$$-2 = -\sqrt{\ln(1) + C}$$

$$-2 = -\sqrt{0 + C}$$

Squaring both sides gives:

$$4 = C$$

Explicit Solution: Substitute $C = 4$ back into our equation with the negative root:

$$y(t) = -\sqrt{\ln(1+t^2) + 4}$$

Problem 29. Find the explicit solution of the initial value problem:

$$(1+t^2)y' - 2ty^2 = 0, \quad y(0) = -2$$

Solution 29. This is a first-order nonlinear differential equation. We can solve it using the method of **Separation of Variables**.

1. Separate the Variables: First, we rearrange the equation to isolate the derivatives:

$$(1+t^2)\frac{dy}{dt} = 2ty^2$$

Now, we move all y terms to one side and all t terms to the other side:

$$\frac{1}{y^2}\,dy = \frac{2t}{1+t^2}\,dt$$

2. Integrate Both Sides:

$$\int y^{-2}\,dy = \int \frac{2t}{1+t^2}\,dt$$

For the left side, we use the power rule. For the right side, we use a u-substitution where $u = 1 + t^2$ and $du = 2t\,dt$:

$$-\frac{1}{y} = \ln|1+t^2| + C$$
$$-\frac{1}{y} = \ln(1+t^2) + C$$

(Since $1 + t^2$ is always positive, we can drop the absolute value bars.)

3. Solve for y explicitly: Multiply by -1 and take the reciprocal:

$$\frac{1}{y} = -(\ln(1+t^2) + C)$$

$$y(t) = \frac{-1}{\ln(1+t^2) + C}$$

4. Apply the Initial Condition $y(0) = -2$: Substitute $t = 0$ and $y = -2$ into the equation to find C:

$$-2 = \frac{-1}{\ln(1+0^2) + C}$$
$$-2 = \frac{-1}{0 + C}$$
$$-2 = -\frac{1}{C}$$

Solving for C, we get $2C = 1$, so $C = \frac{1}{2}$.

Explicit Solution: Substitute $C = \frac{1}{2}$ back into the explicit form:

$$y(t) = \frac{-1}{\ln(1+t^2) + \frac{1}{2}}$$

Section 2.3 - Linear Equations

Problem 30. Solve the initial value problem.

$$ty' = y + 2t^2, y(2) = 10$$

Solution 30. The differential equation is first order and linear, so we want to try to find an integrating factor. We can rewrite in standard form, which gives

$$y' + (-\frac{1}{t})y = 2t$$

Be careful here. In standard form we must have an addition sign on the left side. To find the integrating factor, write

$$p(t) = -\frac{1}{t}$$
$$\int p(t) = \int -\frac{1}{t}dt = -\int \frac{1}{t}dt$$
$$= -\ln|t| = \ln(|t|^{-1})$$

Then the integrating factor is

$$\mu(t) = e^{\ln(|t|^{-1})} = |t|^{-1}$$

We can assume that t > 0 since our initial condition is t = 2. Then our integrating factor is

$$\mu(t) = t^{-1}$$

Multiply both sides by the integrating factor and proceed.

$$t^{-}1y' - t^{-2}y = 2$$
$$\frac{d}{dt}[t^{-1}y] = 2$$
$$t^{-1}y = \int 2dt$$
$$t^{-1}y = 2t + C$$
$$y = 2t^2 + Ct$$

To find the constant, apply the initial condition.

$$10 = 2(2^2) + C(2)$$
$$10 = 8 + 2C$$
$$2 = 2C \Longrightarrow C = 1$$

Then the solution to the initial value problem is

$$y = 2t^2 + t$$

Problem 31. Find the general solution on the given domain.

$$(20 + t)y' + 2y = \frac{3}{2}(20 + t), t > 0$$

Solution 31. The differential equation is first order, linear, so we want to try an integrating factor. First we rewrite in standard form.

$$y' + \frac{2}{20 + t}y = \frac{3}{2}$$

Note that we have subtly used that t is positive to ensure that we do not divide by zero here. To find the integrating factor, note we have

$$p(t) = \frac{2}{20+t}$$

Then we evaluate

$$\begin{aligned} \int p(t)dt &= \int \frac{2}{20+t}dt \\ &= 2\ln|20+t| \\ &= \ln(|20+t|^2) = \ln((20+t)^2) \end{aligned}$$

Then the integrating factor is

$$\mu(t) = e^{\ln((20+t)^2)} = (20+t)^2$$

Multiplying both sides by the integrating factor gives

$$\begin{aligned} (20+t)^2 y' + 2(20+y)y &= \frac{3}{2}(20+t)^2 \\ \frac{d}{dt}[(20+t)^2 y] &= \frac{3}{2}(20+t)^2 \\ (20+t)^2 y &= \int \frac{3}{2}(20+t)^2 dt \\ (20+t)^2 y &= \frac{1}{2}(20+t)^3 + C \\ y &= \frac{1}{2}(20+t) + \frac{C}{(20+t)^2} \end{aligned}$$

Problem 32. Find the general solution to the differential equation

$$y'' - 3y' = 0$$

Solution 32. The ODE is second order and linear, so we write the characteristic equation and solve for roots.

$$\begin{aligned} r^2 - 3r &= 0 \\ r(r-3) &= 0 \\ \Rightarrow r_1 &= 0, r_2 = 3 \end{aligned}$$

Then the general solution is given by

$$y = C_1 e^{0t} + C_2 e^{3t} = C_1 + C_2 e^{3t}$$

Problem 33. Find the general solution to the differential equation

$$y'' - 2y' + 5y = 0$$

Solution 33. The ODE is second order and linear, so we write the characteristic equation and solve for roots.

$$r^2 - 2r + 5 = 0$$
$$\Rightarrow r = 1 \pm 2i$$

Note we have found the roots above by using the quadratic formula. We then can immediately write the general solution

$$y = C_1 e^t \cos(2t) + C_2 e^t \sin(2t)$$

Problem 34. Find the general solution to the differential equation

$$y'' + 6y' + 9y = 0$$

Solution 34. The ODE is second order and linear, so we write the characteristic equation and solve for roots.

$$r^2 + 6r + 9 = 0$$
$$(r+3)(r+3) = 0$$
$$\Rightarrow r_1 = -3, r_2 = -3$$

And we have the case of repeated roots. Then we immediately write the general solution

$$y = C_1 e^{-3t} + C_2 t e^{-3t}$$

Problem 35. Solve the initial value problem

$$4t^2 - ty' = 2y, y(1) = 2$$

Solution 35. Note that this is a first order, linear, differential equation. Thus we choose the method of integrating factors. To write the equation in standard form, write

$$-ty' = 2y - 4t^2$$
$$y' + \frac{2}{t}y = 4t$$

The integrating factor is given by

$$\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln t} = t^2$$

Multiplying both sides by the integrating factor and continue.

$$t^2y' + 2ty = 4t^3$$
$$\frac{d}{dt}[t^2y] = 4t^3$$
$$t^2y = \int 4t^3 dt$$
$$t^2y = t^4 + C$$
$$y = t^2 + \frac{C}{t^2}$$

Then apply the initial condition to find the constant.

$$2 = 1 + C$$
$$\Rightarrow C = 1$$

Then the solution to the initial value is

$$y = t^2 + \frac{1}{t^2}$$

Problem 36. Find the general solution of the differential equation

$$y' = \frac{1 + xy}{x^2}, x > 0$$

Solution 36. The equation is linear, first order, but not separable. So we want to write it in standard form

Problem 37. Find the general solution of the differential equation.

$$t\frac{dy}{dt} + 2y = t^{-3}, t > 0$$

Solution 37.

$$\frac{dy}{dt} + \frac{2}{t}y = t^{-4}$$
$$p(t) = \frac{2}{t}$$
$$\int p(t)dt = \int \frac{2}{t}dt = 2\ln|t| = 2\ln(t)$$
$$\mu = e^{2\ln(t)} = (e^{\ln(t)})^2 = t^2$$
$$t^2\left[\frac{dy}{dt} + \frac{2}{t}y\right] = t^2[t^{-4}]$$
$$t^2\frac{dy}{dt} + 2ty = t^{-2}$$
$$\frac{d}{dt}[t^2y] = t^{-2}$$
$$t^2y = \int t^{-2}dt = -t^{-1} + C$$
$$y = -t^{-3} + Ct^{-2}$$

Problem 38. Solve the initial value problem $ty' = y + t^3\sin(t), \quad y(\pi) = 0$

Solution 38. This is a first order linear equation. Rewriting the differential equation yields:

$$ty' - y = t^3\sin(t)$$

Writing this in standard form gives:

$$y' + \frac{1}{t}y = t^2\sin(t)$$

Finding the integrating factor:

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{-1}{t}dt} = e^{-ln(t)} = e^{\ln(t^{-1})} = t^{-1}$$

Multiplying by the integrating factor yields:

$$t^{-1}(y' - t^{-1}y) = t\sin(t)$$
$$(t^{-1}y)' = t\sin(t)$$

Integrating both sides:

$$\int (t^{-1}y)'dt = \int t\sin(t)dt$$
$$t^{-1}y = -t\cos(t) + \sin(t) + c$$
$$y(t) = -t^2\cos(t) + t\sin(t) + ct$$

Using the initial conditions, $y(0) = \pi$

$$0 = y(\pi) = -\pi^2 cos(\pi) + \pi\sin(\pi) + c(\pi) = c\pi + \pi^2 \Rightarrow c = -\pi$$

The full solution is:

$$y(t) = t\sin(t) - \pi t - t^2\cos(t)$$

Problem 39. Solve the initial value problem $ty' = y + 2t^2\sin(2t), t > 0$

Solution 39. This is a first order linear equation. Rewriting the differential equation yields:

$$ty' - y = 2t^2\sin(2t)$$

Writing this in standard form gives:

$$y' + \frac{1}{t}y = 2t\sin(2t)$$

Finding the integrating factor:

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{-1}{t}dt} = e^{-ln(t)} = e^{\ln(t^{-1})} = t^{-1}$$

Multiplying by the integrating factor yields:

$$t^{-1}(y' - t^{-1}y) = 2\sin(2t)$$
$$(t^{-1}y)' = 2\sin(2t)$$

Integrating both sides:

$$\int (t^{-1}y)'dt = \int 2\sin(2t)dt$$
$$t^{-1}y = -\cos(2t) + c$$
$$y(t) = -t\cos(2t) + ct$$

Problem 40. Find the value of y_0 for which the solution of the initial value problem

$$ty' - y = t^2e^{-t}, \quad y(1) = y_0$$

approaches zero as t goes to infinity.

Solution 40. Here we have a first order linear ODE. First, we get the ODE into standard form. We divide both sides by t:

$$y' - \frac{1}{t}y = te^{-t}.$$

The integrating factor is

$$\mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = \frac{1}{t}.$$

Multiply the equation by the integrating factor equation, and we obtain:

$$\frac{1}{t}y' - \frac{1}{t^2}y = e^{-t}.$$

Notice the left-hand side is the derivative of $\frac{1}{t}y$:

$$\frac{d}{dt}\left(\frac{1}{t}y\right) = e^{-t}.$$

Integrating both sides gives

$$\frac{y}{t} = \int e^{-t}\,dt$$
$$= -e^{-t} + C.$$

Thus,

$$y(t) = -te^{-t} + Ct.$$

Apply the initial condition $y(1) = y_0$:

$$y_0 = -e^{-1} + C,$$

so

$$C = y_0 + e^{-1}.$$

Hence the solution is

$$y(t) = -te^{-t} + (y_0 + e^{-1})t.$$

As $t \to \infty$, we know that $te^{-t} \to 0$. Then we have to focus on the term:

$$(y_0 + e^{-1})t.$$

For the solution to approach zero as $t \to \infty$, this term must vanish. Therefore,

$$y_0 + e^{-1} = 0,$$

and we obtain

$$y_0 = -e^{-1}$$

Problem 41. Find the general solution to the differential equation: $ty' - 2y = t^4 cos(t)$

Solution 41. First, we get the ODE into standard form.

$$y' - \frac{2}{t}y = t^3 cos(t)$$

Now, we use the integrating factor.

$$\mu(t) = e^{\int -\frac{2}{t}dt} = e^{-2ln|t|} = \frac{1}{t^2}$$

$$\begin{aligned} &\Rightarrow t^{-2}y' - 2t^{-3}y = tcos(t) \\ &\Rightarrow (t^{-2}y)' = tcos(t) \\ &\Rightarrow \int (t^{-2}y)'dt = \int tcos(t)dt \\ &\Rightarrow t^{-2}y = tsin(t) + cos(t) + C \\ &\Rightarrow y = t^3 sin(t) + t^2 cos(t) + Ct^2 \end{aligned}$$

Problem 42 (10). Find the general solution of

$$y'' + 4y' + 3y = 0$$

Solution 42. To solve this second-order linear homogeneous equation, we first find the **Characteristic Equation**:

$$r^2 + 4r + 3 = 0$$
$$(r+1)(r+3) = 0$$

Setting each factor to zero, we find the roots:

$$r_1 = -1, \quad r_2 = -3$$

Since the roots are real and distinct, the general solution is in the form $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. Substituting our roots, we get:

$$y(t) = C_1 e^{-t} + C_2 e^{-3t}$$

Problem 43. Find the explicit general solution of the differential equation:

$$ty' - 2y = t^3 e^{-t} - 4$$

Solution 43. We will solve this linear first-order equation using the **Integrating Factor** method.

Put the equation into Standard Form: Divide every term by t to get $y' + p(t)y = g(t)$:

$$y' - \frac{2}{t}y = t^2 e^{-t} - \frac{4}{t}$$

Find the Integrating Factor, $\mu(t)$: Identify $p(t) = -\frac{2}{t}$. The integrating factor is:

$$\begin{aligned} \mu(t) &= e^{\int -\frac{2}{t} dt} \\ &= e^{-2\ln|t|} \\ &= e^{\ln(t^{-2})} \\ &= t^{-2} \quad (\text{or } \frac{1}{t^2}) \end{aligned}$$

Multiply the Standard Form by $\mu(t)$:

$$t^{-2}\left(y' - \frac{2}{t}y\right) = t^{-2}\left(t^2 e^{-t} - \frac{4}{t}\right)$$
$$\frac{d}{dt}[yt^{-2}] = e^{-t} - 4t^{-3}$$

Integrate both sides:

$$\begin{aligned} yt^{-2} &= \int (e^{-t} - 4t^{-3})\, dt \\ yt^{-2} &= -e^{-t} - 4\left(\frac{t^{-2}}{-2}\right) + C \\ yt^{-2} &= -e^{-t} + 2t^{-2} + C \end{aligned}$$

Solve for y explicitly: Multiply the entire equation by t^2 to isolate y:

$$\begin{aligned} y(t) &= t^2(-e^{-t} + 2t^{-2} + C) \\ y(t) &= -t^2 e^{-t} + 2 + Ct^2 \end{aligned}$$

Final Answer:

$$y(t) = -t^2 e^{-t} + 2 + Ct^2$$

Problem 44. Find the explicit general solution of the differential equation:

$$ty' - 2y = t^3 e^{-t} - 4$$

Solution 44. We will solve this linear first-order equation using the **Integrating Factor** method.

Put the equation into Standard Form: Divide every term by t to get $y' + p(t)y = g(t)$:

$$y' - \frac{2}{t}y = t^2 e^{-t} - \frac{4}{t}$$

Find the Integrating Factor, $\mu(t)$: Identify $p(t) = -\frac{2}{t}$. The integrating factor is:

$$\begin{aligned}\mu(t) &= e^{\int -\frac{2}{t}dt} \\ &= e^{-2\ln|t|} \\ &= e^{\ln(t^{-2})} \\ &= t^{-2} \quad (\text{or } \frac{1}{t^2})\end{aligned}$$

Multiply the Standard Form by $\mu(t)$:

$$t^{-2}\left(y' - \frac{2}{t}y\right) = t^{-2}\left(t^2 e^{-t} - \frac{4}{t}\right)$$
$$\frac{d}{dt}[yt^{-2}] = e^{-t} - 4t^{-3}$$

Integrate both sides:

$$\begin{aligned}yt^{-2} &= \int (e^{-t} - 4t^{-3})\, dt \\ yt^{-2} &= -e^{-t} - 4\left(\frac{t^{-2}}{-2}\right) + C \\ yt^{-2} &= -e^{-t} + 2t^{-2} + C\end{aligned}$$

Solve for y explicitly: Multiply the entire equation by t^2 to isolate y:

$$\begin{aligned}y(t) &= t^2(-e^{-t} + 2t^{-2} + C) \\ y(t) &= -t^2 e^{-t} + 2 + Ct^2\end{aligned}$$

Final Answer:

$$y(t) = -t^2 e^{-t} + 2 + Ct^2$$

Problem 45. Find the explicit general solution of the differential equation:

$$y' + 4x^{-1}y = x^{-8}$$

Solution 45. This is a first-order linear differential equation. We solve it using the **Integrating Factor** method.

1. Find the Integrating Factor, $\mu(x)$: The equation is in standard form $y' + p(x)y = g(x)$, where $p(x) = 4x^{-1} = \frac{4}{x}$.

$$\begin{aligned} \mu(x) &= e^{\int p(x)\,dx} \\ &= e^{\int \frac{4}{x} dx} \\ &= e^{4\ln|x|} \\ &= e^{\ln(x^4)} \\ &= x^4 \end{aligned}$$

2. Multiply the differential equation by $\mu(x)$:

$$x^4(y' + 4x^{-1}y) = x^4(x^{-8})$$
$$x^4y' + 4x^3y = x^{-4}$$

The left side is the derivative of the product (x^4y):

$$\frac{d}{dx}[x^4y] = x^{-4}$$

3. Integrate both sides:

$$\int \frac{d}{dx}[x^4y]\,dx = \int x^{-4}\,dx$$
$$x^4y = \frac{x^{-3}}{-3} + C$$
$$x^4y = -\frac{1}{3}x^{-3} + C$$

4. Solve for y explicitly: Divide the entire equation by x^4 (or multiply by x^{-4}) to isolate y:

$$y(x) = x^{-4}\left(-\frac{1}{3}x^{-3} + C\right)$$
$$y(x) = Cx^{-4} - \frac{1}{3}x^{-7}$$

Final Answer:

$$y(x) = Cx^{-4} - \frac{1}{3}x^{-7}$$

This is the general solution of the differential equation on any interval not containing $x = 0$.

Problem 46. Find the explicit general solution of the differential equation:

$$y' - 2t^{-1}y = t^2e^{-t}$$

Solution 46. This is a first-order linear differential equation. We solve it using the **Integrating Factor** method.

Identify $p(t)$ and find the Integrating Factor, $\mu(t)$: The equation is already in standard form $y' + p(t)y = g(t)$, where $p(t) = -2t^{-1} = -\frac{2}{t}$.

$$\begin{aligned}\mu(t) &= e^{\int p(t)\,dt} \\ &= e^{\int -\frac{2}{t}\,dt} \\ &= e^{-2\ln|t|} \\ &= e^{\ln|t|^{-2}} \\ &= t^{-2} \quad (\text{or } \frac{1}{t^2})\end{aligned}$$

We drop the absolute value bars because t^{-2} is always positive for $t \neq 0$.

Multiply the differential equation by $\mu(t)$:

$$\begin{aligned}t^{-2}(y' - 2t^{-1}y) &= t^{-2}(t^2e^{-t}) \\ t^{-2}y' - 2t^{-3}y &= e^{-t}\end{aligned}$$

The left side is now the derivative of the product $(\mu(t)y)$:

$$\frac{d}{dt}[t^{-2}y] = e^{-t}$$

Integrate both sides:

$$\begin{aligned}\int \frac{d}{dt}[t^{-2}y]\,dt &= \int e^{-t}\,dt \\ t^{-2}y &= -e^{-t} + C\end{aligned}$$

Solve for y explicitly: Multiply the entire equation by t^2 to isolate y:

$$\begin{aligned}y(t) &= t^2(-e^{-t} + C) \\ y(t) &= Ct^2 - t^2e^{-t}\end{aligned}$$

Final Answer:

$$y(t) = Ct^2 - t^2e^{-t}$$

Problem 47. Solve the differential equation:

$$y' - \frac{1}{t}y = te^{-t}$$

on the interval $(0, \infty)$.

Solution 47. This is a first-order linear differential equation. We can solve it using the **Integrating Factor** method.

1. Find the Integrating Factor, $\mu(t)$: The equation is already in the standard form $y' + p(t)y = g(t)$, where $p(t) = -\frac{1}{t}$.

$$
\begin{aligned}
\mu(t) &= e^{\int p(t)\,dt} \\
&= e^{\int -\frac{1}{t}\,dt} \\
&= e^{-\ln(t)} \\
&= e^{\ln(t^{-1})} \\
&= t^{-1} = \frac{1}{t}
\end{aligned}
$$

(Note: Since $t > 0$, we do not need absolute value bars for the natural log.)

2. Multiply the differential equation by $\mu(t)$: Multiply every term in the original equation by $\frac{1}{t}$:

$$
\begin{aligned}
\frac{1}{t}\left(y' - \frac{1}{t}y\right) &= \frac{1}{t}(te^{-t}) \\
\frac{1}{t}y' - \frac{1}{t^2}y &= e^{-t}
\end{aligned}
$$

The left side is the derivative of the product $(\frac{1}{t}y)$ by the product rule:

$$\frac{d}{dt}\left[\frac{1}{t}y\right] = e^{-t}$$

3. Integrate both sides:

$$
\begin{aligned}
\int \frac{d}{dt}\left[\frac{1}{t}y\right]\,dt &= \int e^{-t}\,dt \\
\frac{1}{t}y &= -e^{-t} + C
\end{aligned}
$$

4. Solve for y explicitly: Multiply the entire equation by t to isolate y:

$$y(t) = t(-e^{-t} + C)$$

$$y(t) = -te^{-t} + Ct$$

Final Answer:

$$y(t) = -te^{-t} + Ct$$

Problem 48. Find the general solution to the equation:

$$x\frac{dy}{dx} + 3(y + x^2) = \frac{\sin x}{x}, \quad x > 0$$

Solution 48. This is a first-order linear differential equation. We solve it using the **Integrating Factor** method.

1. Put the equation in Standard Form: First, we distribute the 3 and move the x^2 term to the right:

$$x\frac{dy}{dx} + 3y + 3x^2 = \frac{\sin x}{x}$$
$$x\frac{dy}{dx} + 3y = \frac{\sin x}{x} - 3x^2$$

Now, divide the entire equation by x to get y' by itself:

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x$$

Here, $p(x) = \frac{3}{x}$.

2. Find the Integrating Factor, $\mu(x)$:

$$\begin{aligned}\mu(x) &= e^{\int \frac{3}{x}dx} \\ &= e^{3\ln|x|} = e^{\ln(x^3)} \\ &= x^3\end{aligned}$$

3. Multiply by $\mu(x)$ and Integrate: Multiply the standard form equation by x^3:

$$x^3\left(\frac{dy}{dx} + \frac{3}{x}y\right) = x^3\left(\frac{\sin x}{x^2} - 3x\right)$$
$$\frac{d}{dx}[x^3 y] = x\sin x - 3x^4$$

Now, integrate both sides:

$$x^3 y = \int (x\sin x - 3x^4)\, dx$$

To integrate $x\sin x$, we use **Integration by Parts** with $u = x$ and $dv = \sin x\, dx$:

$$\begin{aligned}\int x\sin x\, dx &= -x\cos x - \int (-\cos x)\, dx \\ &= -x\cos x + \sin x\end{aligned}$$

Adding the integral of $-3x^4$:

$$x^3 y = -x\cos x + \sin x - \frac{3}{5}x^5 + C$$

4. Solve for y explicitly: Divide everything by x^3:

$$y(x) = -\frac{\cos x}{x^2} + \frac{\sin x}{x^3} - \frac{3}{5}x^2 + \frac{C}{x^3}$$

General Solution:

$$y(x) = \frac{C}{x^3} - \frac{3}{5}x^2 - \frac{\cos x}{x^2} + \frac{\sin x}{x^3}$$

Section 3.2 - Compartmental Analysis

Problem 49. A population of insects in a certain region has a daily birth rate that equals the square of the current population. Assume that the population's daily death rate is triple the current insect population. On any given day, there is a net migration into the region of 2 million insects. If there are half a million insects initially, write but DO NOT SOLVE an initial value problem which models the population of insects in the region at any time t > 0.

Solution 49. We use the model

$$\frac{dP}{dt} = R_{in} - R_{out}$$

Let t be in days since the initial count of insects. Let P be the number of insects in millions on day t. We are told that the daily birth rate the square of the current population and that 2 million insects enter from outside the region every day, so we have

$$R_{in} = P^2 + 2$$

We are told that the death rate triple the current population, so we have

$$R_{out} = 3P$$

For the initial condition, we are told that we have half a million insects initially. Remember that we set P to be in millions and write

$$P(t) = \frac{1}{2}$$

So the full initial value problem is

$$\frac{dP}{dt} = P^2 + 2 - 3P, P(0) = \frac{1}{2}$$

Problem 50. A 1000-gallon tank originally holds 300 gallons of water solution containing 100 pounds of salt. Then water containing 2 pounds of salt per gallon is poured into the tank at a rate of 5 gallons per minute, and the well-stirred mixture is allowed to leave the tank at a rate of 3 gallons per minute.
(a) How long will it take before the tank begins to overflow?
(b) Set up, BUT DO NOT SOLVE, an initial value problem that models the amount of salt in the tank at all times prior to the moment when the tank overflows.

Solution 50. a) We want to find an equation which gives the volume V of solution in the tank at time t. We are told that the tank originally holds 300 gallons, 5 gallons enter the tank every minute, and 3 gallons exit the tank every minute. Therefore, we write

$$V(t) = 300 + 5t - 3t$$
$$V(t) = 300 + 2t$$

Since the tank can hold a maximum of 1000 gallons, e just need to find the time when V is 1000. Simply plug in and solve.

$$\begin{aligned} 1000 &= 300 + 2t \\ 700 &= 2t \\ t &= 350 \end{aligned}$$

So we know the tank will overflow after 350 minutes.
b) We use the model

$$\frac{dA}{dt} = R_{in} - R_{out}$$

. Using the information given in the problem, we have

$$\begin{aligned} R_{in} &= 5 * 2 = 10 \\ R_{out} &= 3 * \frac{A(t)}{V(t)} = \frac{3A(t)}{300 + 2t} \end{aligned}$$

Plugging these in, we get

$$\frac{dA}{dt} = 10 - \frac{3A(t)}{300 + 2t}$$

Since we are told the tank originally contains 100 pounds of salt, we have the initial condition

$$A(0) = 100$$

. Then the full initial value problem is

$$\frac{dA}{dt} = 10 - \frac{3A(t)}{300 + 2t}, A(0) = 100$$

.

Problem 51. Let P(t) denote the population of fish in a certain lake at time t. Suppose the birth rate of the fish is twice the current fish population and the death rate of the fish equals the square of the current fish population. Also suppose that the fish are harvested at a constant rate h. Write down, but DO NOT SOLVE, a differential equation that models the fish population P(t).

Solution 51. We use the model

$$\frac{dP}{dt} = R_{in} - R_{out}$$

We are told that the the birth rate is twice the current fish population, so we have

$$R_{in} = 2P(t)$$

We are told that the death rate is the square of the population and that the fish are harvested at a constant rate h, so we have

$$R_{out} = (P(t))^2 + h$$

Then a differential equation modeling the fish population is

$$\frac{dP}{dt} = 2P(t) - (P(t))^2 - h$$

Problem 52. A tank initially contains 200 liters of pure water. A mixture of salt and water containing 10 grams per liter of salt enters the tank at a rate of 3 liters per minute, and the well-stirred mixture leaves the tank at a rate of 4 liters per minute. Set up, but DO NOT SOLVE, an initial value problem that models the amount A(t) of salt in the tank at any time t.

Solution 52. We use the model

$$\frac{dA}{dt} = R_{in} - R_{out}$$

Then we have

$$\frac{dA}{dt} = 10 * 3 - \frac{A(t)}{v(t)} * 4$$

$$\frac{dA}{dt} = 30 - \frac{4A(t)}{v(t)}$$

To find the volume at time t, we note that the initial volume is 200 liters, 3 liters enter every minute, and 4 liters exit every minute. This give

$$v(t) = 200 + 3t - 4t = 200 - t$$

Substituting, we have the differential equation

$$\frac{dA}{dt} = 30 - \frac{4A(t)}{200 - t}$$

We are told that the tank initially has only pure water, meaning there is no salt, so the initial value problem is

$$\frac{dA}{dt} = 30 - \frac{4A(t)}{200 - t}, A(0) = 0$$

Problem 53. A 1000 gallon tank originally holds 200 gallons of water solution containing 50 pounds of salt. A water solution containing 2 pounds of salt per gallon is poured into the tank at a rate of 4 gallons per minute. The well-stirred mixture is allowed to drain from the tank at the rate of 2 gallons per minute.
a) How long will it take for the tank to begin to overflow?

b) Set up but DO NOT SOLVE an initial value problem which models the amount A(t) of salt in the tank at time t for the all values of t prior to the moment when the tank overflows.

Solution 53. a) First note the volume of water solution in the tank at time t is given by

$$v(t) = 200 + 4t - 2t = 200 + 2t$$

To find t when the volume of water solution is 100 gallons, just plug in and solve.

$$\begin{aligned} 200 + 2t &= 1000 \\ 2t &= 800 \\ t &= 400 \end{aligned}$$

b) Begin with the model

$$\frac{dA}{dt} = R_{in} - R_{out}$$

.

$$\begin{aligned} \frac{dA}{dt} &= 2 * 4 - \frac{A}{v} * 2 \\ \frac{dA}{dt} &= 8 - \frac{2A}{200 + 2t} \end{aligned}$$

Note that the initial condition is

$$A(0) = 50$$

Then the final answer is

$$\frac{dA}{dt} = 8 - \frac{2A}{200 + 2t}, A(0) = 50$$

.

Problem 54. A 500 gallon tank originally contains 200 gallons of pure water. Then water containing two pounds of salt per gallon is poured into the tank at a rate of four gallons per minute, and the well-stirred mixture leaves at a rate of five gallons per minute. Write but do NOT solve, an initial value problem for the amount Q(t) of salt in the tank at time t.

Solution 54. For the tank problems we model the differential equation using:

$$\begin{aligned} \frac{dQ}{dt} &= R_{in} - R_{out} \\ &= \left(\frac{2lbs}{gal}\right)\left(\frac{4gal}{min}\right) - \left(\frac{Q(t)lbs}{V(t)gal}\right)\left(\frac{5gal}{min}\right) \end{aligned}$$

The volume of solution in the tank in time t is

$$V(t) = 200 - t$$

since the volume starts at 200 gallons and decreases by 1 gallon per minute. Thus,

$$Q' = 8 - \frac{5}{200 - t} Q$$
$$Q(0) = 0$$

Problem 55. A tank initially contains 10 gallons of water in which 2 pounds of salt is dissolved. A mixture containing 3 pounds of salt per gallon of water is pumped into the tank at a rate of 5 gallons per minute. The well-mixed solution is pumped out at a rate of 3 gallons per minute.
(a) Write but DO NOT SOLVE an initial value problem for the amount of salt, Q(t), in the tank at time t.
(b) If the volume of the tank is 200 gallons, find the time t when the tank becomes full.

Solution 55. (a) For the tank problems we model the differential equation using:

$$\begin{aligned} \frac{dQ}{dt} &= R_{in} - R_{out} \\ &= \left(\frac{3lbs}{gal}\right)\left(\frac{5gal}{min}\right) - \left(\frac{Q(t)lbs}{V(t)gal}\right)\left(\frac{3gal}{min}\right) \end{aligned}$$

The volume of solution in the tank in time t is

$$V(t) = 10 + 2t$$

since the volume starts at 10 gallons and increases by 2 gallons per minute. Thus,

$$Q' = 15 - \frac{3}{10 + 2t} Q$$
$$Q(0) = 2$$

(b) At time t, the volume of the mixture in the tank is

$$V = 10 + (5 - 3)t = 10 + 2t$$

When the tank is full we have 200 gallons, so:

$$\begin{aligned} 200 &= 10 + 2t \\ 190 &= 2t \\ \Rightarrow t &= 95\text{min} \end{aligned}$$

Problem 56. A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical. Water containing 0.01g/gal of chemical flows into the pond at a rate of 300gal/h, and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond. Let Q(t) be the amount of the chemical in the pond at time t. Write but do not solve an Initial Value Problem for Q(t).

Solution 56. For the tank problems we model the differential equation using:

$$\begin{aligned}\frac{dQ}{dt} &= R_{in} - R_{out} \\ &= \left(\frac{0.01g}{gal}\right)\left(\frac{300gal}{h}\right) - \left(\frac{Q(t)lbs}{1{,}000{,}000gal}\right)\left(\frac{300gal}{h}\right) \\ Q' &= 3 - \frac{3}{10{,}000}Q\end{aligned}$$

The initial conditions are

$$Q(0) = 0$$

Then the full solution is:

$$\begin{aligned}Q' &= 3 - \frac{3}{10{,}000}Q \\ Q(0) &= 0\end{aligned}$$

Problem 57. One theory of epidemic spread postulates that the time rate of change in the infected population is proportional to the product of the number of individuals who have the disease with the number of disease free individuals. Assuming that the population of mice in a certain meadow has a stable value of one thousand. Use the theory of epidemic spread to write, but do not solve, an initial value problem that models the number $N(t)$ of infected mice at time $t > 0$ if ten mice were initially infected

Solution 57. First, denote that N is the number of mice infected with the disease. Then, the disease free is given by $1000 - N$. Thus,

$$\frac{dN}{dt} \propto N(1000 - N)$$

Which leads to:

$$\frac{dN}{dt} = kN(1000 - N), \quad N(0) = 10$$

Problem 58. A brine solution of salt flows at a constant rate of 10 gallons per minute into a large tank that initially held 1,000 gallons of pure water. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 gallons per minute. The concentration entering the tank is 0.5 pounds per gallon. Write, BUT DO NOT SOLVE, an initial value problem to model the amount $A(t)$ of salt (in pounds) in the tank at time t (in minutes).

Solution 58. To find the rate of change of the amount of salt $\frac{dA}{dt}$, we use the relationship:

$$\frac{dA}{dt} = \text{rate in} - \text{rate out}$$

1. Rate In:

$$\text{rate in} = (10 \text{ gal/min}) \times (0.5 \text{ lb/gal}) = 5 \text{ lb/min}$$

2. Rate Out: The volume of liquid in the tank at time t is the initial volume plus the net flow rate:

$$V(t) = 1000 + (10 - 8)t = 1000 + 2t$$

The concentration of salt exiting the tank is $\frac{A(t)}{V(t)}$. Therefore:

$$\text{rate out} = (8 \text{ gal/min}) \times \frac{A}{1000 + 2t} = \frac{8A}{1000 + 2t}$$

3. Initial Condition: Since the tank initially held pure water, the initial amount of salt is:

$$A(0) = 0$$

Final Initial Value Problem:

$$\frac{dA}{dt} = 5 - \frac{8A}{1000 + 2t}$$
$$A(0) = 0$$

Problem 59. The initial mass of a certain species of fish is 10 million tons. The mass of fish, if left alone, would increase by 4 times the current mass; however, commercial fishing removes fish mass at a constant rate of 13 million tons per year. Write, BUT DO NOT SOLVE, an initial value problem to model the mass $M(t)$ of fish (in million tons) at time t (in years).

Solution 59. To model the mass of the fish population $M(t)$, we use the rate balance equation:

$$\text{Net Rate of Change} = \text{Rate In} - \text{Rate Out}$$

1. Identify the Rate In (Growth): The problem states that the mass increases by 4 times the current mass. This represents a growth rate proportional to the population:

$$\text{Rate In} = 4M$$

2. Identify the Rate Out (Harvesting): Commercial fishing removes mass at a constant rate of 13 million tons per year:

$$\text{Rate Out} = 13$$

3. Identify the Initial Condition: The initial mass of the fish is 10 million tons at time $t = 0$:

$$M(0) = 10$$

Initial Value Problem: Combining these components, we get the following differential equation and initial condition:

$$\frac{dM}{dt} = 4M - 13$$
$$M(0) = 10$$

(Or equivalently: $M'(t) = 4M(t) - 13, \quad M(0) = 10$)

Section 4.2 - Homogeneous Linear Equations: The General Solution

Problem 60. Find the general solution of the differential equation.

$$y'' - 5\pi^2 y = 0$$

Solution 60. The equation is second order, linear, homogeneous, so we immediately write the characteristic equation and find roots.

$$r^2 - 5\pi^2 = 0$$
$$(r + \pi\sqrt{5})(r - \pi\sqrt{5}) = 0$$
$$r_{1,2} = \pm\pi\sqrt{5}$$

Just a note, factoring as a difference of squares here is not necessary, but is a good habit. It will be helpful as we continue in the course. We have the case of real distinct roots, and we immediately write the general solution

$$y(t) = C_1 e^{\pi\sqrt{5}t} + C_2 e^{-\pi\sqrt{5}t}$$

Problem 61. Find the general solution of the differential equation.

$$y'' + y' + y = 0$$

Solution 61. The equation is second order, linear, and homogeneous, so we write the characteristic equation and find roots.

$$r^2 + r + 1 = 0$$
$$r_{1,2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The roots were found using the quadratic formula. We have the case of complex roots in a conjugate pair and immediately write the general solution

$$y(t) = C_1 e^{-\frac{t}{2}} \cos(\frac{\sqrt{3}}{2}t) + C_2 e^{-\frac{t}{2}} \sin(\frac{\sqrt{3}}{2}t)$$

Problem 62. Solve the initial value problem

$$y'' - 4y' + 4y = 0, y(0) = 5, y\left(\frac{\ln(5)}{2}\right) = 0$$

Solution 62. The equation is second order, linear, homogeneous, so we write the characteristic equation and find the roots.

$$r^2 - 4r + 4 = 0$$
$$(r-2)(r-2) = 0$$
$$r_{1,2} = 2$$

And we have the case of real repeated roots. Then we immediately write the general solution

$$y(t) = C_1 e^{2t} + C_2 t e^{2t}$$

To find the constants, we use the initial conditions. There's more than one way to go about this, but I will plug in the condition

$$y(0) = 5$$

first because it gives the first constant directly.

$$5 = C_1 e^{2*0} + C_2 * (0) * e^{2*0}$$
$$5 = C_1$$

Then plug in the other initial condition to find the second constant. Note that we now know the value of the first constant.

$$0 = 5e^{\frac{2\ln(5)}{2}} + C_2\left(\frac{\ln(5)}{2}\right)e^{\frac{2\ln(5)}{2}}$$
$$0 = 5e^{\ln(5)} + C_2\left(\frac{\ln(5)}{2}\right)e^{\ln(5)}$$
$$0 = 5 * 5 + C_2\left(\frac{\ln(5)}{2}\right) * 5$$
$$-25 = C_2\left(\frac{\ln(5)}{2}\right) * 5$$
$$-5 = C_2\left(\frac{\ln(5)}{2}\right)$$
$$-\frac{10}{\ln(5)} = C_2$$

Then the solution to the initial value problem is

$$y(t) = 5e^{2t} - \frac{10}{\ln(5)} t e^{2t}$$

Problem 63. Find the general solution of the differential equation.

$$y'' + 2y' - y = 0$$

Solution 63. The equation is second order, linear, homogeneous, so we immediately write the characteristic equation and find roots.

$$r^2 + 2r - 1 = 0$$
$$r_{1,2} = -1 \pm \sqrt{2}$$

The roots above were found using the quadratic formula. Noting that we have real distinct roots, we immediately write the general solution

$$y = C_1 e^{(-1+\sqrt{2})t} + C_2 e^{(-1-\sqrt{2})t}$$

Problem 64. Find the general solution of the differential equation.

$$y'' + 2y' + 4y = 0$$

Solution 64. The equation is second order, linear, and homogeneous so we immediately write the characteristic equation and find roots.

$$r^2 + 2y' + 4y = 0$$
$$r_{1,2} = -1 \pm \sqrt{3}i$$

The above roots were found using the quadratic formula. We have the case of complex roots and immediately write the general solution

$$y = C_1 e^{-t} \cos(\sqrt{3}t) + C_2 e^{-t} \sin(\sqrt{3}t)$$

Problem 65. Find the general solution of

$$y'' + 4y' + y = 0$$

Solution 65. The characteristic equation is

$$r^2 + 4r + 1 = 0$$

From the quadratic formula, we find roots

$$r = -2 \pm \sqrt{3}$$

Then the general solution is

$$y = C_1 e^{(-2+\sqrt{3})t} + C_2 e^{(-2-\sqrt{3})t}$$

Problem 66. Find the general solution of

$$4y'' + 12y' + 9y = 0$$

Solution 66. The characteristic equation is

$$4r^2 + 12r + 9 = 0$$

Using any method you prefer, this gives the repeated roots

$$r_{1,2} = \frac{-3}{2}$$

Then the general solution is

$$y = C_1 e^{-\frac{3}{2}t} + C_2 t e^{-\frac{3}{2}t}$$

Problem 67. Find the General solution of

$$y'' + 2y' + y = 0$$

and describe the behavior as

$$t \to \infty$$

Solution 67. First, we assume the solutions are in the form:

$$y(t) = e^{rt} \Rightarrow y'(t) = re^{rt} \Rightarrow y''(t) = r^2 e^{rt}$$

$$\begin{aligned} r^2 e^{rt} + 2re^{rt} + e^{rt} &= 0 \\ e^{rt}(r^2 + 2r + 1) &= 0 \\ r^2 + 2r + 1 &= 0 \\ (r+1)^2 &= 0 \\ r &= -1 \text{ (repeated root)} \end{aligned}$$

$$y(t) = (C_1 + C_2 t)e^{-t}$$

We know that

$$\lim_{t \to \infty} e^{-t} = 0$$

so as

$$y(t) \to 0, \text{then } t \to \infty.$$

Problem 68. Find the General solution of

$$y'' + 4y' + 4y = 0$$

and describe the behavior as

$$t \to \infty$$

Solution 68. First, we assume the solutions are in the form:

$$y(t) = e^{rt} \Rightarrow y'(t) = re^{rt} \Rightarrow y''(t) = r^2 e^{rt}$$

$$\begin{aligned} r^2 e^{rt} + 4re^{rt} + 4e^{rt} &= 0 \\ e^{rt}(r^2 + 4r + 4) &= 0 \\ r^2 + 4r + 4 &= 0 \\ (r+2)^2 &= 0 \\ r &= -2 \text{ (repeated root)} \end{aligned}$$

$$y(t) = (C_1 + C_2 t)e^{-2t}$$

Problem 69. Obtain a general solution of the following differential equation:

$$\frac{1}{t}y' + y = e^{t^2/2}$$

Solution 69. First, we write the equation in standard form $y' + p(t)y = g(t)$ by multiplying through by t:

$$y' + ty = te^{t^2/2}$$

Next, we find the **Integrating Factor**, $\mu(t)$:

$$\mu(t) = e^{\int t\,dt} = e^{t^2/2}$$

Multiply the standard form equation by $\mu(t)$:

$$e^{t^2/2}(y' + ty) = e^{t^2/2} \cdot te^{t^2/2}$$
$$\frac{d}{dt}\left(e^{t^2/2}y\right) = te^{t^2}$$

Now, integrate both sides with respect to t:

$$e^{t^2/2}y = \int te^{t^2}\,dt$$
$$e^{t^2/2}y = \frac{1}{2}e^{t^2} + C$$

Finally, solve for $y(t)$ by multiplying by $e^{-t^2/2}$:

$$y(t) = e^{-t^2/2}\left(\frac{1}{2}e^{t^2} + C\right)$$
$$y(t) = \frac{1}{2}e^{t^2/2} + Ce^{-t^2/2}$$

Final Answer:

$$y(t) = \frac{1}{2}e^{t^2/2} + Ce^{-t^2/2}$$

Problem 70. Find the general solution of the differential equation:

$$y'' + 3y' + y = 0$$

Solution 70. This is a second-order linear homogeneous differential equation with constant coefficients. We solve it by finding the roots of the **Characteristic Equation**:

$$r^2 + 3r + 1 = 0$$

Using the quadratic formula $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$:

$$r_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)}$$
$$r_{1,2} = \frac{-3 \pm \sqrt{9-4}}{2}$$
$$r_{1,2} = \frac{-3 \pm \sqrt{5}}{2}$$

Since the roots $r_1 = \frac{-3+\sqrt{5}}{2}$ and $r_2 = \frac{-3-\sqrt{5}}{2}$ are real and distinct, the general solution takes the form $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

General Solution:

$$y(t) = C_1 e^{\left(\frac{-3+\sqrt{5}}{2}\right)t} + C_2 e^{\left(\frac{-3-\sqrt{5}}{2}\right)t}$$

Problem 71. Find the explicit general solution of the differential equation:

$$ty' - 2y = t^3 e^{-t} - 4$$

Solution 71. We will solve this linear first-order equation using the **Integrating Factor** method.

Put the equation into Standard Form: Divide every term by t to get $y' + p(t)y = g(t)$:

$$y' - \frac{2}{t}y = t^2 e^{-t} - \frac{4}{t}$$

Find the Integrating Factor, $\mu(t)$: Identify $p(t) = -\frac{2}{t}$. The integrating factor is:

$$\begin{aligned} \mu(t) &= e^{\int -\frac{2}{t} dt} \\ &= e^{-2\ln|t|} \\ &= e^{\ln(t^{-2})} \\ &= t^{-2} \quad (\text{or } \frac{1}{t^2}) \end{aligned}$$

Multiply the Standard Form by $\mu(t)$:

$$t^{-2}\left(y' - \frac{2}{t}y\right) = t^{-2}\left(t^2 e^{-t} - \frac{4}{t}\right)$$
$$\frac{d}{dt}[yt^{-2}] = e^{-t} - 4t^{-3}$$

Integrate both sides:

$$yt^{-2} = \int (e^{-t} - 4t^{-3})\, dt$$
$$yt^{-2} = -e^{-t} - 4\left(\frac{t^{-2}}{-2}\right) + C$$
$$yt^{-2} = -e^{-t} + 2t^{-2} + C$$

Solve for y explicitly: Multiply the entire equation by t^2 to isolate y:

$$y(t) = t^2(-e^{-t} + 2t^{-2} + C)$$
$$y(t) = -t^2e^{-t} + 2 + Ct^2$$

Final Answer:

$$y(t) = -t^2e^{-t} + 2 + Ct^2$$

Problem 72. Find the general solution of the differential equation:

$$y'' - 4y' - 5y = 0$$

Solution 72. This is a second-order linear homogeneous differential equation with constant coefficients. We solve it by finding the roots of the **Characteristic Equation**:

$$r^2 - 4r - 5 = 0$$

We can solve this by factoring:

$$(r - 5)(r + 1) = 0$$

The roots are $r_1 = 5$ and $r_2 = -1$.

From these roots, we obtain two individual solutions:

$$y_1(t) = e^{5t} \quad \text{and} \quad y_2(t) = e^{-t}$$

To ensure these form a **fundamental system of solutions**, we check the **Wronskian**:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} e^{5t} & e^{-t} \\ 5e^{5t} & -e^{-t} \end{vmatrix} \\ &= (e^{5t})(-e^{-t}) - (e^{-t})(5e^{5t}) \\ &= -e^{4t} - 5e^{4t} \\ &= -6e^{4t} \end{aligned}$$

Since $W \neq 0$ for all real t, y_1 and y_2 form a fundamental system. The general solution is a linear combination of these two solutions.

General Solution:

$$y(t) = C_1e^{5t} + C_2e^{-t}$$

where C_1 and C_2 are arbitrary constants.

Problem 73. Find the general solution of the differential equation:

$$y'' - 5y' + 6y = 0$$

Solution 73. This is a second-order linear homogeneous differential equation with constant coefficients. We begin by finding the roots of the **Characteristic Equation**:

$$r^2 - 5r + 6 = 0$$

We can solve this by factoring:

$$(r - 2)(r - 3) = 0$$

The roots are $r_1 = 2$ and $r_2 = 3$.

Since the roots are real and distinct, the general solution takes the form $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

General Solution:

$$y(t) = C_1 e^{2t} + C_2 e^{3t}$$

Problem 74. Find the general solution of the differential equation:

$$y'' + 6y' + 9y = 0$$

Solution 74. This is a second-order linear homogeneous differential equation with constant coefficients. We solve it by looking at the **Characteristic Equation**:

$$r^2 + 6r + 9 = 0$$

We can factor the quadratic:

$$(r + 3)^2 = 0$$

This gives us a repeated real root, $r = -3$.

When there is a repeated real root r, the general solution takes the form $y(t) = C_1 e^{rt} + C_2 t e^{rt}$.

General Solution:

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}$$

Problem 75. Find the general solution of the differential equation:

$$y'' - 2y' = 0$$

Solution 75. This is a second-order linear homogeneous differential equation with constant coefficients. We solve it by finding the roots of the **Characteristic Equation**:

$$r^2 - 2r = 0$$

We can solve this by factoring out r:

$$r(r-2) = 0$$

The roots are $r_1 = 0$ and $r_2 = 2$.

From these roots, we obtain two linearly independent solutions:

$$y_1(t) = e^{0t} = 1 \quad \text{and} \quad y_2(t) = e^{2t}$$

The general solution is a linear combination of these two solutions.

General Solution:

$$y(t) = C_1 + C_2 e^{2t}$$

where C_1 and C_2 are arbitrary constants.

Problem 76. Find the general solution of the differential equation:

$$y'' - 6y' + 9y = 0$$

Solution 76. This is a second-order linear homogeneous differential equation with constant coefficients. We begin by solving the **Characteristic Equation**:

$$r^2 - 6r + 9 = 0$$

We can solve this by factoring:

$$(r-3)(r-3) = 0 \quad \text{or} \quad (r-3)^2 = 0$$

This gives us a repeated real root, $r = 3$.

When we have a repeated root, the two linearly independent solutions are:

$$y_1(t) = e^{3t} \quad \text{and} \quad y_2(t) = te^{3t}$$

The general solution is a linear combination of these two solutions.

General Solution:

$$y(t) = C_1 e^{3t} + C_2 t e^{3t}$$

where C_1 and C_2 are arbitrary constants.

Problem 77. Find the exact solution to the initial value problem:

$$y'' + 2y' - 8y = 0, \quad y(0) = 3, \quad y'(0) = -6$$

Solution 77. This is a second-order linear homogeneous differential equation with constant coefficients. We solve it by finding the roots of the **Characteristic Equation**:

$$r^2 + 2r - 8 = 0$$

1. Find the Roots: We can solve this by factoring:

$$(r+4)(r-2)=0$$

The roots are $r_1=-4$ and $r_2=2$.

2. Write the General Solution: Based on these real, distinct roots, the general solution is:

$$y(t)=C_1e^{-4t}+C_2e^{2t}$$

3. Apply Initial Conditions: First, find the derivative: $y'(t)=-4C_1e^{-4t}+2C_2e^{2t}$. Using $y(0)=3$:

$$C_1+C_2=3\Longrightarrow C_2=3-C_1$$

Using $y'(0)=-6$:

$$-4C_1+2C_2=-6$$

Substitute $C_2=3-C_1$ into the second equation:

$$\begin{aligned}-4C_1+2(3-C_1)&=-6\\ -4C_1+6-2C_1&=-6\\ -6C_1&=-12\Longrightarrow C_1=2\end{aligned}$$

Now find C_2: $C_2=3-2=1$.

Exact Solution:

$$y(t)=2e^{-4t}+e^{2t}$$

Section 4.3 - Auxiliary Equations with Complex Roots

Problem 78. Find the general solution of

$$y''+2y'+4y=0$$

Solution 78. We propose our solution will be of the form

$$y=e^{rt}$$

The characteristic equation is

$$r^2+2r+4=0$$

Using the quadratic formula we find the complex roots

$$r=-1\pm(i\sqrt{3})t$$

Then the general solution is

$$y=C_1e^{-t}\cos(\sqrt{3}t)+C_2e^{-t}\sin(\sqrt{3}t)$$

Problem 79. Find the General solution of

$$y'' + 4y' + 5y = 0$$

and describe the behavior as

$$t \to \infty$$

Solution 79. Assume that we have the solution in the form:

$$y(t) = e^{rt} \Rightarrow y'(t) = re^{rt} \Rightarrow y''(t) = r^2 e^{rt}$$

$$r^2 e^{rt} + 4re^{rt} + 5e^{rt} = 0$$

$$e^{rt}(r^2 + 4r + 5) = 0$$

(Since $e^r t \neq 0$)

$$r^2 + 4r + 5 = 0$$

(Use the Quadratic Formula)

$$r = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$r = -2 \pm i$$

Thus, the general solution is:

$$y(t) = e^{-2t}(C_1 \text{cos} t + C_2 \text{sin} t)$$

We know that:

$$\lim_{t \to \infty} e^{-2t} = 0$$

and sin(t) and cos(t) are bounded. By using squeeze theorem we obtain:

$$y(t) \to 0 \text{ as } t \to \infty$$

Problem 80. Find the general solution of the following differential equation:

$$y'' + 2y' + 3y = 0$$

Solution 80. See the solution above for the derivation of the auxiliary/characteristic equation. Thus,

$$r^2 + 2r + 3 = 0$$

$$\Rightarrow r = \frac{-2 \pm \sqrt{4 - 12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}i}{2}$$

$$= -1 \pm \sqrt{2}i$$

This yields the solution:

$$y = c_1 e^{-t} \cos(\sqrt{2}t) + c_2 e^{-t} \sin(\sqrt{2}t)$$

Problem 81. Find the general solution of

$$2y'' + 2y' + y = 0$$

Solution 81. By using the solution form $y(t) = e^{rt}$ we obtain the characteristic equation:

$$2r^2 + 2r + 1 = 0$$

Solving this yields:

$$\begin{aligned} r &= \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2(2)} \\ &= \frac{-2 \pm 2i}{4} \\ &= \frac{-1}{2} \pm i\frac{1}{2} \end{aligned}$$

This leads to the general solution:

$$y(t) = c_1 e^{-\frac{t}{2}} cos(\frac{t}{2}) + c_2 e^{-\frac{t}{2}} sin(\frac{t}{2})$$

where c_1, c_2 are arbitrary constants.

Problem 82. Find the general solution of the differential equation:

$$y'' + 36y = 0$$

Solution 82. This is a second-order linear homogeneous differential equation with constant coefficients. We begin by writing the **Characteristic Equation**:

$$\begin{aligned} r^2 + 36 &= 0 \\ r^2 &= -36 \\ r &= \pm\sqrt{-36} \\ r &= \pm 6i \end{aligned}$$

The roots are purely imaginary complex numbers of the form $0 \pm 6i$. For roots of the form $\lambda \pm \mu i$, the general solution is $y(t) = e^{\lambda t}(C_1 \cos \mu t + C_2 \sin \mu t)$. Here, $\lambda = 0$ and $\mu = 6$.

General Solution:

$$y(t) = C_1 \cos(6t) + C_2 \sin(6t)$$

Problem 83. Find the general solution of the differential equation:

$$y'' + 3y' + y = 0$$

Solution 83. This is a second-order linear homogeneous differential equation with constant coefficients. We solve it by finding the roots of the **Characteristic Equation**:

$$r^2 + 3r + 1 = 0$$

Using the quadratic formula $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$:

$$r_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)}$$
$$r_{1,2} = \frac{-3 \pm \sqrt{9-4}}{2}$$
$$r_{1,2} = \frac{-3 \pm \sqrt{5}}{2}$$

Since the roots $r_1 = \frac{-3+\sqrt{5}}{2}$ and $r_2 = \frac{-3-\sqrt{5}}{2}$ are real and distinct, the general solution takes the form $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

General Solution:

$$y(t) = C_1 e^{\left(\frac{-3+\sqrt{5}}{2}\right)t} + C_2 e^{\left(\frac{-3-\sqrt{5}}{2}\right)t}$$

Problem 84 (10). Solve the initial value problem

$$y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Solution 84. **1. Find the Characteristic Equation and Roots:**

$$r^2 + 9 = 0$$
$$r^2 = -9$$
$$r = \pm 3i$$

This gives us complex roots of the form $\lambda \pm \mu i$ where $\lambda = 0$ and $\mu = 3$.

2. General Solution: For purely imaginary roots, the general solution is:

$$y(t) = C_1 \cos(3t) + C_2 \sin(3t)$$

3. Apply Initial Conditions: To find C_1 and C_2, we first find the derivative $y'(t)$:

$$y'(t) = -3C_1 \sin(3t) + 3C_2 \cos(3t)$$

Using $y(0) = 1$:

$$1 = C_1 \cos(0) + C_2 \sin(0)$$
$$1 = C_1(1) + 0 \Longrightarrow C_1 = 1$$

Using $y'(0) = 1$:

$$1 = -3C_1 \sin(0) + 3C_2 \cos(0)$$
$$1 = 0 + 3C_2(1) \Longrightarrow C_2 = \frac{1}{3}$$

Final Answer:

$$y(t) = \cos(3t) + \frac{1}{3}\sin(3t)$$

Problem 85. Find the general solution of the differential equation:

$$y'' + 36y = 0$$

Solution 85. This is a second-order linear homogeneous differential equation with constant coefficients. We begin by writing the **Characteristic Equation**:

$$r^2 + 36 = 0$$
$$r^2 = -36$$
$$r = \pm\sqrt{-36}$$
$$r = \pm 6i$$

The roots are purely imaginary complex numbers of the form $0 \pm 6i$. For roots of the form $\lambda \pm \mu i$, the general solution is $y(t) = e^{\lambda t}(C_1 \cos\mu t + C_2 \sin\mu t)$. Here, $\lambda = 0$ and $\mu = 6$.

General Solution:

$$y(t) = C_1 \cos(6t) + C_2 \sin(6t)$$

Problem 86. Find the general solution of the differential equation:

$$y'' + 36y = 0$$

Solution 86. This is a second-order linear homogeneous differential equation with constant coefficients. We begin by writing the **Characteristic Equation**:

$$r^2 + 36 = 0$$
$$r^2 = -36$$
$$r = \pm\sqrt{-36}$$
$$r = \pm 6i$$

The roots are purely imaginary complex numbers of the form $0 \pm 6i$. For roots of the form $\lambda \pm \mu i$, the general solution is $y(t) = e^{\lambda t}(C_1 \cos\mu t + C_2 \sin\mu t)$. Here, $\lambda = 0$ and $\mu = 6$.

General Solution:

$$y(t) = C_1 \cos(6t) + C_2 \sin(6t)$$

Problem 87. Find the general solution of the differential equation:

$$y'' + 2y' + 5y = 0$$

Solution 87. This is a second-order linear homogeneous differential equation with constant coefficients. We begin by solving the **Characteristic Equation**:

$$r^2 + 2r + 5 = 0$$

Using the quadratic formula $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$
\begin{aligned}
r &= \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} \\
&= \frac{-2 \pm \sqrt{4 - 20}}{2} \\
&= \frac{-2 \pm \sqrt{-16}}{2} \\
&= \frac{-2 \pm 4i}{2} \\
&= -1 \pm 2i
\end{aligned}
$$

The roots are complex numbers of the form $\lambda \pm \mu i$, where $\lambda = -1$ and $\mu = 2$. These roots correspond to the complex-valued solutions:

$$y_1(t) = e^{(-1+2i)t} = e^{-t}(\cos(2t) + i\sin(2t))$$

$$y_2(t) = e^{(-1-2i)t} = e^{-t}(\cos(2t) - i\sin(2t))$$

By the **Superposition Principle**, we can form real-valued solutions by taking linear combinations of y_1 and y_2:

$$\tilde{y}_1(t) = \frac{y_1(t) + y_2(t)}{2} = e^{-t}\cos(2t)$$

$$\tilde{y}_2(t) = \frac{y_1(t) - y_2(t)}{2i} = e^{-t}\sin(2t)$$

Since the Wronskian $W(\tilde{y}_1, \tilde{y}_2)(t) = 2e^{-2t} \neq 0$, these form a fundamental set of solutions.

General Solution:

$$y(t) = C_1 e^{-t}\cos(2t) + C_2 e^{-t}\sin(2t)$$

where C_1 and C_2 are arbitrary constants.

Problem 88. Find the general solution of the differential equation:

$$y'' + 25y = 0$$

Solution 88. This is a second-order linear homogeneous differential equation with constant coefficients. We solve it by using the **Characteristic Equation**:

$$r^2 + 25 = 0$$

Solve for r:

$$
\begin{aligned}
r^2 &= -25 \\
r &= \pm\sqrt{-25} \\
r &= \pm 5i
\end{aligned}
$$

The roots are purely imaginary complex numbers of the form $0 \pm 5i$. When the roots of the characteristic equation are complex ($\lambda \pm \mu i$), the general solution is $y(t) = e^{\lambda t}(C_1 \cos(\mu t) + C_2 \sin(\mu t))$. Here, $\lambda = 0$ and $\mu = 5$.

General Solution:

$$y(t) = C_1 \cos(5t) + C_2 \sin(5t)$$

Problem 89. Find the general solution to the given differential equation:

$$y'' + 4y' + 5y = 0$$

Solution 89. We begin by solving the **Characteristic Equation**:

$$r^2 + 4r + 5 = 0$$

1. Find the Roots: Using the quadratic formula $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$\begin{aligned} r &= \frac{-4 \pm \sqrt{16 - 20}}{2} \\ &= \frac{-4 \pm \sqrt{-4}}{2} \\ &= \frac{-4 \pm 2i}{2} = -2 \pm i \end{aligned}$$

2. Write the General Solution: The roots are complex ($\lambda \pm \mu i$) with $\lambda = -2$ and $\mu = 1$. The general solution for complex roots follows the form $y(t) = e^{\lambda t}(C_1 \cos(\mu t) + C_2 \sin(\mu t))$.

General Solution:

$$y(t) = C_1 e^{-2t} \cos(t) + C_2 e^{-2t} \sin(t)$$

Section 4.4 - Nonhomogeneous Equations: The Method of Undetermined Coefficients

Problem 90. Find the general solution of the differential equation

$$y''' - 3y' + 2y = 2e^{3t}$$

The homogeneous solution is given at the end of the solution, if you prefer. It may not be given on an exam.

Solution 90. First, we note that the general solution will be of the form

$$y = y_h + y_p$$

We propose that

$$y_p = Ae^{3t}$$

Then we take the necessary derivatives and substitute.

$$y_p = Ae^{3t}$$
$$y_p' = 3Ae^{3t}$$
$$y_p'' = 9Ae^{3t}$$

Substituting the derivatives into the DE, we have

$$9Ae^{3t} - 3 * (3Ae^{3t}) + 2 * (Ae^{3t}) = 2e^{3t}$$
$$2Ae^{3t} = 2e^{3t}$$
$$2A = 2$$
$$A = 1$$
$$\Longrightarrow y_p(t) = e^{3t}$$

We can find

$$y_h = C1e^t + C_2e^{2t}$$

Then the general solution is

$$y = C_1e^t + C_2e^{2t} + e^{3t}$$

Problem 91. Find the solution of the initial value problem:

$$y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$$

Solution 91. **Step 1: Solve the associated homogeneous equation.**
To find the complementary solution $y_c(t)$, we solve the homogeneous equation $y'' - 2y' + y = 0$ using the characteristic equation:

$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0$$

This gives a repeated root $r = 1$ with multiplicity 2. Therefore, the complementary solution is:

$$y_c(t) = C_1e^t + C_2te^t$$

Step 2: Find a particular solution of the nonhomogeneous equation.
The forcing term is $g(t) = te^t + 4$. We split this into $g_1(t) = te^t$ and $g_2(t) = 4$.

- For $g_1(t) = te^t$, the exponent $\alpha = 1$ is a root of the characteristic equation with multiplicity $s = 2$.
  So we have $g_1(t) = te^t$.
  Here s =(multiplicity of $\alpha = 1$ as a root of the characteristic equation) = 2.
  Our trial form is then $Y_1(t) = t^s(A_0t^n + A + 1t^{n-1} + \cdots + A_N)e^{\alpha t} = t^2(A_0t + A_1)e^t$.

- For $g_2(t) = 4$, the exponent $\alpha = 0$ is not a root ($s = 0$).
  So we have $g_2(t) = 4$.
  So here $s =$(multiplicity of 0 as a root of the characteristic equation) = 0
  Our trial form is then $Y_2(t) = t^s(B_0 t^n + B + 1t^{n-1} + \cdots + B_n = B_0$.

Combining these, our trial particular solution is $y_p(t) = Y_1(t) + Y_2(t) = (A_0 t^3 + A_1 t^2)e^t + B_0$. We now need to determine our coefficients A_0, A_1 and B_0.
We calculate the needed derivatives:

$$\begin{aligned} y_p' &= (3A_0 t^2 + 2A_1 t)e^t + (A_0 t^3 + A_1 t^2)e^t \\ &= [A_0 t^3 + (A_1 + 3A_0)t^2 + 2A_1 t]e^t \\ y_p'' &= [3A_0 t^2 + 2(A_1 + 3A_0)t + 2A_1]e^t + [A_0 t^3 + (A_1 + 3A_0)t^2 + 2A_1 t]e^t \\ &= [A_0 t^3 + (A_1 + 6A_0)t^2 + (4A_1 + 6A_0)t + 2A_1]e^t \end{aligned}$$

Substituting y_p into the differential equation and simplifying, we get:

$$6A_0 te^t + 2A_1 e^t + B_0 = (1)te^t + (0)e^t + 4$$

Matching coefficients, we find $6A_0 = 1 \Rightarrow A_0 = \frac{1}{6}$, $2A_1 = 0 \Rightarrow A_1 = 0$, and $B_0 = 4$.

$$y_p(t) = \frac{1}{6}t^3 e^t + 4$$

Step 3: Write the general solution.
The general solution is the sum of the complementary and particular solutions:

$$y(t) = C_1 e^t + C_2 te^t + \frac{1}{6}t^3 e^t + 4$$

Step 4: Determine the constants using initial conditions.
Applying $y(0) = 1$:

$$1 = C_1(1) + C_2(0) + 0 + 4 \Longrightarrow C_1 = -3$$

Now we find $y'(t) = C_1 e^t + C_2(e^t + te^t) + \frac{1}{2}t^2 e^t + \frac{1}{6}t^3 e^t$. Applying $y'(0) = 1$:

$$1 = C_1 + C_2 \Longrightarrow 1 = -3 + C_2 \Longrightarrow C_2 = 4$$

The final solution is:

$$y(t) = -3e^t + 4te^t + \frac{1}{6}t^3 e^t + 4$$

Problem 92. Find the general solution to the differential equation:

$$3y'' + y' - 2y = 2\cos(x)$$

Solution 92. **Step 1: Solve the associated homogeneous equation.**
To find the complementary solution $y_c(x)$, we solve the homogeneous equation $3y'' + y' - 2y = 0$ using the characteristic equation:

$$3r^2 + r - 2 = 0$$
$$(3r - 2)(r + 1) = 0$$

Setting each factor to zero, we find the roots:

$$r_1 = \frac{2}{3}, \quad r_2 = -1$$

Since the roots are real and distinct, the complementary solution is:

$$y_c(x) = C_1 e^{\frac{2}{3}x} + C_2 e^{-x}$$

Step 2: Find a particular solution of the nonhomogeneous equation.
We have $g(x) = 2\cos(x)$. Here the coefficient of x in the cosine term is 1 and the coefficient α of the $e^{\alpha x}$ term is 0. Since the complex number $\alpha + i\beta = 0 + 1i = i$ is not a root of our characteristic equation, the multiplicity is $s = 0$.
Our trial form is then

$$y_p(x) = x^s(A\cos(x) + B\sin(x)) = A\cos(x) + B\sin(x)$$

We now need to determine the coefficients A and B. We calculate the needed derivatives:

$$y_p' = -A\sin(x) + B\cos(x)$$
$$y_p'' = -A\cos(x) - B\sin(x)$$

Substituting y_p, y_p', and y_p'' into the differential equation $3y'' + y' - 2y = 2\cos(x)$:

$$3(-A\cos(x) - B\sin(x)) + (-A\sin(x) + B\cos(x)) - 2(A\cos(x) + B\sin(x)) = 2\cos(x)$$
$$(-3A + B - 2A)\cos(x) + (-3B - A - 2B)\sin(x) = 2\cos(x)$$
$$(-5A + B)\cos(x) + (-A - 5B)\sin(x) = 2\cos(x)$$

Matching the coefficients for $\cos(x)$ and $\sin(x)$:

$$-5A + B = 2$$
$$-A - 5B = 0 \Longrightarrow A = -5B$$

Substituting $A = -5B$ into the first equation:

$$-5(-5B) + B = 2 \Longrightarrow 26B = 2 \Longrightarrow B = \frac{1}{13}$$

Then, $A = -5\left(\frac{1}{13}\right) = -\frac{5}{13}$. So, the particular solution is:

$$y_p(x) = -\frac{5}{13}\cos(x) + \frac{1}{13}\sin(x)$$

Step 3: Write the general solution.
The general solution is the sum of the complementary and particular solutions:

$$y(x) = y_c(x) + y_p(x) = C_1 e^{\frac{2}{3}x} + C_2 e^{-x} - \frac{5}{13}\cos(x) + \frac{1}{13}\sin(x)$$

Section 4.5 - The Superposition Principle and Undetermined Coefficients Revisited

Problem 93. Given that $y_1(t) = e^{3t}$ and $y_2(t) = e^{2t}$ form a fundamental set of solutions for

$$y'' - 5y' + 6y = 0$$

Use the method of undetermined coefficients to solve the differential equation

$$y'' - 5y + 6y = 3e^{2t}$$

Solution 93. Since we are given a fundamental set of solutions e^{3t} and e^{2t}, the homogeneous solution is

$$y_c(t) = C_1 e^{3t} + C_2 e^{2t}.$$

Note that the general solution is obtained by

$$y(t) = y_h(t) + y_p(t)$$

We know our initial guess would be $y_p(t) = Ae^{2t}$. However, this duplicates our solution. Then we need to multiply our initial guess by t. Thus we try

$$y_p(t) = Ate^{2t}.$$

We then compute the derivatives:

$$\begin{aligned} y_p'(t) &= A(e^{2t} + 2te^{2t}) \\ &= Ae^{2t}(1 + 2t) \\ y_p''(t) &= A(2e^{2t} + 2e^{2t} + 4te^{2t}) \\ &= Ae^{2t}(4 + 4t). \end{aligned}$$

Substitute into the left-hand side $y'' - 5y' + 6y$:

$$\begin{aligned} y_p'' - 5y_p' + 6y_p &= Ae^{2t}(4 + 4t) - 5(Ae^{2t}(1 + 2t)) + 6(Ate^{2t}) \\ &= Ae^{2t}[(4 + 4t) - 5(1 + 2t) + 6t] \\ &= Ae^{2t}[4 + 4t - 5 - 10t + 6t] \\ &= Ae^{2t}[-1 + 0 \cdot t] \\ &= -Ae^{2t}. \end{aligned}$$

We want this to equal the right-hand side $3e^{2t}$, so

$$
\begin{aligned}
-Ae^{2t} &= 3e^{2t} \\
-A &= 3 \\
A &= -3.
\end{aligned}
$$

Therefore,

$$y_p(t) = -3te^{2t}.$$

The general solution is

$$y(t) = C_1e^{3t} + C_2e^{2t} - 3te^{2t}$$

Problem 94. Use the method of undetermined coefficients to solve the IVP

$$y'' - 6y' + 9y = 4e^t, \quad y(0) = 0, \quad y'(0) = 2$$

Solution 94. Since the differential equation is linear with constant coefficients, we first solve the homogeneous equation. The characteristic equation is

$$
\begin{aligned}
r^2 - 6r + 9 &= 0 \\
(r-3)^2 &= 0 \\
r &= 3
\end{aligned}
$$

Hence the homogeneous solution is

$$y_h(t) = C_1e^{3t} + C_2te^{3t}$$

Note that the general solution is obtained by

$$y(t) = y_h(t) + y_p(t).$$

For the forcing term $4e^t$, we try a particular solution of the form

$$y_p(t) = Ae^t,$$

since e^t does not duplicate the homogeneous solution.

We then compute the derivatives:

$$
\begin{aligned}
y_p'(t) &= Ae^t \\
y_p''(t) &= Ae^t.
\end{aligned}
$$

Substitute into the ODE yields:

$$
\begin{aligned}
Ae^t - 6(Ae^t) + 9(Ae^t) &= 4e^t \\
(1 - 6 + 9)Ae^t &= 4e^t \\
4Ae^t &= 4e^t \\
4A &= 4 \\
A &= 1.
\end{aligned}
$$

Therefore,

$$y_p(t) = e^t.$$

So the general solution is

$$y(t) = C_1e^{3t} + C_2te^{3t} + e^t = (C_1 + C_2t)e^{3t} + e^t$$

Now apply the initial conditions. First,

$$y(0) = 0 \Rightarrow (C_1 + C_2(0))e^0 + e^0 = 0 \Rightarrow C_1 + 1 = 0 \Rightarrow C_1 = -1.$$

Next compute $y'(t)$. Using the product rule on $(C_1 + C_2t)e^{3t}$,

$$\begin{aligned} y'(t) &= \frac{d}{dt}((C_1 + C_2t)e^{3t}) + \frac{d}{dt}(e^t) \\ &= (C_2 + 3(C_1 + C_2t))e^{3t} + e^t \\ &= (3C_1 + C_2 + 3C_2t)e^{3t} + e^t. \end{aligned}$$

Now use $y'(0) = 2$:

$$y'(0) = (3C_1 + C_2)e^0 + e^0 = 3C_1 + C_2 + 1 = 2.$$

Substitute $C_1 = -1$:

$$3(-1) + C_2 + 1 = 2 \Rightarrow -2 + C_2 = 2 \Rightarrow C_2 = 4.$$

Therefore the explicit solution of the IVP is

$$y(t) = (-1 + 4t)e^{3t} + e^t.$$