Exam 2 Review Sheet

Variation of Parameters

Problem 1. Find the general solution of

$$4y'' + 4y' + y = \frac{e^{-t/2}}{t^2},\ t > 0$$

Problem 2. Find the general solution of

$$y'' + 4y' + 4y = t^{-2}e^{-2t},\ t > 0$$

Problem 3. Find the general solution of

$$y'' + 6y' + 9y = \frac{e^{-3t}}{t^3}$$

Problem 4. Given that $y(t) = c_1 t + c_2 e^{-t}$ is the general solution of the homogeneous equation $(t+1)y'' + ty' - y = 0$ on the interval $t > -1$, find the general solution of the nonhomogeneous equation

$$(t+1)y'' + ty' - y = (t+1)^2$$

for $t > -1$.

Problem 5. Find the general solution of $y'' - 2y' + y = t^{-1}e^t$ on the interval $t > 0$.

Problem 6. Find the general solution of

$$y'' - 6y' + 9y = t^{-3}e^{3t}, \qquad t > 0.$$

Problem 7. Find the general solution of

$$4y'' - 4y' + y = \frac{8e^{t/2}}{t}, \qquad 0 < t < \infty.$$

Problem 8. Find the general solution of

$$y'' + y = \frac{1}{\sin t}, \qquad 0 < t < \pi.$$

Problem 9. Using variation of parameters, find the general solution of

$$t^2y'' - 4ty' + 6y = 3t, \qquad t > 0.$$

Problem 10. Use Variation of Parameters to find the general solution to the following problem:

$$y'' - 2y' + y = \frac{e^t}{t^2}$$

Problem 11. Use Variation of Parameters to find the general solution of the ODE

$$ty'' + (1-2t)y' + (t-1)y = te^t, \quad t > 0$$

given that $y_1 = e^t$ and $y_2 = e^t \ln t$ are linearly independent solutions of the associated homogeneous equation.

Problem 12. Find a particular solution of the differential equation

$$y'' - 2t^{-1}y' + 2t^{-2}y = 3t^2, \quad t > 0,$$

given that $y_1(t) = t$ and $y_2(t) = t^2$ are solutions to the corresponding homogeneous differential equation.

Problem 13. Use variation of parameters to find a particular solution of the differential equation

$$y'' - 4t^{-1}y' + 6t^{-2}y = t^2, \quad t > 0,$$

given that $y_1(t) = t^2$ and $y_2(t) = t^3$ are solutions of the corresponding homogeneous differential equation.

Problem 14. Use variation of parameters to find a particular solution of the differential equation

$$y'' - 4t^{-1}y' + 6t^{-2}y = t^2, \quad t > 0,$$

given that $y_1(t) = t^2$ and $y_2(t) = t^3$ are solutions of the corresponding homogeneous differential equation.

Variable Coefficient Equations, Cauchy-Euler Equations

Problem 15. Find the general solution of

$$t^2y'' + 3ty' + y = 0, \ t > 0$$

Problem 16. Find the general solution of

$$x^2y'' - 2xy' + 2y = 0, \ 0 < x < \infty$$

Problem 17. Solve the initial value problem

$$t^2y'' - 2ty' + 2y = 3t^2, \quad y(1) = 0, \quad y'(1) = 4.$$

Problem 18. Find the general solution of the differential equation

$$x^2y'' + 7xy' + 13y = 0, \; x > 0$$

Problem 19. Find the general solution of the differential equation

$$x^2y'' + 7xy' + 13y = 0, \; x > 0$$

Problem 20. Find the general solution of the differential equation

$$t^2y'' - 6y = 0$$

Problem 21. Find the general solution of the differential equation

$$t^2y'' - 2ty' + 3y = 0.$$

Problem 22. Find the solution of

$$t^2y'' + 6ty' + 6y = 0$$

satisfying $y(1) = 0$ and $y'(1) = 10$.

Problem 23. Solve the initial value problem

$$t^2y'' - 3ty' + 4y = 0, \qquad y(1) = 4, \qquad y'(1) = 5.$$

Problem 24. Solve the following initial value problem:

$$t^2y'' - 5ty' + 9y = 0, \quad y(1) = 1, \quad y'(1) = 4$$

Problem 25. Find the general solution of the differential equation:

$$x^2y'' - xy' - 3y = 0$$

Problem 26. Find the general solution of the differential equation:

$$4t^2y'' + 8ty' + y = 0$$

Problem 27. Find the general solution of the differential equation

$$t^2y'' + 7ty' + 5y = 0, \quad t > 0.$$

Spring-Mass Systems

Problem 28. (Please use 32 ft per seconds squared as the acceleration of gravity in this problem.) A body weighing 8 pounds hangs from a vertical spring attached to the ceiling. At its equilibrium position, the body stretches the spring 1/2 ft from its natural length. The body is started in motion from the equilibrium position with an initial velocity of 4 ft/s in the downward direction.
(a) Assume there is no damping and the body is acted on by a downward external force of

$$F(t) = 3\cos(2t)$$

pounds. Set up, but do not solve, an initial value problem describing the motion of the body.
(b) If the given downward external force is replaced by a force of $3\cos(\omega t)$ pounds, find the value of the frequency ω which will cause resonance.

Problem 29. A certain spring hangs vertically from a rigid support. When a 3 pound mass is attached to the end of the spring, the mass stretches the spring 2 feet. Suppose the mass is pulled down 2 additional feet from the rest position and then released. Assuming air resistance (or the damping force) at any instant is equal to twice the instantaneous velocity of the mass, write BUT DO NOT SOLVE, an initial value problem describing the motion of the mass. (Please use 32 ft per seconds squared as the acceleration of gravity in this problem.)

Problem 30. (a) A 20 kilogram body hangs from a vertical spring attached to a rigid support. At its equilibrium position, the body stretches the spring 50 centimeters beyond its natural length. The body is acted on by an external force of

$$10\cos(2t)$$

Newtons and moves in a medium with a damping constant of 100 Newton seconds per meter. If the body is set in motion from its equilibrium position with an upward velocity of 20 centimeters per second, SET UP, BUT DO NOT SOLVE, an initial value problem describing the motion of the body. (Please use 9.8 meters per second per second as the acceleration of gravity in this problem.)

(b) If the given downward external force is replaced by a force of

$$10\cos(\omega t)$$

Newtons, find the value of the frequency

$$\omega$$

which will cause resonance or explain why there is no such frequency.

Problem 31. A body weighing 64 pounds stretches a spring 2 feet beyond its natural length. The body moves in a medium that impacts a viscous damping force of 8 pounds when the speed of the body is 3 feet per second. The body is pulled down an additional 1 foot and is set in motion with an initial upward velocity of 5 feet per second. Set up, but DO NOT SOLVE, an initial value problem that models the motion of the body. Note: Use

$$32 \text{ ft/sec}^2$$

as the acceleration of gravity in this problem.

Problem 32. (a) A 5 kilogram body hangs from a vertical spring attached to a rigid support. At its equilibrium position, the body stretches the spring 20 centimeters beyond its natural length. The body is acted upon by a downward external force of

$$10\sin(t/2)$$

newtons and there is no damping. If the body is set in motion from a position 10 centimeters below its equilibrium position with an upward initial velocity of 30 centimeters per second, set up, BUT DO NOT SOLVE, an initial value problem that describes the motion of the body. [In the following problem, assume that the acceleration of gravity is 9.8 meters per second per second.]

(b) If the given downward external force is replaced by

$$4\cos(\omega t)$$

newtons, find the value of the frequency omega which will cause resonance or explain why there is no such frequency.

Problem 33. A spring hangs vertically from a rigid support. When a 2 pound box is attached to the end of the spring, the box stretches the spring 0.5 feet and then comes to rest. Suppose the box is lifted up 2 feet from the rest position and then released. Assuming the air resistance (or the damping force) at any instant is equal to half the instantaneous velocity of the box, write BUT DO NOT SOLVE, an initial value problem modeling the motion of the box.

Problem 34. A mass weighing 8 lb hangs from a vertical spring. The spring stretches 4 inches from its natural length at equilibrium. The medium provides a damping constant 20 lb·s/ft. The mass is displaced 2 ft upward and released. An external force $0.001\cos(\omega t)$ acts downward. (a) Set up the initial value problem. (b) Does a resonant frequency exist?

Problem 35. (Use 32 ft/sec^2 as the acceleration due to gravity.) A spring hanging vertically from a rigid support is stretched 6 inches by a mass that weighs 8 lb. The mass is in a medium with damping constant 0.25 lb $\cdot\, s$/ft and is acted on by an external downward force $4\cos(2t)$ lb. Suppose that the mass is displaced 2 ft upward and then released. **Set up, but do not solve,** the initial value problem describing the motion.

Problem 36. A spring hangs vertically from a rigid support. When an 8 pound box is attached to the end of the spring, the box stretches the spring 4 inches and then comes to rest. Suppose the box is given a downward displacement of 6 inches from the rest position and then released with no initial velocity.

(a) Assuming there is no damping and that the box is acted on by an external force $6\cos(4t)$ pounds, formulate but **do not solve** the initial value problem modeling the motion.

(b) If the external force is replaced by $12\sin(\omega t)$ pounds, find the value of $\omega > 0$ for which resonance occurs.

Problem 37. A spring hangs vertically from a rigid support. When a 4 pound box is attached to the end of the spring, the box stretches the spring 0.75 ft and then comes to rest. Suppose the box is pushed down 3 ft below the rest position and then released. Assuming air resistance (the damping force) at any instant is equal to twice the instantaneous velocity of the box, write but do not solve an initial value problem modeling the motion of the box.

Problem 38. A body weighing 8 pounds hangs from a vertical spring attached to the ceiling. At its equilibrium position, the body stretches the spring $1/2$ feet from its natural length. The body is started in motion from the equilibrium position with an initial velocity 4 feet/second in the downward direction. Let $g = 32$ ft/s^2.

(a) Assume that there is no damping and the body is acted on by a downward external force $3\cos(2t)$ pounds, set up, BUT DO NOT SOLVE, an initial value problem that models the motion of the body.

(b) If the given downward external force is replaced by a force of $3\cos(\omega t)$ pounds, find the value of the frequency ω which will cause resonance or explain why there is no such a frequency.

Problem 39. A mass of 2 kg stretches a spring 10 cm. The mass is pulled down 20 cm from the equilibrium position and then released with a downward initial velocity of 2 m/s. To make this problem easier, assume $g = 10$ m/s^2. If there is no air resistance, write but DO NOT SOLVE an initial value problem which models the motion of the mass.

Problem 40. A 2 kg mass is attached to a spring hanging from a ceiling. The mass causes the spring to stretch 0.2 m before coming to rest at equilibrium. The damping constant for the system is 4 Ns/m. At time $t = 0$, the mass is pulled down 50 cm below the equilibrium position and released with a upward initial velocity of 3 m/s. For simplicity, set the gravitational acceleration to $g = 10$ m/s^2. If the mass is acted upon by an external upward force at time t of $8\cos(4t)$ N, set up BUT DO NOT SOLVE an initial value problem that models the motion of the mass.

Problem 41. A 1 kg mass is attached to a spring hanging from a ceiling. The box causes the spring to stretch 10 cm before coming to rest at equilibrium. At time $t = 0$, the box is pulled down 20 cm below the equilibrium position and released with a downward initial velocity of 2 m/s. Ignore air resistance and, for simplicity set the gravitational acceleration to $g = 10$ m/s^2. If no external forces act on the box, set up BUT DO NOT SOLVE an initial value problem that models the motion of the box.

Problem 42. A 1 kg mass is attached to a spring hanging from a ceiling. The box causes the spring to stretch 10 cm before coming to rest at equilibrium. At time $t = 0$, the box is pulled down 20 cm below the equilibrium position and released with a downward initial velocity of 2 m/s. Ignore air resistance and, for simplicity set the gravitational acceleration to $g = 10$ m/s^2. If no external forces act on the box, set up BUT DO NOT SOLVE an initial value problem that models the motion of the box.

Higher Order Differential Equations: Method of Undetermined Coefficients

Problem 43. Solve the differential equation

$$y''' - y = 7e^t$$

Problem 44. Find the general solution of

$$y^{(4)} - 3y'' - 4y = e^{-t}$$

on the interval

$$-\infty < t < \infty$$

.

Problem 45. Find the general solutions of the following differential equations:

(a) $y^{(4)} + 5y'' + 4y = 0$

(b) $y^{(4)} + 5y'' + 4y = 8t^2 + t$

Problem 46. Find the general solution of the differential equation

$$y^{(4)} + 4y'' + 4y = 0.$$

Problem 47. Find the general solution of the differential equation

$$y^{(5)} + 4y''' = 0.$$

Problem 48. Find the general solution of the differential equation

$$y^{(6)} - 16y'' = 0.$$

Problem 49. Find the general solution of the following differential equations.

(a) $y''' - y'' - y' + y = 0$

(b) $t^2y'' + y = 0$

Problem 50. Find the general solution of

$$y^{(4)} + 3y'' - 4y = 2t + e^{2t}.$$

Problem 51. (a) Find the general solution of

$$y^{(5)} - 6y''' + 9y' = 0.$$

Problem 52. Use Method of Undetermined Coefficients to find the general solution of the following problem:

$$y'' - 9y = \sin t$$

Problem 53. Recall that the Laplace transform $F(s)$ of a function $f(t)$ is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \quad s > 0.$$

Use this definition to find the Laplace transform $F(s)$ of the function

$$f(t) = \begin{cases} 0 & \text{when } 0 \le t \le 1 \\ 1 & \text{when } t > 1 \end{cases}$$

and state the values of s for which your answer is valid.

Problem 54. Use the Method of Undetermined Coefficients to find the general solution of

$$y''' - y' = 7e^t.$$

Problem 55. Find the general solution of the differential equation

$$y'' - 3y' + 2y = 2e^t.$$

Problem 56. Find the general solution of the differential equation

$$y^{(5)} + 2y^{(4)} + 2y''' = 0.$$

Problem 57. Find the general solution of the differential equation

$$y'' + 4y' + 4y = e^t.$$

Problem 58. Find the general solution of the differential equation

$$y^{(4)} + 4y'' - 5y = 0.$$

Problem 59. Find the general solution of the differential equation

$$y'' + 4y = \cos(t) - 3\sin(t).$$

Problem 60. Find the general solution of the differential equation

$$y^{(5)} - 3y^{(4)} + 2y''' = 0.$$

Problem 61. Find the general solution of the differential equation

$$y'' + 4y = \cos(t) - 3\sin(t).$$

Problem 62. Find the general solution of the differential equation

$$y^{(5)} - 3y^{(4)} + 2y''' = 0.$$

Laplace Transforms

Problem 63. Use the definition of the Laplace transform, $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$, to find the Laplace transform of the function

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 1, \\ 1 & \text{if } t \geq 1. \end{cases}$$

For which values of s is the Laplace transform of f defined?

Problem 64. Use the definition of the Laplace transform,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

for those values of s for which the improper integral converges, to compute the Laplace transform of the function given by

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ t-1 & \text{if } 1 \leq t < \infty. \end{cases}$$

Problem 65. Use the definition of the Laplace transform,

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$$

for those values of

$$s$$

for which the improper integral converges, to find the Laplace transform of the function

$$f(t) = te^{at}$$

where a is a real constant. For which values of s is the Laplace transform of f defined?

Problem 66. Use the definition of the Laplace transform,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

for those values of s for which this improper integral converges, to find the Laplace transform of the function

$$f(t) = \begin{cases} t & \text{if } 0 \leq t < 2, \\ e^t & \text{if } t \geq 2. \end{cases}$$

For which values of s is the Laplace transform of f defined?

Problem 67. Use the definition of the Laplace transform,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

for those values of s for which this improper integral converges, to find the Laplace transform of the function

$$f(t) = \begin{cases} \pi & \text{if } 0 \leq t < \pi, \\ t & \text{if } \pi \leq t < \infty. \end{cases}$$

For which values of s is the Laplace transform of f defined?

Problem 68. Recall the definition of the Laplace transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

(a) Use the definition to find the Laplace transform of $f(t) = t$.

(b) For which values of s is the transform defined?

Problem 69. Using the definition of the Laplace transform, compute the Laplace transform of

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t \leq 2 \\ 2, & t > 2 \end{cases}$$

and determine for which s the transform is defined.

Problem 70. Recall the definition of the Laplace transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

(a) Use the definition to find $\mathcal{L}\{f\}(s)$ for

$$f(t) = \begin{cases} 2, & 0 \leq t \leq 1, \\ e^t, & t > 1. \end{cases}$$

(b) For which values of s is the Laplace transform defined?

Problem 71. Using the definition of the Laplace transform, find $\mathcal{L}\{f(t)\}(s)$ if

$$f(t) = \begin{cases} 7e^{4t} & 0 < t < 3 \\ 4 & t > 3 \end{cases}$$

You must also specify the domain of the transform.

Inverse Laplace Transforms

Problem 72. Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s-2)(s-3)(s-6)}$$

Problem 73. Find the inverse Laplace transform of

$$\frac{3s+5}{s^2-4s+5}.$$

Problem 74. Find the inverse Laplace transform of

$$F(s) = \frac{s^2+s+2}{s^3+s}.$$

Problem 75. Find the inverse Laplace transform of

$$F(s) = \frac{s+1}{s^2-s-6}.$$

Problem 76. Find the inverse Laplace transform of

$$F(s) = \frac{5s+1}{s^2-2s+3}.$$

Problem 77. Find the inverse Laplace transform of

$$F(s) = \frac{5s}{s^2-s-6}.$$

Problem 78. Find the inverse Laplace transform of

$$H(s) = \frac{1}{s(s^2+1)}$$

Problem 79. Find the inverse Laplace transform of

$$F(s) = \frac{s+11}{(s-1)(s+3)}$$

Problem 80. Determine the inverse Laplace transform of

$$F(s) = \frac{5}{(s-1)(s^2+4)}.$$

Problem 81. Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+9)}$$

Problem 82. Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+9)}$$

Exam 2 Review Sheet

Variation of Parameters

Problem 1. Find the general solution of

$$4y'' + 4y' + y = \frac{e^{-t/2}}{t^2},\ t > 0$$

Solution 1. First, we find the homogeneous solution. Using methods we have already covered, we find the homogeneous solution

$$y_h = C_1 y_1 + C_2 y_2 = C_1 e^{-t/2} + C_2 t e^{-t/2}$$

And we turn to the method of variation of parameters. The vital preliminary step is to write the differential equation in the required standard form. This enforces that we have the correct g(t) on the right side. Here, standard form is

$$y'' + y' + \frac{1}{4}y = \frac{e^{-t/2}}{4t^2}$$

We have that the particular solution will be of the form

$$y_p = u_1 y_1 + u_2 y_2$$

where

$$u_1 = \int \frac{g(t)W_2}{W}dt,\ u_2 = \int \frac{g(t)W_2}{W}dt$$

As always start by obtaining the Wronskian

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} e^{-t/2} & te^{-t/2} \\ -\frac{1}{2}e^{-t/2} & (1-\frac{t}{2})e^{-t/2} \end{bmatrix} = (1-\frac{t}{2})e^{-t} + \frac{t}{2}e^{-t} = e^{-t}$$

Then we quickly find the corresponding

$$W_1 = \det\begin{bmatrix} 0 & y_2 \\ 1 & y_2' \end{bmatrix} = -y_2 = -te^{-t/2},\ W_2 = \det\begin{bmatrix} y_1 & 0 \\ y_1' & 1 \end{bmatrix} = y_1 = e^{-t/2}$$

Finally, note

$$g(t) = \frac{e^{-t/2}}{4t^2}$$

We plug in and integrate.

$$u_1 = \int \frac{(\frac{e^{-t/2}}{4t^2})(te^{-t/2})}{e^{-t}} dt = -\frac{1}{4}\int \frac{e^{-}t}{te^{-t}} dt = -\frac{1}{4}\int \frac{1}{t} dt = -\frac{1}{4} ln(t)$$

Note above we have used that t is positive to disregard the absolute value.

$$u_2 = \int \frac{(\frac{e^{-t/2}}{4t^2})(e^{-t/2})}{e^{-t}} dt = \frac{1}{4}\int \frac{e^{-t}}{t^2 e^{-t}} dt = \frac{1}{4}\int \frac{1}{t^2} dt = -\frac{1}{4t} =$$

So we have the particular solution

$$y_p = -\frac{1}{4} ln(t) e^{-t/2} - \frac{1}{4t} te^{-t/2} = -\frac{1}{4} ln(t) e^{-t/2} - \frac{1}{4} e^{-t/2}$$

Finally, we have the general solution

$$y = y_h + y_p = C_1 e^{-t/2} + C_2 te^{-t/2} + \left(-\frac{1}{4} ln(t) e^{-t/2} - \frac{1}{4} e^{-t/2}\right)$$
$$= C_3 e^{-t/2} + C_2 te^{-t/2} - \frac{1}{4} ln(t) e^{-t/2}$$

Note above that we have set

$$C_3 = C_1 - \frac{1}{4}$$

to further reduce.

Problem 2. Find the general solution of

$$y'' + 4y' + 4y = t^{-2} e^{-2t},\ t > 0$$

Solution 2. First, we find the homogeneous solution. We solve the homogeneous equation

$$y'' + 4y' + 4y = 0.$$

The corresponding characteristic equation is

$$r^2 + 4r + 4 = (r + 2)^2 = 0.$$

Thus we have a repeated root

$$r = -2$$

and the homogeneous solution is

$$y_h = C_1 e^{-2t} + C_2 te^{-2t}.$$

Next, we turn to the method of variation of parameters. The differential equation is already in standard form, so the right-hand side is

$$g(t) = t^{-2}e^{-2t} = \frac{e^{-2t}}{t^2}.$$

The particular solution is of the form

$$y_p = u_1y_1 + u_2y_2$$

where

$$y_1 = e^{-2t}$$

and

$$y_2 = te^{-2t}$$

We start by computing the Wronskian.

$$W = \det\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{bmatrix} = e^{-2t}(1-2t)e^{-2t} - (te^{-2t})(-2e^{-2t})$$
$$= e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}.$$

Now we compute

u_1 and u_2

$$u_1 = \int \frac{-y_2 g(t)}{W} = \int \frac{-(te^{-2t})(\frac{e^{-2t}}{t^2})}{e^{-4t}} = \int \frac{-\frac{e^{-4t}}{t}}{e^{-4t}} = \int -\frac{1}{t} = -\ln(t)$$

$$u_2 = \int \frac{y_1 g(t)}{W} = \int \frac{(e^{-2t})(\frac{e^{-2t}}{t^2})}{e^{-4t}} = \int \frac{\frac{e^{-4t}}{t^2}}{e^{-4t}} = \int \frac{1}{t^2} = -\frac{1}{t}$$

(Since t > 0, we omit the absolute value in the natural log.

The particular solution is:

$$y_p = u_1y_1 + u_2y_2 = -\ln(t)e^{-2t} + \left(-\frac{1}{t}\right)te^{-2t} = -e^{-2t}\ln(t) - e^{-2t}.$$

Finally, the general solution is:

$$y = y_h + y_p = C_1e^{-2t} + C_2te^{-2t} - e^{-2t}\ln(t) - e^{-2t}.$$

We can combine the terms

$$C_1e^{-2t}$$

and

$$-e^{-2t}$$

by letting

$$C_3 = C_1 - 1$$

Then the final solution is

$$y = C_3 e^{-2t} + C_2 t e^{-2t} - e^{-2t}\ln(t).$$

Problem 3. Find the general solution of

$$y'' + 6y' + 9y = \frac{e^{-3t}}{t^3}$$

Solution 3. First, we find the homogeneous solution. We solve the homogeneous equation

$$y'' + 6y' + 9y = 0.$$

The corresponding characteristic equation is

$$r^2 + 6r + 9 = (r+3)^2 = 0.$$

Thus we have a repeated root

$$r = -3$$

and the homogeneous solution is

$$y_h = C_1 e^{-3t} + C_2 t e^{-3t}.$$

Next, we turn to the method of variation of parameters. The differential equation is already in standard form, so the right-hand side is

$$g(t) = \frac{e^{-3t}}{t^3}.$$

The particular solution is of the form

$$y_p = u_1 y_1 + u_2 y_2$$

We start by computing the Wronskian.

$$W = \det\begin{bmatrix} e^{-3t} & te^{-3t} \\ -3e^{-3t} & (1-3t)e^{-3t} \end{bmatrix} = e^{-3t}(1-3t)e^{-3t} - (te^{-3t})(-3e^{-3t}) = e^{-6t}.$$

Then we find the corresponding

$$W_1 \text{ and } W_2$$

$$W_1 = \det\begin{bmatrix} 0 & y_2 \\ 1 & y_2' \end{bmatrix} = -y_2 = -te^{-3t}$$

$$W_2 = \det\begin{bmatrix} y_1 & 0 \\ y_1' & 1 \end{bmatrix} = y_1 = e^{-3t}$$

Now we compute

$$u_1 \text{ and } u_2$$

by integrating:

$$u_1 = \int \frac{g(t)W_1}{W}dt = \int \frac{(\frac{e^{-3t}}{t^3})(-te^{-3t})}{e^{-6t}}dt = \int \frac{-\frac{e^{-6t}}{t^2}}{e^{-6t}}dt = \int -\frac{1}{t^2}dt = \frac{1}{t}$$

$$u_2 = \int \frac{g(t)W_2}{W}dt = \int \frac{(\frac{e^{-3t}}{t^3})(e^{-3t})}{e^{-6t}}dt = \int \frac{\frac{e^{-6t}}{t^3}}{e^{-6t}}dt = \int \frac{1}{t^3}dt = -\frac{1}{2t^2}$$

The particular solution is:

$$y_p = u_1y_1 + u_2y_2 = \left(\frac{1}{t}\right)e^{-3t} + \left(-\frac{1}{2t^2}\right)te^{-3t} = \frac{e^{-3t}}{t} - \frac{e^{-3t}}{2t} = \frac{e^{-3t}}{2t}.$$

Finally, the general solution is:

$$y = y_h + y_p = C_1e^{-3t} + C_2te^{-3t} + \frac{e^{-3t}}{2t}.$$

Problem 4. Given that $y(t) = c_1t + c_2e^{-t}$ is the general solution of the homogeneous equation $(t+1)y'' + ty' - y = 0$ on the interval $t > -1$, find the general solution of the nonhomogeneous equation

$$(t+1)y'' + ty' - y = (t+1)^2$$

for $t > -1$.

Solution 4. First, we identify the homogeneous solution and its components from the given information:

$$y_h = C_1t + C_2e^{-t} \Longrightarrow y_1 = t, \quad y_2 = e^{-t}.$$

Next, we put the differential equation into standard form by dividing by $(t+1)$ to find $g(t)$:

$$y'' + \frac{t}{t+1}y' - \frac{1}{t+1}y = t+1.$$

Thus, $g(t) = t + 1$.

We start by computing the Wronskian:

$$W = \det\begin{bmatrix} t & e^{-t} \\ 1 & -e^{-t} \end{bmatrix} = t(-e^{-t}) - (1)(e^{-t}) = -(t+1)e^{-t}.$$

Then we find the corresponding W_1 and W_2:

$$W_1 = \det\begin{bmatrix} 0 & y_2 \\ 1 & y_2' \end{bmatrix} = -y_2 = -e^{-t}$$

$$W_2 = \det\begin{bmatrix} y_1 & 0 \\ y_1' & 1 \end{bmatrix} = y_1 = t$$

Now we compute u_1 and u_2 by integrating:

$$u_1 = \int \frac{g(t)W_1}{W} dt = \int \frac{(t+1)(-e^{-t})}{-(t+1)e^{-t}} dt = \int 1 dt = t$$

$$u_2 = \int \frac{g(t)W_2}{W} dt = \int \frac{(t+1)(t)}{-(t+1)e^{-t}} dt = \int -te^{t} dt$$

Using integration by parts for u_2:

$$u_2 = -te^t + \int e^t dt = -te^t + e^t = (1-t)e^t$$

The particular solution is:

$$y_p = u_1 y_1 + u_2 y_2 = (t)(t) + (1-t)e^t(e^{-t}) = t^2 + 1 - t.$$

Finally, the general solution is:

$$y = y_h + y_p = C_1 t + C_2 e^{-t} + t^2 - t + 1.$$

And if we prefer, we can set

$$C_3 = C_1 - 1$$

Then we can write

$$y = C_3 t + C_2 e^{-t} + t^2 + 1$$

Problem 5. Find the general solution of $y'' - 2y' + y = t^{-1}e^t$ on the interval $t > 0$.

Solution 5. First, we find the homogeneous solution. We solve the homogeneous equation

$$y'' - 2y' + y = 0.$$

The corresponding characteristic equation is

$$r^2 - 2r + 1 = (r-1)^2 = 0.$$

Thus we have a repeated root

$$r = 1$$

and the homogeneous solution is

$$y_h = C_1 e^t + C_2 te^t.$$

Next, we use the method of variation of parameters. The differential equation is already in standard form, so the right-hand side is

$$g(t) = t^{-1}e^t = \frac{e^t}{t}.$$

The particular solution is of the form

$$y_p = u_1 y_1 + u_2 y_2$$

where

$$y_1 = e^t \text{ and } y_2 = te^t$$

We start by computing the Wronskian:

$$W = \det\begin{bmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{bmatrix} = e^t(1+t)e^t - (te^t)(e^t) = e^{2t} + te^{2t} - te^{2t} = e^{2t}.$$

Then we find the corresponding

$$W_1 \text{ and } W_2$$

$$W_1 = \det\begin{bmatrix} 0 & y_2 \\ 1 & y_2' \end{bmatrix} = -y_2 = -te^t$$

$$W_2 = \det\begin{bmatrix} y_1 & 0 \\ y_1' & 1 \end{bmatrix} = y_1 = e^t$$

Now we compute

$$u_1 \text{ and } u_2$$

.

$$u_1 = \int \frac{g(t)W_1}{W} dt = \int \frac{(\frac{e^t}{t})(-te^t)}{e^{2t}} dt = \int \frac{-e^{2t}}{e^{2t}} dt = \int -1 dt = -t$$

$$u_2 = \int \frac{g(t)W_2}{W} dt = \int \frac{(\frac{e^t}{t})(e^t)}{e^{2t}} dt = \int \frac{\frac{e^{2t}}{t}}{e^{2t}} dt = \int \frac{1}{t} dt = \ln t$$

We neglect the absolute value in the natural log as the we specify that t is positive.

The particular solution is:

$$y_p = u_1 y_1 + u_2 y_2 = (-t)e^t + (\ln t)te^t = -te^t + te^t \ln t.$$

Finally, the general solution is:

$$y = y_h + y_p = C_1 e^t + C_2 te^t - te^t + te^t \ln t.$$

Since $-te^t$ is a part of the homogeneous solution, we can absorb it into the second constant.

$$y = C_1e^t + C_2te^t + te^t\ln t.$$

Problem 6. Find the general solution of

$$y'' - 6y' + 9y = t^{-3}e^{3t}, \qquad t > 0.$$

Solution 6. First solve the homogeneous equation

$$y'' - 6y' + 9y = 0.$$

Let $y = e^{rt}$. Then

$$\begin{aligned} r^2 - 6r + 9 &= 0 \\ (r-3)^2 &= 0 \end{aligned}$$

Thus,

$$y_h(t) = c_1e^{3t} + c_2te^{3t}.$$

Using variation of parameters, we assume the solution has the form:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where $y_1 = e^{3t}$ and $y_2 = te^{3t}$.

The Wronskian is

$$W(y_1, y_2) = \begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (3t+1)e^{3t} \end{vmatrix} = e^{6t}.$$

Recall that our $g(t) = t^{-3}e^{3t}$. Then,

$$u_1' = \frac{W_1g(t)}{W} = -\frac{y_2g(t)}{W}, \qquad u_2' = \frac{W_2g(t)}{W} = \frac{y_1g(t)}{W}.$$

Thus,

$$\begin{aligned} u_1' &= -\frac{te^{3t}t^{-3}e^{3t}}{e^{6t}} = -t^{-2} \\ u_1 &= \frac{1}{t} \end{aligned}$$

and

$$\begin{aligned} u_2' &= \frac{e^{3t}t^{-3}e^{3t}}{e^{6t}} = t^{-3} \\ u_2 &= -\frac{1}{2t^2} \end{aligned}$$

Hence our particular solution becomes,

$$y_p = \frac{1}{t}e^{3t} - \frac{1}{2t^2}(te^{3t}) = \frac{1}{2t}e^{3t}.$$

Therefore the general solution is

$$y(t) = c_1e^{3t} + c_2te^{3t} + \frac{1}{2t}e^{3t}.$$

Problem 7. Find the general solution of

$$4y'' - 4y' + y = \frac{8e^{t/2}}{t}, \qquad 0 < t < \infty.$$

Solution 7. First, write the equation in standard form:

$$y'' - y' + \frac{1}{4}y = 2t^{-1}e^{t/2}.$$

First solve the homogeneous equation:

$$y'' - y' + \frac{1}{4}y = 0.$$

By assuming the solution is in the form $y = e^{rt}$, the characteristic equation is

$$\begin{aligned} r^2 - r + \frac{1}{4} &= 0 \\ 4r^2 - 4r + 1 &= 0 \\ (2r-1)^2 &= 0 \end{aligned}$$

Thus $r = \frac{1}{2}$. This leads to the homogeneous solution:

$$y_h = c_1e^{t/2} + c_2te^{t/2}.$$

with

$$y_1 = e^{t/2}, \qquad y_2 = te^{t/2}.$$

We assume that the particular solution is given in the form:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

The Wronskian is given by:

$$W(y_1, y_2) = \begin{vmatrix} e^{\frac{t}{2}} & te^{\frac{t}{2}} \\ \frac{1}{2}e^{\frac{t}{2}} & e^{\frac{t}{2}} + \frac{1}{2}te^{\frac{t}{2}} \end{vmatrix} = e^t.$$

Recall that $g(t) = 2t^{-1}e^{t/2}$, then:

$$u_1' = -\frac{(2t^{-1}e^{t/2})(te^{t/2})}{e^t} = -2$$
$$u_2' = \frac{(2t^{-1}e^{t/2})(e^{t/2})}{e^t} = \frac{2}{t}.$$

Integrating gives

$$u_1 = -2t, \qquad u_2 = 2\ln t.$$

Therefore, the particular solution is given by:

$$y_p = (-2t)e^{t/2} + (2\ln t)(te^{t/2}).$$

Hence the general solution is

$$y = c_1 e^{t/2} + c_2 te^{t/2} + (-2t)e^{t/2} + (2\ln t)(te^{t/2}).$$

Problem 8. Find the general solution of

$$y'' + y = \frac{1}{\sin t}, \qquad 0 < t < \pi.$$

Solution 8. First solve the homogeneous equation

$$y'' + y = 0.$$

Assuming a solution in the form $y = e^{rt}$, the characteristic equation is

$$r^2 + 1 = 0 \Rightarrow r = \pm i$$

so,

$$y_1 = \cos t, \qquad y_2 = \sin t.$$

And the associated homogeneous solution is given by:

$$y_h = C_1 \cos t + C_2 \sin t.$$

Now we find the particular solution using variation of parameters,

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

The Wronskian is given by:

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1.$$

Also,

$$W_1 = \begin{vmatrix} 0 & \sin t \\ 1 & \cos t \end{vmatrix} = \sin t.$$

$$W_2 = \begin{vmatrix} \cos t & 0 \\ -\sin t & 1 \end{vmatrix} = \cos t.$$

Then,

$$u_1' = \frac{W_1 g(t)}{W} = -\frac{\sin t}{\sin t} = -1$$

$$u_2' = \frac{W_2 g(t)}{W} = \frac{\cos t}{\sin t}$$

so,

$$u_1 = -t, \qquad u_2 = \ln|\sin t|$$

Note that the domain is given by $0 < t < \pi$, which means that $\sin t > 0$, so we do not need the absolute value sign inside the natural log.

Hence,

$$y_p = -t\cos t + (\ln(\sin t))\sin t.$$

Therefore, the general solution is given by:

$$y = C_1\cos t + C_2\sin t - t\cos t + (\ln(\sin t))\sin t.$$

Problem 9. Using variation of parameters, find the general solution of

$$t^2 y'' - 4ty' + 6y = 3t, \qquad t > 0.$$

Solution 9. First solve the homogeneous equation:

$$t^2 y'' - 4ty' + 6y = 0.$$

By assuming the solution is in the form $y = t^m$. We can substitute this to obtain the characteristic equation:

$$\begin{aligned} m(m-1) - 4m + 6 &= 0 \\ m^2 - 5m + 6 &= 0 \\ (m-2)(m-3) &= 0 \end{aligned}$$

Hence,

$$y_1 = t^2, \qquad y_2 = t^3.$$

and the associated homogeneous solution is given by:

$$y_h = C_1 t^2 + C_2 t^3.$$

Now, we find the particular solution using variation of parameters: Assume the particular solution is in the form:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

The Wronskian is

$$W = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = t^4.$$

with,

$$W_1 = \begin{vmatrix} 0 & t^3 \\ 1 & 3t^2 \end{vmatrix} = -t^3.$$

$$W_2 = \begin{vmatrix} t^2 & 0 \\ 2t & 1 \end{vmatrix} = t^2.$$

And g(t) is given by:

$$g(t) = \frac{3t}{t^2} = \frac{3}{t}.$$

Compute

$$u_1' = \frac{W_1 g(t)}{W} = -\frac{t^3(3/t)}{t^4} = -\frac{3}{t^2},$$

$$u_2' = \frac{W_2 g(t)}{W} = \frac{t^2(3/t)}{t^4} = \frac{3}{t^3}.$$

Integrating yields:

$$u_1 = \frac{3}{t}, \qquad u_2 = -\frac{1}{2t^2}.$$

Thus,

$$y_p = \left(\frac{3}{t}\right)t^2 + \left(-\frac{1}{2t^2}\right)t^3$$

Then, the particular solution is given by:

$$y_p = 3t - \frac{1}{2}t = \frac{5}{2}t.$$

Hence, the general solution is:

$$y = C_1 t^2 + C_2 t^3 + \frac{5}{2}t.$$

Problem 10. Use Variation of Parameters to find the general solution to the following problem:

$$y'' - 2y' + y = \frac{e^t}{t^2}$$

Solution 10. First, we find the complementary solution. We solve the homogeneous equation

$$y_c'' - 2y_c' + y_c = 0.$$

The corresponding characteristic equation is

$$r^2 - 2r + 1 = (r-1)^2 = 0.$$

Thus, we have a repeated root $r = 1$. The complementary solution is

$$y_c(t) = C_1 e^t + C_2 t e^t.$$

Next, we turn to the method of variation of parameters. The differential equation is already in standard form, so the right-hand side is

$$g(t) = \frac{e^t}{t^2}.$$

The particular solution is of the form

$$y_p = u_1 y_1 + u_2 y_2$$

where

$$y_1 = e^t$$

and

$$y_2 = te^t.$$

We start by computing the Wronskian:

$$W = \det \begin{bmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{bmatrix} = e^t(1+t)e^t - (te^t)(e^t) = e^{2t} + te^{2t} - te^{2t} = e^{2t}.$$

Then we find the corresponding

W_1 and W_2

$$W_1 = \det \begin{bmatrix} 0 & y_2 \\ 1 & y_2' \end{bmatrix} = -y_2 = -te^t$$

$$W_2 = \det \begin{bmatrix} y_1 & 0 \\ y_1' & 1 \end{bmatrix} = y_1 = e^t$$

Now we compute

u_1 and u_2

by integrating:

$$u_1 = \int \frac{g(t)W_1}{W} dt = \int \frac{\left(\frac{e^t}{t^2}\right)(-te^t)}{e^{2t}} dt = \int \frac{-\frac{e^{2t}}{t}}{e^{2t}} dt = \int -\frac{1}{t} dt = -\ln t$$

$$u_2 = \int \frac{g(t)W_2}{W}dt = \int \frac{\left(\frac{e^t}{t^2}\right)(e^t)}{e^{2t}}dt = \int \frac{\frac{e^{2t}}{t^2}}{e^{2t}}dt = \int \frac{1}{t^2}dt = -\frac{1}{t}$$

(Since $t > 0$, we omit the absolute value in the natural log).

The particular solution is:

$$y_p = u_1y_1 + u_2y_2 = (-\ln t)e^t + \left(-\frac{1}{t}\right)te^t = -e^t\ln t - e^t.$$

Finally, the general solution is:

$$y = y_c + y_p = C_1e^t + C_2te^t - e^t\ln t - e^t.$$

We can combine the terms C_1e^t and $-e^t$ by letting $C_3 = C_1 - 1$. Then the final solution is:

$$y(t) = C_3e^t + C_2te^t - e^t\ln t.$$

Problem 11. Use Variation of Parameters to find the general solution of the ODE

$$ty'' + (1-2t)y' + (t-1)y = te^t, \quad t > 0$$

given that $y_1 = e^t$ and $y_2 = e^t\ln t$ are linearly independent solutions of the associated homogeneous equation.

Solution 11. First, we write the differential equation in standard form by dividing by the leading coefficient t:

$$y'' + \left(\frac{1-2t}{t}\right)y' + \left(\frac{t-1}{t}\right)y = e^t.$$

From this, we identify the right-hand side function $g(t) = e^t$.

Next, we compute the Wronskian W of the homogeneous solutions $y_1 = e^t$ and $y_2 = e^t\ln t$:

$$W = \det\begin{bmatrix} e^t & e^t\ln t \\ e^t & e^t\ln t + \frac{1}{t}e^t \end{bmatrix} = e^t\left(e^t\ln t + \frac{1}{t}e^t\right) - (e^t\ln t)(e^t) = e^{2t}\ln t + \frac{1}{t}e^{2t} - e^{2t}\ln t$$
$$= \frac{1}{t}e^{2t}.$$

We then find W_1 and W_2 by replacing the columns of the identity-like matrix with $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (to be used with $g(t)$ later):

$$W_1 = \det\begin{bmatrix} 0 & e^t\ln t \\ 1 & e^t\ln t + \frac{1}{t}e^t \end{bmatrix} = -e^t\ln t, \quad W_2 = \det\begin{bmatrix} e^t & 0 \\ e^t & 1 \end{bmatrix} = e^t.$$

Now we compute u_1 and u_2 by integrating:

$$u_1 = \int \frac{g(t)W_1}{W} dt = \int \frac{e^t(-e^t \ln t)}{\frac{1}{t}e^{2t}} dt = \int -t\ln t\, dt.$$

Using integration by parts with $u = \ln t$ and $dv = -t\,dt$:

$$u_1 = -\frac{1}{2}t^2 \ln t + \int \frac{t^2}{2} \cdot \frac{1}{t} dt = -\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2.$$

For u_2:

$$u_2 = \int \frac{g(t)W_2}{W} dt = \int \frac{e^t(e^t)}{\frac{1}{t}e^{2t}} dt = \int t\, dt = \frac{1}{2}t^2.$$

The particular solution y_p is:

$$y_p = u_1 y_1 + u_2 y_2 = \left(-\frac{1}{2}t^2 \ln t + \frac{1}{4}t^2\right)e^t + \left(\frac{1}{2}t^2\right)e^t \ln t = -\frac{1}{2}t^2 e^t \ln t + \frac{1}{4}t^2 e^t + \frac{1}{2}t^2 e^t \ln t$$
$$= \frac{1}{4}t^2 e^t.$$

Finally, the general solution is:

$$y(t) = C_1 e^t + C_2 e^t \ln t + \frac{1}{4}t^2 e^t.$$

Problem 12. Find a particular solution of the differential equation

$$y'' - 2t^{-1}y' + 2t^{-2}y = 3t^2, \quad t > 0,$$

given that $y_1(t) = t$ and $y_2(t) = t^2$ are solutions to the corresponding homogeneous differential equation.

Solution 12. We use the method of variation of parameters to find the particular solution y_p. We assume a solution of the form:

$$y_p = u_1 y_1 + u_2 y_2 = u_1(t) \cdot t + u_2(t) \cdot t^2.$$

First, we check that the differential equation is in standard form. The coefficient of y'' is 1, so the right-hand side is $g(t) = 3t^2$.

Next, we compute the Wronskian of the homogeneous solutions:

$$W = \det\begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix} = t(2t) - (t^2)(1) = 2t^2 - t^2 = t^2.$$

Now we calculate u_1 and u_2 using the variation of parameters formulas:

$$u_1' = \frac{-y_2 g(t)}{W} = \frac{-(t^2)(3t^2)}{t^2} = -3t^2$$

Integrating u_1' gives:

$$u_1 = \int -3t^2 dt = -t^3.$$

For u_2:

$$u_2' = \frac{y_1 g(t)}{W} = \frac{(t)(3t^2)}{t^2} = 3t$$

Integrating u_2' gives:

$$u_2 = \int 3t dt = \frac{3}{2}t^2.$$

The particular solution y_p is:

$$y_p = u_1 y_1 + u_2 y_2 = (-t^3)(t) + \left(\frac{3}{2}t^2\right)(t^2)$$

$$y_p = -t^4 + \frac{3}{2}t^4 = \frac{1}{2}t^4.$$

Problem 13. Use variation of parameters to find a particular solution of the differential equation

$$y'' - 4t^{-1}y' + 6t^{-2}y = t^2, \quad t > 0,$$

given that $y_1(t) = t^2$ and $y_2(t) = t^3$ are solutions of the corresponding homogeneous differential equation.

Solution 13. We use the method of variation of parameters to find the particular solution y_p. We assume a solution of the form:

$$y_p = u_1 y_1 + u_2 y_2 = u_1 t^2 + u_2 t^3.$$

First, we verify the differential equation is in standard form. Since the coefficient of y'' is 1, the right-hand side function is $g(t) = t^2$.

Next, we compute the Wronskian of the homogeneous solutions:

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix} = t^2(3t^2) - (t^3)(2t) = 3t^4 - 2t^4 = t^4.$$

Now we calculate u_1 and u_2 using the variation of parameters formulas:

$$u_1' = \frac{-y_2 g(t)}{W} = \frac{-(t^3)(t^2)}{t^4} = -t.$$

Integrating u_1' gives:

$$u_1 = \int -t\, dt = -\frac{1}{2}t^2.$$

For u_2:

$$u_2' = \frac{y_1 g(t)}{W} = \frac{(t^2)(t^2)}{t^4} = 1.$$

Integrating u_2' gives:

$$u_2 = \int 1\, dt = t.$$

The particular solution y_p is:

$$y_p = u_1 y_1 + u_2 y_2 = \left(-\frac{1}{2}t^2\right)(t^2) + (t)(t^3)$$

$$y_p = -\frac{1}{2}t^4 + t^4 = \frac{1}{2}t^4.$$

Problem 14. Use variation of parameters to find a particular solution of the differential equation

$$y'' - 4t^{-1}y' + 6t^{-2}y = t^2, \quad t > 0,$$

given that $y_1(t) = t^2$ and $y_2(t) = t^3$ are solutions of the corresponding homogeneous differential equation.

Solution 14. We use the method of variation of parameters to find the particular solution y_p. We assume a solution of the form:

$$y_p = u_1 y_1 + u_2 y_2 = u_1 t^2 + u_2 t^3.$$

First, we verify the differential equation is in standard form. Since the coefficient of y'' is 1, the right-hand side function is $g(t) = t^2$.

Next, we compute the Wronskian of the homogeneous solutions:

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det\begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix} = t^2(3t^2) - (t^3)(2t) = 3t^4 - 2t^4 = t^4.$$

Now we calculate u_1 and u_2 using the variation of parameters formulas:

$$u_1' = \frac{-y_2 g(t)}{W} = \frac{-(t^3)(t^2)}{t^4} = -t.$$

Integrating u_1' gives:

$$u_1 = \int -t\, dt = -\frac{1}{2}t^2.$$

For u_2:

$$u_2' = \frac{y_1 g(t)}{W} = \frac{(t^2)(t^2)}{t^4} = 1.$$

Integrating u_2' gives:

$$u_2 = \int 1\, dt = t.$$

The particular solution y_p is:

$$y_p = u_1 y_1 + u_2 y_2 = \left(-\frac{1}{2}t^2\right)(t^2) + (t)(t^3)$$

$$y_p = -\frac{1}{2}t^4 + t^4 = \frac{1}{2}t^4.$$

Variable Coefficient Equations, Cauchy-Euler Equations

Problem 15. Find the general solution of

$$t^2 y'' + 3ty' + y = 0,\ t > 0$$

Solution 15. First, we recognize that this is a Cauchy–Euler equation. For such equations, we try a solution of the form

$$y = t^r.$$

We compute the derivatives

$$y' = rt^{r-1}, \qquad y'' = r(r-1)t^{r-2}.$$

Substituting these into the differential equation gives

$$t^2(r(r-1)t^{r-2}) + 3t(rt^{r-1}) + t^r = 0.$$

Simplifying, we obtain

$$r(r-1)t^r + 3rt^r + t^r = 0.$$

Factoring out t^r yields

$$t^r(r(r-1) + 3r + 1) = 0.$$

Thus we obtain the characteristic equation

$$r(r-1) + 3r + 1 = 0.$$

Simplifying gives

$$r^2 + 2r + 1 = (r+1)^2 = 0$$

Therefore we have a repeated root

$$r = -1.$$

For a repeated root in a Cauchy–Euler equation, the homogeneous solution takes the form

$$y_h = C_1 t^r + C_2 t^r \ln(t).$$

Substituting $r = -1$ gives

$$y = C_1 t^{-1} + C_2 t^{-1} \ln(t).$$

Problem 16. Find the general solution of

$$x^2 y'' - 2xy' + 2y = 0, \ 0 < x < \infty$$

Solution 16. First, we recognize that this is a Cauchy–Euler equation. For such equations, we try a solution of the form

$$y = x^r.$$

We compute the derivatives

$$y' = r x^{r-1}, \qquad y'' = r(r-1)x^{r-2}.$$

Substituting these into the differential equation gives

$$x^2(r(r-1)x^{r-2}) - 2x(rx^{r-1}) + 2x^r = 0.$$

Simplifying, we obtain

$$r(r-1)x^r - 2rx^r + 2x^r = 0.$$

Factoring out x^r yields

$$x^r(r(r-1) - 2r + 2) = 0.$$

Thus we obtain the characteristic equation

$$r(r-1) - 2r + 2 = 0.$$

Simplifying gives

$$r^2 - r - 2r + 2 = 0$$

$$r^2 - 3r + 2 = 0.$$

Factoring the quadratic equation, we find

$$(r-1)(r-2) = 0.$$

Therefore we have two distinct real roots

$$r_1 = 1, \quad r_2 = 2.$$

For distinct real roots in a Cauchy–Euler equation, the general solution takes the form

$$y = C_1 x^{r_1} + C_2 x^{r_2}.$$

Substituting $r_1 = 1$ and $r_2 = 2$ gives

$$y = C_1 x + C_2 x^2.$$

Problem 17. Solve the initial value problem

$$t^2 y'' - 2ty' + 2y = 3t^2, \quad y(1) = 0, \quad y'(1) = 4.$$

Solution 17. First, we find the homogeneous solution. This is a Cauchy–Euler equation, so we solve the homogeneous equation

$$t^2 y'' - 2ty' + 2y = 0.$$

We assume a solution of the form

$$y = t^r$$

Substituting this into the equation, we get the characteristic equation

$$r(r-1) - 2r + 2 = 0.$$

Simplifying this expression yields

$$r^2 - 3r + 2 = 0.$$

Factoring the quadratic gives

$$(r-1)(r-2) = 0.$$

Thus we obtain the roots

$$r = 1, \quad r = 2.$$

Therefore the homogeneous solution is

$$y_h = C_1 t + C_2 t^2.$$

Next, we find a particular solution. Since the right-hand side is

$$3t^2$$

and

$$t^2$$

is already a solution to the homogeneous equation, we try

$$y_p = At^2 \ln(t)$$

We compute the derivatives:

$$y_p' = 2At\ln(t) + At,$$

$$y_p'' = 2A\ln(t) + 2A + A = 2A\ln(t) + 3A.$$

Substituting into the differential equation

$$t^2y'' - 2ty' + 2y = 3t^2$$

:

$$t^2(2A\ln(t) + 3A) - 2t(2At\ln(t) + At) + 2(At^2\ln(t)) = 3t^2.$$

$$2At^2\ln(t) + 3At^2 - 4At^2\ln(t) - 2At^2 + 2At^2\ln(t) = 3t^2.$$

$$\Longrightarrow At^2 = 3t^2$$

Thus

$$A = 3$$

and our particular solution is

$$y_p = 3t^2\ln(t)$$

The general solution is

$$y = C_1t + C_2t^2 + 3t^2\ln(t).$$

Finally, we use the initial conditions to find the constants. Using

$$y(1) = 0$$

$$y(1) = C_1(1) + C_2(1)^2 + 3(1)^2\ln(1) = C_1 + C_2 = 0.$$

Now we find the derivative.

$$y' = C_1 + 2C_2t + 6t\ln(t) + 3t.$$

Using

$$y'(1) = 4$$

$$y'(1) = C_1 + 2C_2 + 6(1)\ln(1) + 3 = C_1 + 2C_2 + 3 = 4,$$

which simplifies to

$$C_1 + 2C_2 = 1.$$

Solving the system

$$C_1 + C_2 = 0$$

$$C_1 + 2C_2 = 1$$

$$\Longrightarrow C_2 = 1, \quad C_1 = -1$$

Therefore the solution to the initial value problem is

$$y = -t + t^2 + 3t^2\ln(t).$$

Problem 18. Find the general solution of the differential equation

$$x^2y'' + 7xy' + 13y = 0,\ x > 0$$

Solution 18. First, we recognize that this is a Cauchy–Euler equation. For such equations, we try a solution of the form

$$y = x^m.$$

We compute the derivatives:

$$y' = mx^{m-1},$$

$$y'' = m(m-1)x^{m-2}.$$

Substituting these into the differential equation gives

$$x^2(m(m-1)x^{m-2}) + 7x(mx^{m-1}) + 13x^m = 0.$$

Simplifying by factoring out

$$x^m$$

yields

$$x^m(m(m-1) + 7m + 13) = 0.$$

Since

$$x > 0$$

we solve the characteristic equation

$$m^2 - m + 7m + 13 = 0,$$

which simplifies to

$$m^2 + 6m + 13 = 0.$$

To find the roots, we use the quadratic formula:

$$m = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2}.$$

Thus we obtain the complex roots

$$m = -3 \pm 2i.$$

For complex roots of the form

$$\alpha \pm i\beta$$

the general solution to the Cauchy-Euler equation is

$$y = x^\alpha(C_1\cos(\beta\ln x) + C_2\sin(\beta\ln x)).$$

Substituting

$$\alpha = -3$$

and

$$\beta = 2$$

gives

$$y = x^{-3}(C_1\cos(2\ln x) + C_2\sin(2\ln x)).$$

Problem 19. Find the general solution of the differential equation

$$x^2y'' + 7xy' + 13y = 0,\ x > 0$$

Solution 19. First, we recognize that this is a Cauchy–Euler equation. For such equations, we try a solution of the form

$$y = x^m.$$

We compute the derivatives:

$$y' = mx^{m-1},$$

$$y'' = m(m-1)x^{m-2}.$$

Substituting these into the differential equation gives

$$x^2(m(m-1)x^{m-2}) + 7x(mx^{m-1}) + 13x^m = 0.$$

Simplifying by factoring out

$$x^m$$

yields

$$x^m(m(m-1) + 7m + 13) = 0.$$

Since

$$x > 0$$

we solve the characteristic equation

$$m^2 - m + 7m + 13 = 0,$$

which simplifies to

$$m^2 + 6m + 13 = 0.$$

To find the roots, we use the quadratic formula:

$$m = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2}.$$

Thus we obtain the complex roots

$$m = -3 \pm 2i.$$

For complex roots of the form

$$\alpha \pm i\beta$$

the general solution to the Cauchy-Euler equation is

$$y = x^{\alpha}(C_1\cos(\beta\ln x) + C_2\sin(\beta\ln x)).$$

Substituting

$$\alpha = -3$$

and

$$\beta = 2$$

gives

$$y = x^{-3}(C_1\cos(2\ln x) + C_2\sin(2\ln x)).$$

Problem 20. Find the general solution of the differential equation

$$t^2y'' - 6y = 0$$

Solution 20. First, we recognize that this is a Cauchy–Euler equation. For such equations, we try a solution of the form

$$y = t^m.$$

We compute the derivatives:

$$y' = mt^{m-1},$$

$$y'' = m(m-1)t^{m-2}.$$

Substituting these into the differential equation gives

$$t^2(m(m-1)t^{m-2}) - 6t^m = 0.$$

Factoring yields

$$t^m(m(m-1) - 6) = 0.$$

We solve the characteristic equation

$$m^2 - m - 6 = 0.$$

Factoring gives

$$(m-3)(m+2)=0.$$

Thus we obtain the distinct real roots

$$m=3, \quad m=-2.$$

For distinct real roots

$$m_1 \text{ and } m_2$$

the general solution to the Cauchy–Euler equation is

$$y=C_1t^{m_1}+C_2t^{m_2}.$$

Thus the general solution is

$$y=C_1t^3+C_2t^{-2}.$$

Problem 21. Find the general solution of the differential equation

$$t^2y''-2ty'+3y=0.$$

Solution 21. First, we recognize that this is a Cauchy–Euler equation. For such equations, we try a solution of the form

$$y=t^m.$$

We compute the derivatives:

$$y'=mt^{m-1},$$

$$y''=m(m-1)t^{m-2}.$$

Substituting these into the differential equation gives

$$t^2(m(m-1)t^{m-2})-2t(mt^{m-1})+3t^m=0.$$

Factoring yields

$$t^m(m(m-1)-2m+3)=0.$$

We solve the characteristic equation

$$m^2-m-2m+3=0,$$

$$m^2-3m+3=0.$$

Using the quadratic formula to find the roots:

$$m=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(3)}}{2(1)}=\frac{3\pm\sqrt{9-12}}{2}=\frac{3\pm\sqrt{-3}}{2}.$$

Thus we obtain the complex roots

$$m = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}.$$

For complex roots of the form

$$\alpha \pm i\beta$$

the general solution to the Cauchy–Euler equation is

$$y = t^{\alpha}[C_1\cos(\beta\ln t) + C_2\sin(\beta\ln t)].$$

Thus the general solution is

$$y = t^{3/2}\left[C_1\cos\left(\frac{\sqrt{3}}{2}\ln t\right) + C_2\sin\left(\frac{\sqrt{3}}{2}\ln t\right)\right].$$

Problem 22. Find the solution of

$$t^2y'' + 6ty' + 6y = 0$$

satisfying $y(1) = 0$ and $y'(1) = 10$.

Solution 22. Use the Cauchy–Euler substitution $y = t^m$.

Substituting gives

$$m(m-1) + 6m + 6 = 0$$

so

$$m^2 + 5m + 6 = 0$$

and

$$(m+2)(m+3) = 0.$$

Thus $m = -2$ or $m = -3$, which leads to the associated homogeneous solution:

$$y = c_1t^{-2} + c_2t^{-3}.$$

Then

$$y' = -2c_1t^{-3} - 3c_2t^{-4}.$$

Applying the initial conditions gives

$$0 = c_1 + c_2,$$

$$10 = -2c_1 - 3c_2.$$

Solving yields

$$c_1 = 10, \qquad c_2 = -10.$$

Hence

$$y(t) = 10t^{-2} - 10t^{-3}.$$

Problem 23. Solve the initial value problem

$$t^2y'' - 3ty' + 4y = 0, \qquad y(1) = 4, \qquad y'(1) = 5.$$

Solution 23. Assume the solution is in the form $y = t^m$. Substitution gives

$$\begin{aligned} m(m-1) - 3m + 4 &= 0 \\ m^2 - 4m + 4 &= 0 \\ (m-2)^2 &= 0 \end{aligned}$$

Hence,

$$y = c_1t^2 + c_2t^2\ln t.$$

To solve for the unknown constants c_1, c_2, we compute the derivative:

$$y' = 2c_1t + 2c_2t\ln t + c_2t.$$

First, using $y(1) = 4$:

$$4 = c_1 + c_2\ln 1 \Rightarrow c_1 = 4.$$

Now, we use $y'(1) = 5$:

$$5 = 2c_1 + 2c_2\ln 1 + c_2 \Rightarrow 2c_1 + c_2 = 5 \Rightarrow c_2 = -3$$

Therefore,

$$y(t) = 4t^2 - 3t^2\ln t.$$

Problem 24. Solve the following initial value problem:

$$t^2y'' - 5ty' + 9y = 0, \quad y(1) = 1, \quad y'(1) = 4$$

Solution 24. This is a Cauchy-Euler equation. To solve the homogeneous equation

$$t^2y'' - 5ty' + 9y = 0,$$

we assume a solution of the form $y = t^r$. Substituting this into the equation, we obtain the characteristic equation:

$$r(r-1) - 5r + 9 = 0.$$

Simplifying this gives:

$$r^2 - 6r + 9 = 0.$$

Factoring the quadratic equation, we get $(r-3)^2 = 0$, which yields a repeated root:

$$r_1 = r_2 = 3.$$

Therefore, the general solution for $t > 0$ is:

$$y(t) = C_1 t^3 + C_2 t^3 \ln t.$$

Next, we use the initial conditions to solve for C_1 and C_2. First, we use $y(1) = 1$:

$$y(1) = C_1(1)^3 + C_2(1)^3 \ln(1) = C_1 + 0 = 1 \Longrightarrow C_1 = 1.$$

Now we find the derivative $y'(t)$:

$$y'(t) = 3C_1 t^2 + 3C_2 t^2 \ln t + C_2 t^2.$$

Using the second initial condition $y'(1) = 4$:

$$y'(1) = 3C_1(1)^2 + 3C_2(1)^2 \ln(1) + C_2(1)^2 = 3C_1 + C_2 = 4.$$

Substituting $C_1 = 1$ into the equation:

$$3(1) + C_2 = 4 \Longrightarrow C_2 = 1.$$

The solution to the initial value problem is:

$$y(t) = t^3 + t^3 \ln t.$$

Problem 25. Find the general solution of the differential equation:

$$x^2 y'' - xy' - 3y = 0$$

Solution 25. This is a Cauchy-Euler equation. To solve it, we solve the characteristic equation of the form $[ar^2 + (b - a)r + c] = 0$. Here, $a = 1$, $b = -1$, and $c = -3$.

$$r^2 + (-1 - 1)r - 3 = 0$$

$$r^2 - 2r - 3 = 0$$

Factoring the quadratic equation gives:

$$(r - 3)(r + 1) = 0$$

Thus, the roots are:

$$r = 3, \quad r = -1$$

Therefore, the general solution is:

$$y(x) = C_1 x^3 + C_2 x^{-1}$$

Problem 26. Find the general solution of the differential equation:

$$4t^2 y'' + 8ty' + y = 0$$

Solution 26. This is a Cauchy-Euler equation. We solve the characteristic equation $[ar^2 + (b-a)r + c] = 0$. Here, $a = 4$, $b = 8$, and $c = 1$.

$$4r^2 + (8-4)r + 1 = 0$$

$$4r^2 + 4r + 1 = 0$$

We solve for r using the quadratic formula:

$$r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 4 \cdot 1}}{2 \cdot 4} = \frac{-4 \pm \sqrt{16-16}}{8} = -\frac{4}{8} = -\frac{1}{2}$$

Since we have a repeated root $r = -1/2$, the general solution for $t > 0$ is:

$$y(t) = C_1 t^{-1/2} + C_2 t^{-1/2} \ln t$$

Problem 27. Find the general solution of the differential equation

$$t^2 y'' + 7ty' + 5y = 0, \quad t > 0.$$

Solution 27. This is a Cauchy-Euler equation. To solve it, we assume a solution of the form $y = t^m$. We then find the first and second derivatives:

$$y' = mt^{m-1}, \quad y'' = m(m-1)t^{m-2}.$$

Substituting these into the original differential equation:

$$t^2[m(m-1)t^{m-2}] + 7t[mt^{m-1}] + 5[t^m] = 0.$$

Factoring out t^m:

$$t^m[m(m-1) + 7m + 5] = 0.$$

Since $t > 0$, we solve the characteristic equation:

$$m^2 - m + 7m + 5 = 0$$

$$m^2 + 6m + 5 = 0.$$

Factoring the quadratic equation gives:

$$(m+5)(m+1) = 0.$$

The roots are $m = -5$ and $m = -1$.

Therefore, the general solution for $t > 0$ is:

$$y = C_1 t^{-5} + C_2 t^{-1}.$$

Spring-Mass Systems

Problem 28. (Please use 32 ft per seconds squared as the acceleration of gravity in this problem.) A body weighing 8 pounds hangs from a vertical spring attached to the ceiling. At its equilibrium position, the body stretches the spring 1/2 ft from its natural length. The body is started in motion from the equilibrium position with an initial velocity of 4 ft/s in the downward direction.
(a) Assume there is no damping and the body is acted on by a downward external force of

$$F(t) = 3\cos(2t)$$

pounds. Set up, but do not solve, an initial value problem describing the motion of the body.
(b) If the given downward external force is replaced by a force of $3\cos(\omega t)$ pounds, find the value of the frequency ω which will cause resonance.

Solution 28. **(a)** We begin by determining the constants appearing in the mass–spring model,

$$my'' + \gamma y' + ky = F(t)$$

The weight of the body is 8 pounds. Since weight is given by

$$W = mg$$

we obtain the mass

$$m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4} \text{ slugs.}$$

Next we determine the spring constant using Hooke's Law. At equilibrium the spring is stretched a half foot by the weight of the body, so

$$ky_0 = mg \Longrightarrow k\left(\frac{1}{2}\right) = 8 \Longrightarrow k = 16$$

Because there is no damping, the damping constant is $\gamma = 0$.

Substituting the values obtained above gives

$$\frac{1}{4}y'' + 16y = 3\cos(2t).$$

Finally, we determine the initial conditions. The body begins at the equilibrium position, so

$$y(0) = 0.$$

The initial velocity is 4 ft/sec in the downward direction. Taking downward as the positive direction gives

$$y'(0) = 4.$$

Thus the required initial value problem describing the motion is

$$\frac{1}{4}y'' + 16y = 3\cos(2t), \qquad y(0) = 0, \qquad y'(0) = 4.$$

(b) From part **(a)**, the differential equation is

$$\frac{1}{4}y'' + 16y = 3\cos(\omega t).$$

To determine the resonance frequency, we need to find when the particular solution will duplicate the homogeneous solution. First consider the corresponding homogeneous equation

$$\frac{1}{4}y'' + 16y = 0.$$

The characteristic equation is

$$\frac{1}{4}r^2 + 16 = 0,$$

which gives

$$r = \pm 8i.$$

Thus the homogeneous solution is

$$y_h = C_1\cos(8t) + C_2\sin(8t).$$

The natural frequency of the system is therefore

$$\omega = 8.$$

as then we would have a particular solution of the form

$$y_p = t(A\cos(8t) + B\sin(8t))$$

for some coefficients which I do not need to find. So resonance occurs for

$$\omega = 8$$

Problem 29. A certain spring hangs vertically from a rigid support. When a 3 pound mass is attached to the end of the spring, the mass stretches the spring 2 feet. Suppose the mass is pulled down 2 additional feet from the rest position and then released. Assuming air resistance (or the damping force) at any instant is equal to twice the instantaneous velocity of the mass, write BUT DO NOT SOLVE, an initial value problem describing the motion of the mass. (Please use 32 ft per seconds squared as the acceleration of gravity in this problem.)

Solution 29. We begin by determining the constants appearing in the mass–spring model,

$$my'' + \gamma y' + ky = F(t).$$

The weight of the mass is 3 pounds. Since weight is given by

$$W = mg$$

we obtain the mass

$$m = \frac{W}{g} = \frac{3}{32} \text{ slugs.}$$

Next, we determine the spring constant using Hooke's Law. At equilibrium, the spring is stretched 2 feet by the weight of the mass, so

$$kL = mg \Longrightarrow k(2) = 3 \Longrightarrow k = \frac{3}{2}.$$

The problem states that the damping force is equal to twice the instantaneous velocity. Therefore, the damping constant is

$$\gamma = 2.$$

There is no external force mentioned, so

$$F(t) = 0.$$

Substituting the values obtained above into the differential equation gives

$$\frac{3}{32}y'' + 2y' + \frac{3}{2}y = 0.$$

Finally, we determine the initial conditions. The mass is pulled down 2 feet from the rest (equilibrium) position. Taking downward as the positive direction, we have

$$y(0) = 2.$$

Since the mass is "released," the initial velocity is zero:

$$y'(0) = 0.$$

Thus, the initial value problem describing the motion is

$$\frac{3}{32}y'' + 2y' + \frac{3}{2}y = 0, \qquad y(0) = 2, \qquad y'(0) = 0.$$

Problem 30. (a) A 20 kilogram body hangs from a vertical spring attached to a rigid support. At its equilibrium position, the body stretches the spring 50 centimeters beyond its natural length. The body is acted on by an external force of

$$10\cos(2t)$$

Newtons and moves in a medium with a damping constant of 100 Newton seconds per meter. If the body is set in motion from its equilibrium position with an upward velocity of 20 centimeters per second, SET UP, BUT DO NOT SOLVE, an initial value problem describing the motion of the body. (Please use 9.8 meters per second per second as the acceleration of gravity in this problem.)

(b) If the given downward external force is replaced by a force of

$$10\cos(\omega t)$$

Newtons, find the value of the frequency

$$\omega$$

which will cause resonance or explain why there is no such frequency.

Solution 30. (a) We begin by determining the constants appearing in the mass–spring model,

$$my'' + \gamma y' + ky = F(t).$$

The mass of the body is given directly as

$$m = 20 \text{ kg}.$$

Next, we determine the spring constant using Hooke's Law. At equilibrium, the weight of the mass is balanced by the spring force. The weight is

$$W = mg = 20(9.8) = 196 \text{ N}$$

The stretch is

$$50 \text{ cm} = 0.5 \text{ m}$$

so

$$kL = mg \Longrightarrow k(0.5) = 196 \Longrightarrow k = 392 \text{ N/m}.$$

The problem states the damping constant is

$$\gamma = 100 \text{ Ns/m}.$$

The external force acting on the body is given as

$$F(t) = 10\cos(2t).$$

Substituting these values into the differential equation gives

$$20y'' + 100y' + 392y = 10\cos(2t).$$

Finally, we determine the initial conditions. The body is set in motion from its equilibrium position, so

$$y(0) = 0.$$

The initial velocity is upward at

$$20 \text{ cm/s} = 0.2 \text{ m/s}$$

Taking downward as the positive direction, an upward velocity is negative:

$$y'(0) = -0.2.$$

Thus, the initial value problem is

$$20y'' + 100y' + 392y = 10\cos(2t), \qquad y(0) = 0, \qquad y'(0) = -0.2.$$

(b) Observe that the roots of the characteristic equation will be complex, with negative real part. Specifically, we have

$$r = -\frac{5}{2} \pm i\frac{\sqrt{1335}}{10}$$

though note you do not need to go so far as actually finding said roots, you just need to observe that the real part of r is nonzero.

$$Re(r) < 0$$

Then the homogeneous solution will be of the form

$$y_h = e^{Re(r)}(C_1\cos(Im(r)t) + C_2\sin(Im(r)t))$$

The homogeneous solution is clearly bounded. For the particular solution, we will have that it is of the form

$$A\cos(\omega t) + B\sin(\omega t)$$

which is also bounded. Thus, we do not have resonance for any given frequency omega. Note behavior will clearly change with no damping.

Problem 31. A body weighing 64 pounds stretches a spring 2 feet beyond its natural length. The body moves in a medium that impacts a viscous damping force of 8 pounds when the speed of the body is 3 feet per second. The body is pulled down an additional 1 foot and is set in motion with an initial upward velocity of 5 feet per second. Set up, but DO NOT SOLVE, an initial value problem that models the motion of the body. Note: Use

$$32 \text{ ft/sec}^2$$

as the acceleration of gravity in this problem.

Solution 31. We begin by determining the constants appearing in the mass–spring model,

$$my'' + \gamma y' + ky = F(t).$$

The weight of the body is 64 pounds. Since weight is given by

$$W = mg$$

we obtain the mass

$$m = \frac{W}{g} = \frac{64}{32} = 2 \text{ slugs.}$$

Next, we determine the spring constant. Using Hooke's Law. At equilibrium, the weight of the body is balanced by the spring force, so

$$kL = W \Longrightarrow k(2) = 64 \Longrightarrow k = 32 \text{ lb/ft.}$$

The damping force is proportional to the velocity,

$$F_d = \gamma y'$$

We are given that the damping force is 8 pounds when the speed is 3 feet per second, so

$$8 = \gamma(3) \Longrightarrow \gamma = \frac{8}{3} \text{ lb-sec/ft.}$$

Since no external force is mentioned, we have

$$F(t) = 0$$

Substituting these values into the differential equation gives

$$2y'' + \frac{8}{3}y' + 32y = 0.$$

Finally, we determine the initial conditions. The body is pulled down 1 foot from the equilibrium position. Taking downward as the positive direction, we have

$$y(0) = 1.$$

The body is set in motion with an initial upward velocity of 5 feet per second. Since upward is the negative direction, we have

$$y'(0) = -5.$$

Thus, the initial value problem describing the motion is

$$2y'' + \frac{8}{3}y' + 32y = 0, \qquad y(0) = 1, \qquad y'(0) = -5.$$

Problem 32. (a) A 5 kilogram body hangs from a vertical spring attached to a rigid support. At its equilibrium position, the body stretches the spring 20 centimeters beyond its natural length. The body is acted upon by a downward external force of

$$10\sin(t/2)$$

newtons and there is no damping. If the body is set in motion from a position 10 centimeters below its equilibrium position with an upward initial velocity of 30 centimeters per second, set up, BUT DO NOT SOLVE, an initial value problem that describes the motion of the body. [In the following problem, assume that the acceleration of gravity is 9.8 meters per second per second.]

(b) If the given downward external force is replaced by

$$4\cos(\omega t)$$

newtons, find the value of the frequency omega which will cause resonance or explain why there is no such frequency.

Solution 32. (a) We begin by determining the constants appearing in the mass–spring model,

$$my'' + \gamma y' + ky = F(t).$$

The mass of the body is given directly as

$$m = 5 \text{ kg}.$$

Next, we determine the spring constant using Hooke's Law. At equilibrium, the weight of the body is balanced by the spring force. The weight is

$$W = mg = 5(9.8) = 49 \text{ N}$$

The stretch is

$$20 \text{ cm} = 0.2 \text{ m}$$

so

$$kL = mg \Longrightarrow k(0.2) = 49 \Longrightarrow k = 245 \text{ N/m}.$$

The problem states there is no damping, so the damping constant is

$$\gamma = 0.$$

The external force acting on the body is given as

$$F(t) = 10\sin(t/2).$$

Substituting these values into the differential equation gives

$$5y'' + 245y = 10\sin(t/2).$$

Finally, we determine the initial conditions. The body is set in motion from 10 centimeters below equilibrium. Taking downward as the positive direction, we have

$$y(0) = 10 \text{ cm} = 0.1 \text{ m}.$$

The initial velocity is upward at

$$30 \text{ cm/s} = 0.3 \text{ m/s}$$

Since upward is the negative direction, we have

$$y'(0) = -0.3.$$

Thus, the initial value problem describing the motion is

$$5y'' + 245y = 10\sin(t/2), \qquad y(0) = 0.1, \qquad y'(0) = -0.3.$$

(b) From our solution to part (a), we can see that the homogeneous solution will be

$$y_h = C_1\cos(7t) + C_2\sin(7t)$$

Therefore, the value of the frequency that will cause resonance is

$$\omega = 7.$$

as then the particular solution will be of the form

$$y_p = t(A\cos(7t) + B\sin(7t))$$

which is unbounded for increasing t.

Problem 33. A spring hangs vertically from a rigid support. When a 2 pound box is attached to the end of the spring, the box stretches the spring 0.5 feet and then comes to rest. Suppose the box is lifted up 2 feet from the rest position and then released. Assuming the air resistance (or the damping force) at any instant is equal to half the instantaneous velocity of the box, write BUT DO NOT SOLVE, an initial value problem modeling the motion of the box.

Solution 33. We begin by determining the constants appearing in the mass–spring model,

$$my'' + \gamma y' + ky = F(t).$$

First, we find the mass m. Given the weight $W = 2$ lb and using the acceleration due to gravity $g = 32$ ft/s^2:

$$m = \frac{W}{g} = \frac{2}{32} = \frac{1}{16} \text{ slugs.}$$

Next, we determine the spring constant k using Hooke's Law. At the rest position, the weight is balanced by the spring force. The stretch is given as $L = 0.5$ ft:

$$kL = W \Longrightarrow k(0.5) = 2 \Longrightarrow k = 4 \text{ lb/ft.}$$

The problem states the damping force is equal to half the instantaneous velocity, so the damping constant is:

$$\gamma = 0.5 = \frac{1}{2}.$$

There is no mentioned external driving force, so:

$$F(t) = 0.$$

Substituting these values into the differential equation gives:

$$\frac{1}{16}y'' + \frac{1}{2}y' + 4y = 0.$$

Finally, we determine the initial conditions. Let the downward direction be positive. The box is lifted up 2 feet from the rest position, which is the negative direction:

$$y(0) = -2.$$

The box is "released," which implies the initial velocity is zero:

$$y'(0) = 0.$$

Thus, the initial value problem modeling the motion of the box is:

$$\frac{1}{16}y'' + \frac{1}{2}y' + 4y = 0, \qquad y(0) = -2, \qquad y'(0) = 0.$$

Problem 34. A mass weighing 8 lb hangs from a vertical spring. The spring stretches 4 inches from its natural length at equilibrium. The medium provides a damping constant 20 lb·s/ft. The mass is displaced 2 ft upward and released. An external force $0.001\cos(\omega t)$ acts downward. (a) Set up the initial value problem. (b) Does a resonant frequency exist?

Solution 34. Let $u(t)$ be the displacement of the mass from its equilibrium position.

We can model this system by a second order differential equation in the form:

$$mu'' + \gamma u' + ku = F(t).$$

Since, $mg = 8$ and $g = 32\ ft/s^2 \Rightarrow m = \frac{8}{32} = \frac{1}{4}$

The spring stretch is

$$4 \text{ in} = \frac{1}{3} \text{ ft.}$$

Using Hooke's law, we can find the spring constant. Given that $mg = kL$, we obtain

$$k = \frac{8}{1/3} = 24.$$

The damping constant is given in the question as $\gamma = 20$.

The forcing function is

$$F(t) = 0.001\cos(\omega t).$$

Thus, the IVP is

$$\frac{1}{4}u'' + 20u' + 24u = 0.001\cos(\omega t)$$

with initial conditions

$$u(0) = -2, \qquad u'(0) = 0.$$

(b) Resonance does not occur because the system has damping, so the amplitude remains bounded.

Problem 35. (Use 32 ft/sec^2 as the acceleration due to gravity.) A spring hanging vertically from a rigid support is stretched 6 inches by a mass that weighs 8 lb. The mass is in a medium with damping constant 0.25 lb $\cdot$ s/ft and is acted on by an external downward force $4\cos(2t)$ lb. Suppose that the mass is displaced 2 ft upward and then released. **Set up, but do not solve,** the initial value problem describing the motion.

Solution 35. Let $y(t)$ denote the displacement from the equilibrium position.

The spring mass system can be modeled by the second order linear differential equation:

$$my'' + \gamma y' + ky = F(t).$$

Since the weight is 8 lb, $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$.

The spring stretch is 6 in $= \frac{1}{2}$ ft. Using $ky_0 = mg$, we obtain $k = \frac{8}{1/2} = 16$.

The damping constant is given as $\gamma = 0.25 = \frac{1}{4}$.

The forcing function is $F(t) = 4\cos(2t)$.

Thus, the IVP is

$$\frac{1}{4}y'' + \frac{1}{4}y' + 16y = 4\cos(2t)$$

with initial conditions

$$y(0) = -2, \qquad y'(0) = 0.$$

Problem 36. A spring hangs vertically from a rigid support. When an 8 pound box is attached to the end of the spring, the box stretches the spring 4 inches and then comes to rest. Suppose the box is given a downward displacement of 6 inches from the rest position and then released with no initial velocity.

(a) Assuming there is no damping and that the box is acted on by an external force $6\cos(4t)$ pounds, formulate but **do not solve** the initial value problem modeling the motion.

(b) If the external force is replaced by $12\sin(\omega t)$ pounds, find the value of $\omega > 0$ for which resonance occurs.

Solution 36. Let $x(t)$ denote the displacement from the equilibrium position. The spring mass system can be modeled by the second order linear differential equation:

$$mx'' + \gamma x' + kx = F(t).$$

The weight is 8 lb, so $m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$. The damping coefficient is 0. The stretch is 4 inches $= \frac{1}{3}$ ft. Using $ky_0 = mg$, The initial displacement is 6 inches $= \frac{1}{2}$ ft downward.

$$8 = k\left(\frac{1}{3}\right) \Rightarrow k = 24.$$

Thus, the IVP becomes

$$\begin{aligned}\frac{1}{4}x'' + 24x &= F(t)\\ &= 6\cos(4t)\end{aligned}$$

with $x(0) = \frac{1}{2}, \qquad x'(0) = 0.$

(b) The homogeneous equation is

$$\begin{aligned}\frac{1}{4}x'' + 24x &= 0\\ x'' + 96x &= 0.\end{aligned}$$

Thus, the homogeneous solution is:

$$x_h = C_1\cos(\sqrt{96}t) + C_2\sin(\sqrt{96}t)$$

Thus, resonance occurs when

$$\omega = \sqrt{96}.$$

Problem 37. A spring hangs vertically from a rigid support. When a 4 pound box is attached to the end of the spring, the box stretches the spring 0.75 ft and then comes to rest. Suppose the box is pushed down 3 ft below the rest position and then released. Assuming air resistance (the damping force) at any instant is equal to twice the instantaneous velocity of the box, write but do not solve an initial value problem modeling the motion of the box.

Solution 37. Let $x(t)$ be the displacement from equilibrium. We can model this using a second order linear differential equation in the form:

$$mx'' + \gamma x' + kx = 0.$$

The weight is 4 lb, so $m = \frac{w}{g} = \frac{4}{32} = \frac{1}{8}$.

The damping constant is $\gamma = 2$.

The spring constant satisfies Hooke's Law which is given by:

$$ky_0 = mg.$$

Since the y_0 is $0.75 = \frac{3}{4}$ ft, then $k = \frac{4}{3/4} = \frac{16}{3}$.

Thus, the IVP is

$$\frac{1}{8}x'' + 2x' + \frac{16}{3}x = 0 \qquad x(0) = 3, \qquad x'(0) = 0.$$

Problem 38. A body weighing 8 pounds hangs from a vertical spring attached to the ceiling. At its equilibrium position, the body stretches the spring $1/2$ feet from its natural length. The body is started in motion from the equilibrium position with an initial velocity 4 feet/second in the downward direction. Let $g = 32$ ft/s^2.

(a) Assume that there is no damping and the body is acted on by a downward external force $3\cos(2t)$ pounds, set up, BUT DO NOT SOLVE, an initial value problem that models the motion of the body.

(b) If the given downward external force is replaced by a force of $3\cos(\omega t)$ pounds, find the value of the frequency ω which will cause resonance or explain why there is no such a frequency.

Solution 38. (a) Let $u(t)$ denote the displacement of the body from its equilibrium position, where the downward direction is positive. The general equation for a spring-mass system is:

$$mu'' + \gamma u' + ku = F(t)$$

First, we determine the constants:

- **Mass (m):** Given weight $w = 8$ lbs and $g = 32$ ft/s^2, we have

$$m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4} \text{ slugs}$$

- **Spring Constant (k):** Using Hooke's Law at equilibrium, $mg = kL$. With a stretch of $L = 1/2$ ft,

$$k = \frac{mg}{L} = \frac{8}{1/2} = 16 \text{ lb/ft}$$

- **Damping (γ):** Since there is no damping, $\gamma = 0$.

- **External Force ($F(t)$):** $F(t) = 3\cos(2t)$.

The body starts from the **equilibrium position** ($u(0) = 0$) with an **initial velocity** of 4 ft/s downward ($u'(0) = 4$). The Initial Value Problem (IVP) is:

$$\frac{1}{4}u'' + 16u = 3\cos(2t), \quad u(0) = 0, \quad u'(0) = 4$$

(b) Pure resonance occurs in an undamped system when the frequency of the external force, ω, matches the natural frequency of the system, ω_0. The natural frequency is calculated as:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{1/4}} = \sqrt{64} = 8$$

Thus, resonance occurs when:

$$\omega = 8$$

Problem 39. A mass of 2 kg stretches a spring 10 cm. The mass is pulled down 20 cm from the equilibrium position and then released with a downward initial velocity of 2 m/s. To make this problem easier, assume $g = 10$ m/s^2. If there is no air resistance, write but DO NOT SOLVE an initial value problem which models the motion of the mass.

Solution 39. Let $y(t)$ denote the displacement from the equilibrium position, where the downward direction is positive. The general equation for a spring-mass system is:

$$my'' + \gamma y' + ky = F(t)$$

First, we determine the constants:

- **Mass (m):** Given mass $m = 2$ kg.
- **Spring Constant (k):** Using Hooke's Law at equilibrium, $mg = ky_0$. Note that 10 cm $= \frac{1}{10}$ m.

$$(2\text{ kg})(10\text{ m/s}^2) = k\left(\frac{1}{10}\text{ m}\right) \Longrightarrow 20 = \frac{k}{10} \Longrightarrow k = 200\text{ N/m}$$

- **Damping (γ):** Since there is no air resistance, $\gamma = 0$.
- **External Force ($F(t)$):** There is no mentioned external force, so $F(t) = 0$.

Substituting these into the differential equation gives:

$$2y'' + 200y = 0$$

Dividing by 2, we simplify the equation to:

$$y'' + 100y = 0$$

Next, we determine the initial conditions. Note that 20 cm $= \frac{1}{5}$ m.

- **Initial Displacement:** The mass is pulled down $\frac{1}{5}$ m, so $y(0) = \frac{1}{5}$.

- **Initial Velocity:** The mass is released with a downward velocity of 2 m/s, so $y'(0) = 2$.

Thus, the Initial Value Problem (IVP) is:

$$y'' + 100y = 0, \quad y(0) = \frac{1}{5}, \quad y'(0) = 2$$

Problem 40. A 2 kg mass is attached to a spring hanging from a ceiling. The mass causes the spring to stretch 0.2 m before coming to rest at equilibrium. The damping constant for the system is 4 Ns/m. At time $t = 0$, the mass is pulled down 50 cm below the equilibrium position and released with a upward initial velocity of 3 m/s. For simplicity, set the gravitational acceleration to $g = 10$ m/s^2. If the mass is acted upon by an external upward force at time t of $8\cos(4t)$ N, set up BUT DO NOT SOLVE an initial value problem that models the motion of the mass.

Solution 40. Let $y(t)$ denote the displacement of the mass from its equilibrium position, where the downward direction is positive. The general equation for a damped spring-mass system is:

$$my'' + \gamma y' + ky = F(t)$$

First, we determine the constants:

- **Mass (m):** $m = 2$ kg.

- **Spring Constant (k):** Using the equilibrium condition $mg = k\Delta L$ where $\Delta L = 0.2$ m:

$$k = \frac{mg}{\Delta L} = \frac{(2)(10)}{0.2} = \frac{20}{0.2} = 100 \text{ N/m}$$

- **Damping Coefficient (γ):** Given as $\gamma = 4$ Ns/m.

- **External Force ($F(t)$):** An upward force is negative in our coordinate system. Thus, $F(t) = -8\cos(4t)$.

Substituting these values into the differential equation:

$$2y'' + 4y' + 100y = -8\cos(4t)$$

Next, we determine the initial conditions. Note that 50 cm = 0.5 m.

- **Initial Displacement:** The mass is pulled down 0.5 m, so $y(0) = 0.5$.

- **Initial Velocity:** The mass is released with an **upward** initial velocity of 3 m/s. Since upward is the negative direction, $y'(0) = -3$.

The Initial Value Problem (IVP) is:

$$2y'' + 4y' + 100y = -8\cos(4t), \quad y(0) = 0.5, \quad y'(0) = -3$$

Problem 41. A 1 kg mass is attached to a spring hanging from a ceiling. The box causes the spring to stretch 10 cm before coming to rest at equilibrium. At time $t = 0$, the box is pulled down 20 cm below the equilibrium position and released with a downward initial velocity of 2 m/s. Ignore air resistance and, for simplicity set the gravitational acceleration to $g = 10$ m/s^2. If no external forces act on the box, set up BUT DO NOT SOLVE an initial value problem that models the motion of the box.

Solution 41. Let $x(t)$ denote the displacement of the box from its equilibrium position, where the downward direction is positive. The general equation for a spring-mass system is:

$$mx'' + \gamma x' + kx = F_{ext}(t)$$

First, we determine the constants:

- **Mass (m):** $m = 1$ kg.
- **Spring Constant (k):** Using the equilibrium condition $mg = k \cdot$ stretch. Note that 10 cm $= 0.1$ m $= \frac{1}{10}$ m.

$$(1\text{ kg})(10\text{ m/s}^2) = k\left(\frac{1}{10}\text{ m}\right) \Longrightarrow 10 = \frac{k}{10} \Longrightarrow k = 100\text{ N/m}$$

- **Damping Coefficient (γ):** Since we ignore air resistance, $\gamma = 0$.
- **External Force (F_{ext}):** No external forces act on the box, so $F_{ext} = 0$.

Substituting these values into the differential equation gives:

$$1x'' + 0x' + 100x = 0 \Longrightarrow x'' + 100x = 0.$$

Next, we determine the initial conditions. Note that 20 cm $= 0.2$ m.

- **Initial Displacement:** The box is pulled down 0.2 m below equilibrium, so $x(0) = 0.2$.
- **Initial Velocity:** The box is released with a downward initial velocity of 2 m/s, so $x'(0) = 2$.

The Initial Value Problem (IVP) is:

$$x'' + 100x = 0, \quad x(0) = 0.2, \quad x'(0) = 2$$

Problem 42. A 1 kg mass is attached to a spring hanging from a ceiling. The box causes the spring to stretch 10 cm before coming to rest at equilibrium. At time $t = 0$, the box is pulled down 20 cm below the equilibrium position and released with a downward initial velocity of 2 m/s. Ignore air resistance and, for simplicity set the gravitational acceleration to $g = 10$ m/s^2. If no external forces act on the box, set up BUT DO NOT SOLVE an initial value problem that models the motion of the box.

Solution 42. Let $x(t)$ denote the displacement of the box from its equilibrium position, where the downward direction is positive. The general equation for a spring-mass system is:

$$mx'' + \gamma x' + kx = F_{ext}(t)$$

First, we determine the constants:

- **Mass (m):** $m = 1$ kg.
- **Spring Constant (k):** Using the equilibrium condition $mg = k \cdot$ stretch. Note that 10 cm $= 0.1$ m $= \frac{1}{10}$ m.

$$(1\text{ kg})(10\text{ m/s}^2) = k\left(\frac{1}{10}\text{ m}\right) \Longrightarrow 10 = \frac{k}{10} \Longrightarrow k = 100\text{ N/m}$$

- **Damping Coefficient (γ):** Since we ignore air resistance, $\gamma = 0$.
- **External Force (F_{ext}):** No external forces act on the box, so $F_{ext} = 0$.

Substituting these values into the differential equation gives:

$$1x'' + 0x' + 100x = 0 \Longrightarrow x'' + 100x = 0.$$

Next, we determine the initial conditions. Note that 20 cm $= 0.2$ m.

- **Initial Displacement:** The box is pulled down 0.2 m below equilibrium, so $x(0) = 0.2$.
- **Initial Velocity:** The box is released with a downward initial velocity of 2 m/s, so $x'(0) = 2$.

The Initial Value Problem (IVP) is:

$$x'' + 100x = 0, \quad x(0) = 0.2, \quad x'(0) = 2$$

Higher Order Differential Equations: Method of Undetermined Coefficients

Problem 43. Solve the differential equation

$$y''' - y = 7e^t$$

Solution 43. First, we find the homogeneous solution. Using methods we have already covered, we solve the homogeneous equation

$$y''' - y = 0.$$

The corresponding characteristic equation is

$$r^3 - 1 = 0.$$

Using the difference of cubes formula, we find

$$(r-1)(r^2 + r + 1) = 0.$$

Thus we obtain the roots

$$r = 1, \quad r = \frac{-1 \pm i\sqrt{3}}{2}.$$

Therefore the homogeneous solution is

$$y_h = C_1 e^t + C_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + C_3 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right).$$

Next we turn to the method of undetermined coefficients to find a particular solution. Since the right-hand side is $7e^t$, we would normally try a particular solution of the form

$$y_p = Ae^t.$$

However, since e^t already appears in the homogeneous solution, we multiply by t to obtain the modified guess

$$y_p = Ate^t.$$

We now compute the derivatives:

$$y_p' = A(1+t)e^t,$$

$$y_p'' = A(2+t)e^t,$$

$$y_p''' = A(3+t)e^t.$$

Substituting into the differential equation gives

$$y_p''' - y_p = A(3+t)e^t - Ate^t = 3Ae^t.$$

We set this equal to the right-hand side:

$$3Ae^t = 7e^t.$$

Thus

$$A = \frac{7}{3}.$$

Therefore the particular solution is

$$y_p = \frac{7}{3} te^t.$$

Finally, the general solution is

$$y = y_h + y_p$$

$$y = C_1 e^t + C_2 e^{-t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + C_3 e^{-t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) + \frac{7}{3} t e^t.$$

Problem 44. Find the general solution of

$$y^{(4)} - 3y'' - 4y = e^{-t}$$

on the interval

$$-\infty < t < \infty$$

.

Solution 44. First, we find the homogeneous solution. We solve the homogeneous equation

$$y^{(4)} - 3y'' - 4y = 0.$$

The corresponding characteristic equation is

$$r^4 - 3r^2 - 4 = 0.$$

We can factor this like so.

$$(r^2 - 4)(r^2 + 1) = 0.$$

$$(r + 2)(r - 2)(r^2 + 1) = 0]$$

This gives us the roots

$$r = \pm 2, \quad r = \pm i.$$

Therefore the homogeneous solution is

$$y_h = C_1 e^{2t} + C_2 e^{-2t} + C_3 \cos(t) + C_4 \sin(t).$$

Next, we use the method of undetermined coefficients to find a particular solution. Since the right-hand side is

$$e^{-t}$$

and this function does not appear in the homogeneous solution, we guess a particular solution of the form

$$y_p = A e^{-t}.$$

We now compute the derivatives:

$$y_p' = -A e^{-t},$$

$$y_p'' = Ae^{-t},$$

$$y_p''' = -Ae^{-t},$$

$$y_p^{(4)} = Ae^{-t}.$$

Substituting these into the differential equation gives

$$y_p^{(4)} - 3y_p'' - 4y_p = Ae^{-t} - 3(Ae^{-t}) - 4(Ae^{-t}) = -6Ae^{-t}.$$

We set this equal to the right-hand side:

$$-6Ae^{-t} = e^{-t}.$$

Thus

$$A = -\frac{1}{6}.$$

Therefore the particular solution is

$$y_p = -\frac{1}{6}e^{-t}.$$

Finally, the general solution is the sum of the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = C_1e^{2t} + C_2e^{-2t} + C_3\cos(t) + C_4\sin(t) - \frac{1}{6}e^{-t}.$$

Problem 45. Find the general solutions of the following differential equations:

(a) $y^{(4)} + 5y'' + 4y = 0$

(b) $y^{(4)} + 5y'' + 4y = 8t^2 + t$

Solution 45. **(a)** First, we find the homogeneous solution for

$$y^{(4)} + 5y'' + 4y = 0$$

We assume a solution of the form

$$y = e^{rt}$$

which leads to the characteristic equation

$$r^4 + 5r^2 + 4 = 0.$$

We can factor this as

$$(r^2 + 4)(r^2 + 1) = 0.$$

This gives the roots

$$r^2 = -4 \Longrightarrow r = \pm 2i,$$

$$r^2 = -1 \Longrightarrow r = \pm i.$$

Therefore, the general solution to the homogeneous equation is

$$y_h = C_1\cos(2t) + C_2\sin(2t) + C_3\cos(t) + C_4\sin(t).$$

(b) Now we find the general solution for

$$y^{(4)} + 5y'' + 4y = 8t^2 + t$$

Since the homogeneous solution is already found in part (a), we use the method of undetermined coefficients to find a particular solution. Based on the right-hand side, we guess

$$y_p = At^2 + Bt + C.$$

We now compute the derivatives:

$$y_p' = 2At + B,$$

$$y_p'' = 2A,$$

$$y_p''' = 0,$$

$$y_p^{(4)} = 0.$$

Substituting these into the differential equation gives

$$(0) + 5(2A) + 4(At^2 + Bt + C) = 8t^2 + t,$$

$$4At^2 + 4Bt + (10A + 4C) = 8t^2 + t.$$

Matching the coefficients of like terms:

$$4A = 8 \Longrightarrow A = 2,$$

$$4B = 1 \Longrightarrow B = \frac{1}{4},$$

$$10A + 4C = 0 \Longrightarrow 10(2) + 4C = 0 \Longrightarrow 20 + 4C = 0 \Longrightarrow C = -5.$$

Therefore, the particular solution is

$$y_p = 2t^2 + \frac{1}{4}t - 5.$$

Finally, the general solution is

$$y = y_h + y_p$$

$$y = C_1\cos(2t) + C_2\sin(2t) + C_3\cos(t) + C_4\sin(t) + 2t^2 + \frac{1}{4}t - 5.$$

Problem 46. Find the general solution of the differential equation

$$y^{(4)} + 4y'' + 4y = 0.$$

Solution 46. First, we find the homogeneous solution. We assume a solution of the form

$$y = e^{rt}$$

which leads to the characteristic equation

$$r^4 + 4r^2 + 4 = 0.$$

We can factor this as

$$(r^2 + 2)^2 = 0.$$

This gives us the roots

$$r^2 = -2 \Longrightarrow r = \pm i\sqrt{2}.$$

Since the factor

$$(r^2 + 2)$$

is squared, these roots are repeated.

Therefore, the general solution to the differential equation is

$$y = C_1\cos(\sqrt{2}t) + C_2\sin(\sqrt{2}t) + C_3 t\cos(\sqrt{2}t) + C_4 t\sin(\sqrt{2}t).$$

Problem 47. Find the general solution of the differential equation

$$y^{(5)} + 4y''' = 0.$$

Solution 47. First, we find the homogeneous solution. We assume a solution of the form

$$y = e^{rt}$$

which leads to the characteristic equation

$$r^5 + 4r^3 = 0.$$

We can factor as

$$r^3(r^2 + 4) = 0.$$

From

$$r^3 = 0$$

we have the triple root at

$$r_{1,2,3} = 0$$

From

$$r^2 + 4 = 0$$

,

$$r^2 = -4 \Longrightarrow r_{4,5} = \pm 2i.$$

This corresponds to the complex solutions: Therefore, the general solution to the differential equation is

$$y = C_1 + C_2 t + C_3 t^2 + C_4 \cos(2t) + C_5 \sin(2t).$$

Problem 48. Find the general solution of the differential equation

$$y^{(6)} - 16y'' = 0.$$

Solution 48. First, we find the homogeneous solution. We assume a solution of the form

$$y = e^{rt}$$

which leads to the characteristic equation

$$r^6 - 16r^2 = 0.$$

We can factor this as

$$r^2(r^4 - 16) = 0$$

$$r^2(r^2 - 4)(r^2 + 4) = 0$$

$$r^2(r - 2)(r + 2)(r^2 + 4) = 0.$$

This gives us the following roots:
1. From

$$r^2 = 0$$

we have

$$r_{1,2} = 0$$

2. From

$$(r - 2)(r + 2) = 0$$

we have

$r_3 = 2$ and $r_4 = -2$

3. From

$$r^2 + 4 = 0$$

we have

$$r^2 = -4 \Longrightarrow r_{5,6} = \pm 2i$$

Therefore, the general solution to the differential equation is

$$y = C_1 + C_2 t + C_3 e^{2t} + C_4 e^{-2t} + C_5 \cos(2t) + C_6 \sin(2t).$$

Problem 49. Find the general solution of the following differential equations.

(a) $y''' - y'' - y' + y = 0$

(b) $t^2 y'' + y = 0$

Solution 49. (a) By assuming the solution is of the form $y = e^{rt}$, we can obtain the characteristic equation:

$$r^3 - r^2 - r + 1 = 0.$$

By using factoring by grouping we can simplify this polynomial:

$$\begin{aligned} r^3 - r^2 - r + 1 &= (r^3 - r^2) - (r - 1) \\ &= r^2(r - 1) - (r - 1) \\ &= (r - 1)(r^2 - 1) \\ &= (r - 1)^2(r + 1) \end{aligned}$$

Thus, the roots are

$$r = 1 \text{ (double)}, \qquad r = -1.$$

Therefore the general solution is given by:

$$y = c_1 e^t + c_2 t e^t + c_3 e^{-t}.$$

(b) By assuming the solution is of the form $y = t^m$, we can obtain the characteristic equation:

$$\begin{aligned} m(m - 1) + 1 &= 0 \\ m^2 - m + 1 &= 0 \end{aligned}$$

Using the quadratic formula we obtain the roots:

$$m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

Therefore the general solution is given by:

$$y = c_1 t^{1/2} \cos\left(\frac{\sqrt{3}}{2}\ln|t|\right) + c_2 t^{1/2} \sin\left(\frac{\sqrt{3}}{2}\ln|t|\right).$$

Problem 50. Find the general solution of

$$y^{(4)} + 3y'' - 4y = 2t + e^{2t}.$$

Solution 50. First solve the homogeneous equation

$$y^{(4)} + 3y'' - 4y = 0.$$

We assume the solution form $y = e^{rt}$. Then we obtain the characteristic equation:

$$r^4 + 3r^2 - 4 = 0$$
$$(r^2 + 4)(r^2 - 1) = 0$$

Thus,

$$r = \pm 1, \qquad r = \pm 2i.$$

Hence,

$$y_h = c_1 e^t + c_2 e^{-t} + c_3 \cos(2t) + c_4 \sin(2t).$$

We use the method of undetermined coefficients to find out particular solution. We make our guess of:

$$y_p = At + B + Ce^{2t}.$$

Which leads to:

$$y_p = At + B + Ce^{2t}$$
$$y_p' = A + 2Ce^{2t}$$
$$y_p'' = 4Ce^{2t}$$
$$y_p''' = 8Ce^{2t}$$
$$y_p'''' = 16Ce^{2t}$$

Substituting into the differential equation yields

$$16Ce^{2t} + 3(4Ce^{2t}) - 4(At + B + Ce^{2t}) = 2t + e^{2t}$$
$$-4At - 4B + 24Ce^{2t} = 2t + e^{2t}$$

Matching Coefficients leads to:

$$-4At = 2t$$
$$-4B = 0$$
$$24Ce^{2t} = e^{2t}$$

Then,

$$A = -\frac{1}{2}, \qquad B = 0, \qquad C = \frac{1}{24}.$$

Then our particular solution becomes:

$$y_p = -\frac{1}{2}t + \frac{1}{24}e^{2t}.$$

And the general solution is

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos(2t) + c_4 \sin(2t) - \frac{1}{2}t + \frac{1}{24}e^{2t}.$$

Problem 51. (a) Find the general solution of

$$y^{(5)} - 6y''' + 9y' = 0.$$

Solution 51. Assume a solution of the form: $y = e^{rt}$. Then,

$$\begin{aligned} r^5 - 6r^3 + 9r &= 0 \\ r(r^4 - 6r^2 + 9) &= 0 \\ r(r^2 - 3)^2 &= 0 \end{aligned}$$

Hence the roots are given by:

$$r = 0, \quad r = \pm\sqrt{3} \quad \text{(double root)}.$$

Thus, the general solution is given by:

$$y = C_1 + C_2 e^{\sqrt{3}t} + C_3 e^{-\sqrt{3}t} + C_4 t e^{\sqrt{3}t} + C_5 t e^{-\sqrt{3}t}.$$

Problem 52. Use Method of Undetermined Coefficients to find the general solution of the following problem:

$$y'' - 9y = \sin t$$

Solution 52. First, we find the complementary solution. We solve the homogeneous equation

$$y_c'' - 9y_c = 0.$$

The corresponding characteristic equation is

$$r^2 - 9 = 0.$$

Solving for r, we find $r = \pm 3$. Thus, the complementary solution is

$$y_c(t) = C_1 e^{3t} + C_2 e^{-3t}.$$

Next, we find the particular solution $Y(t)$ using the Method of Undetermined Coefficients. Since the right-hand side is $\sin t$, we assume a solution of the form:

$$Y(t) = A\sin t + B\cos t.$$

Taking the first and second derivatives:

$$Y' = A\cos t - B\sin t$$

$$Y'' = -A\sin t - B\cos t$$

Substituting these into the original differential equation $y'' - 9y = \sin t$:

$$(-A\sin t - B\cos t) - 9(A\sin t + B\cos t) = \sin t$$

Simplifying the left side, we get:

$$-10A\sin t - 10B\cos t = \sin t$$

By comparing the coefficients on both sides, we solve for A and B:

- For $\sin t$: $-10A = 1 \Longrightarrow A = -\frac{1}{10}$
- For $\cos t$: $-10B = 0 \Longrightarrow B = 0$

So, the particular solution is:

$$Y(t) = -\frac{1}{10}\sin t.$$

Finally, the general solution is the sum of the complementary and particular solutions:

$$y(t) = y_c(t) + Y(t)$$

$$y(t) = C_1 e^{3t} + C_2 e^{-3t} - \frac{1}{10}\sin t.$$

Problem 53. Recall that the Laplace transform $F(s)$ of a function $f(t)$ is given by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \quad s > 0.$$

Use this definition to find the Laplace transform $F(s)$ of the function

$$f(t) = \begin{cases} 0 & \text{when } 0 \le t \le 1 \\ 1 & \text{when } t > 1 \end{cases}$$

and state the values of s for which your answer is valid.

Solution 53. We use the definition of the Laplace transform and split the integral based on the piecewise definition of $f(t)$:

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^1 e^{-st}(0)dt + \int_1^\infty e^{-st}(1)dt$$

The first integral is simply 0:

$$\int_0^1 0\,dt = 0$$

For the second integral, we evaluate the improper integral using a limit:

$$\int_1^\infty e^{-st}\,dt = \lim_{q\to\infty} \int_1^q e^{-st}\,dt$$

Integrating e^{-st} with respect to t:

$$\lim_{q\to\infty} \left[-\frac{1}{s}e^{-st}\right]_1^q = \lim_{q\to\infty} \left[-\frac{1}{s}e^{-sq} - \left(-\frac{1}{s}e^{-s(1)}\right)\right]$$

$$= \lim_{q\to\infty} \left[-\frac{1}{s}e^{-sq} + \frac{1}{s}e^{-s}\right]$$

For the limit to converge, the exponent $-sq$ must be negative as $q \to \infty$, which requires $s > 0$. Under this condition, $e^{-sq} \to 0$:

$$0 + \frac{1}{s}e^{-s} = \frac{1}{s}e^{-s}$$

The Laplace transform is:

$$F(s) = \frac{1}{s}e^{-s}$$

This answer is valid for:

$$s > 0 \quad \text{(or the interval } (0, \infty))$$

Problem 54. Use the Method of Undetermined Coefficients to find the general solution of

$$y''' - y' = 7e^t.$$

Solution 54. First, we find the homogeneous solution y_h. We solve the characteristic equation:

$$r^3 - r = 0.$$

Factoring the equation, we get:

$$r(r^2 - 1) = r(r-1)(r+1) = 0.$$

The roots are $r = 0,1,-1$. Thus, the homogeneous solution is:

$$y_h = C_1 + C_2e^t + C_3e^{-t}.$$

Next, we find a particular solution y_p. Since the right-hand side is $7e^t$ and e^t is already a part of the homogeneous solution, we assume a particular solution of the form:

$$y_p = Ate^t.$$

We compute the derivatives:

$$y_p' = Ae^t + Ate^t$$

$$y_p'' = Ae^t + Ae^t + Ate^t = 2Ae^t + Ate^t$$

$$y_p''' = 2Ae^t + Ae^t + Ate^t = 3Ae^t + Ate^t.$$

Substituting these into the original differential equation $y''' - y' = 7e^t$:

$$(3Ae^t + Ate^t) - (Ae^t + Ate^t) = 7e^t.$$

Simplifying the left side, we get:

$$2Ae^t = 7e^t.$$

Comparing the coefficients, we solve for A:

$$2A = 7 \Longrightarrow A = \frac{7}{2}.$$

So, the particular solution is:

$$y_p = \frac{7}{2}te^t.$$

Finally, the general solution is the sum of the homogeneous and particular solutions:

$$y(t) = C_1 + C_2e^t + C_3e^{-t} + \frac{7}{2}te^t.$$

Problem 55. Find the general solution of the differential equation

$$y'' - 3y' + 2y = 2e^t.$$

Solution 55. First, we find the homogeneous solution. We solve the characteristic equation:

$$r^2 - 3r + 2 = 0.$$

Factoring the quadratic equation gives:

$$(r - 2)(r - 1) = 0.$$

The roots are $r = 2$ and $r = 1$. Thus, the homogeneous solution is:

$$y_h = C_1e^{2t} + C_2e^t.$$

Next, we find a particular solution y_p. Since the right-hand side is $2e^t$ and e^t is already a part of the homogeneous solution, we assume a particular solution of the form:

$$y_p = Ate^t.$$

We compute the derivatives of y_p:

$$y_p' = Ae^t + Ate^t$$

$$y_p'' = Ae^t + (Ae^t + Ate^t) = 2Ae^t + Ate^t.$$

Substituting these into the original differential equation $y'' - 3y' + 2y = 2e^t$:

$$(2Ae^t + Ate^t) - 3(Ae^t + Ate^t) + 2(Ate^t) = 2e^t.$$

Grouping the te^t and e^t terms:

$$(Ate^t - 3Ate^t + 2Ate^t) + (2Ae^t - 3Ae^t) = 2e^t$$

Simplifying the left side:

$$-Ae^t = 2e^t.$$

Comparing the coefficients, we solve for A:

$$-A = 2 \Longrightarrow A = -2.$$

So, the particular solution is:

$$y_p = -2te^t.$$

Finally, the general solution is the sum of the homogeneous and particular solutions:

$$y(t) = C_1e^{2t} + C_2e^t - 2te^t.$$

Problem 56. Find the general solution of the differential equation

$$y^{(5)} + 2y^{(4)} + 2y''' = 0.$$

Solution 56. We begin by finding the characteristic equation for the given homogeneous differential equation:

$$r^5 + 2r^4 + 2r^3 = 0.$$

We can factor out r^3:

$$r^3(r^2 + 2r + 2) = 0.$$

This gives us the roots:

- $r^3 = 0 \Longrightarrow r_1 = r_2 = r_3 = 0$ (a triple root at zero).
- $r^2 + 2r + 2 = 0$. Using the quadratic formula:

$$r_{4,5} = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

The roots are $0,0,0, -1 + i$, and $-1 - i$.

The general solution is formed by combining the solutions corresponding to each root:

- For the triple root $r = 0$, the solutions are C_1e^{0t}, C_2te^{0t}, and $C_3t^2e^{0t}$, which simplify to $C_1, C_2t,$ and C_3t^2.
- For the complex roots $r = -1 \pm i$, the solutions are $C_4e^{-t}\text{cos}t$ and $C_5e^{-t}\text{sin}t$.

Thus, the general solution is:

$$y(t) = C_1 + C_2t + C_3t^2 + C_4e^{-t}\text{cos}t + C_5e^{-t}\text{sin}t.$$

Problem 57. Find the general solution of the differential equation

$$y'' + 4y' + 4y = e^t.$$

Solution 57. First, we find the homogeneous solution y_h. We solve the characteristic equation:

$$r^2 + 4r + 4 = 0.$$

Factoring the quadratic equation gives:

$$(r + 2)^2 = 0 \Longrightarrow r = -2 \text{ (repeated roots)}.$$

Thus, the homogeneous solution is:

$$y_h = C_1e^{-2t} + C_2te^{-2t}.$$

Next, we find a particular solution y_p using the Method of Undetermined Coefficients (MUC). Since the term $g(t) = e^t$ is not contained in y_h, we assume a particular solution of the form:

$$y_p = Ae^t.$$

The derivatives of y_p are:

$$y_p' = Ae^t \quad \text{and} \quad y_p'' = Ae^t.$$

Substituting these into the original differential equation $y'' + 4y' + 4y = e^t$:

$$(Ae^t) + 4(Ae^t) + 4(Ae^t) = e^t.$$

$$9Ae^t = e^t \Longrightarrow 9A = 1 \Longrightarrow A = \frac{1}{9}.$$

So, the particular solution is:

$$y_p = \frac{1}{9}e^t.$$

Finally, the general solution is the sum of the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = C_1 e^{-2t} + C_2 t e^{-2t} + \frac{1}{9} e^t.$$

Problem 58. Find the general solution of the differential equation

$$y^{(4)} + 4y'' - 5y = 0.$$

Solution 58. We begin by assuming a solution of the form $y = e^{rt}$. Substituting this into the differential equation gives the characteristic equation:

$$r^4 + 4r^2 - 5 = 0.$$

This is a quadratic equation in terms of r^2. We can factor it as follows:

$$(r^2 + 5)(r^2 - 1) = 0.$$

Setting each factor to zero, we solve for the roots r:

- $r^2 + 5 = 0 \Longrightarrow r^2 = -5 \Longrightarrow r = \pm\sqrt{5}i$
- $r^2 - 1 = 0 \Longrightarrow (r+1)(r-1) = 0 \Longrightarrow r = -1, \quad r = 1$

The roots of the characteristic equation are $\pm\sqrt{5}i$, -1, and 1.

The general solution is formed by combining the linearly independent solutions corresponding to these roots:

- For the complex roots $r = \pm\sqrt{5}i$, the solutions are $\cos(\sqrt{5}t)$ and $\sin(\sqrt{5}t)$.
- For the real roots $r = -1$ and $r = 1$, the solutions are e^{-t} and e^t.

Thus, the general solution is:

$$y = C_1 \cos(\sqrt{5}t) + C_2 \sin(\sqrt{5}t) + C_3 e^{-t} + C_4 e^t.$$

Problem 59. Find the general solution of the differential equation

$$y'' + 4y = \cos(t) - 3\sin(t).$$

Solution 59. First, we find the homogeneous solution y_h. We solve the characteristic equation:

$$r^2 + 4 = 0 \Longrightarrow r^2 = -4 \Longrightarrow r = \pm 2i.$$

The homogeneous solution is:

$$y_h = C_1 \cos(2t) + C_2 \sin(2t).$$

Next, we find a particular solution y_p using the Method of Undetermined Coefficients. Since the right-hand side is $\cos(t) - 3\sin(t)$ and there is no overlap with y_h, we assume:

$$y_p = A\cos t + B\sin t.$$

We find the derivatives:

$$y_p' = -A\sin t + B\cos t$$

$$y_p'' = -A\cos t - B\sin t.$$

Substituting these into the original differential equation $y'' + 4y = \cos(t) - 3\sin(t)$:

$$[-A\cos t - B\sin t] + 4[A\cos t + B\sin t] = \cos t - 3\sin t.$$

Simplifying the left side by grouping terms:

$$3A\cos t + 3B\sin t = 1\cos t - 3\sin t.$$

By matching coefficients, we solve for A and B:

- For $\cos t$: $3A = 1 \Longrightarrow A = \frac{1}{3}$
- For $\sin t$: $3B = -3 \Longrightarrow B = -1$

So, the particular solution is:

$$y_p = \frac{1}{3}\cos t - \sin t.$$

Finally, the general solution is the sum of the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{3}\cos t - \sin t.$$

Problem 60. Find the general solution of the differential equation

$$y^{(5)} - 3y^{(4)} + 2y''' = 0.$$

Solution 60. We begin by finding the characteristic equation for the given homogeneous differential equation by assuming a solution of the form $y = e^{rt}$:

$$r^5 - 3r^4 + 2r^3 = 0.$$

We can factor out r^3:

$$r^3(r^2 - 3r + 2) = 0.$$

Further factoring the quadratic expression gives:

$$r^3(r - 2)(r - 1) = 0.$$

This provides the following roots for r:

- $r = 0$ with multiplicity 3 (a triple root).

- $r = 1$ with multiplicity 1.
- $r = 2$ with multiplicity 1.

The general solution is formed by combining the linearly independent solutions corresponding to each root:

- For the triple root $r = 0$, the solutions are $C_1, C_2t,$ and C_3t^2.
- For the root $r = 1$, the solution is C_4e^t.
- For the root $r = 2$, the solution is C_5e^{2t}.

Thus, the general solution is:

$$y = C_1 + C_2t + C_3t^2 + C_4e^t + C_5e^{2t}.$$

Problem 61. Find the general solution of the differential equation

$$y'' + 4y = \cos(t) - 3\sin(t).$$

Solution 61. First, we find the homogeneous solution y_h. We solve the characteristic equation:

$$r^2 + 4 = 0 \Longrightarrow r^2 = -4 \Longrightarrow r = \pm 2i.$$

The homogeneous solution is:

$$y_h = C_1\cos(2t) + C_2\sin(2t).$$

Next, we find a particular solution y_p using the Method of Undetermined Coefficients. Since the right-hand side is $\cos(t) - 3\sin(t)$ and there is no overlap with y_h, we assume:

$$y_p = A\cos t + B\sin t.$$

We find the derivatives:

$$y_p' = -A\sin t + B\cos t$$

$$y_p'' = -A\cos t - B\sin t.$$

Substituting these into the original differential equation $y'' + 4y = \cos(t) - 3\sin(t)$:

$$[-A\cos t - B\sin t] + 4[A\cos t + B\sin t] = \cos t - 3\sin t.$$

Simplifying the left side by grouping terms:

$$3A\cos t + 3B\sin t = 1\cos t - 3\sin t.$$

By matching coefficients, we solve for A and B:

- For $\cos t$: $3A = 1 \Longrightarrow A = \frac{1}{3}$

- For $\sin t$: $3B = -3 \Rightarrow B = -1$

So, the particular solution is:

$$y_p = \frac{1}{3}\cos t - \sin t.$$

Finally, the general solution is the sum of the homogeneous and particular solutions:

$$y = y_h + y_p$$

$$y = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{3}\cos t - \sin t.$$

Problem 62. Find the general solution of the differential equation

$$y^{(5)} - 3y^{(4)} + 2y''' = 0.$$

Solution 62. We begin by finding the characteristic equation for the given homogeneous differential equation by assuming a solution of the form $y = e^{rt}$:

$$r^5 - 3r^4 + 2r^3 = 0.$$

We can factor out r^3:

$$r^3(r^2 - 3r + 2) = 0.$$

Further factoring the quadratic expression gives:

$$r^3(r-2)(r-1) = 0.$$

This provides the following roots for r:

- $r = 0$ with multiplicity 3 (a triple root).
- $r = 1$ with multiplicity 1.
- $r = 2$ with multiplicity 1.

The general solution is formed by combining the linearly independent solutions corresponding to each root:

- For the triple root $r = 0$, the solutions are $C_1, C_2t,$ and C_3t^2.
- For the root $r = 1$, the solution is C_4e^t.
- For the root $r = 2$, the solution is C_5e^{2t}.

Thus, the general solution is:

$$y = C_1 + C_2t + C_3t^2 + C_4e^t + C_5e^{2t}.$$

Laplace Transforms

Problem 63. Use the definition of the Laplace transform, $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$, to find the Laplace transform of the function

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 1, \\ 1 & \text{if } t \ge 1. \end{cases}$$

For which values of s is the Laplace transform of f defined?

Solution 63. We use the definition of the Laplace transform by splitting the integral at the point where the function's definition changes:

$$\mathcal{L}\{f(t)\} = \int_0^1 t\,e^{-st}dt + \int_1^\infty 1 \cdot e^{-st}dt.$$

First, we evaluate the integral $\int_0^1 t\,e^{-st}dt$ using integration by parts. Let $u = t$ and $dv = e^{-st}dt$. Then $du = dt$ and $v = -\frac{1}{s}e^{-st}$.

$$\int_0^1 t\,e^{-st}dt = \left[-\frac{t}{s}e^{-st}\right]_0^1 - \int_0^1 -\frac{1}{s}e^{-st}dt$$

$$= \left(-\frac{1}{s}e^{-s} - 0\right) + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_0^1$$

$$= -\frac{e^{-s}}{s} - \frac{1}{s^2}(e^{-s} - 1) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}.$$

Next, we evaluate the improper integral $\int_1^\infty e^{-st}\,dt$:

$$\int_1^\infty e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s}\right).$$

This limit converges to $\frac{e^{-s}}{s}$ if and only if $s > 0$.

Adding the two parts together, we find:

$$\mathcal{L}\{f(t)\} = \left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}\right) + \left(\frac{e^{-s}}{s}\right).$$

Simplifying the expression, the terms $\frac{e^{-s}}{s}$ cancel out:

$$\mathcal{L}\{f(t)\} = \frac{1 - e^{-s}}{s^2},\ s > 0$$

Problem 64. Use the definition of the Laplace transform,

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

for those values of s for which the improper integral converges, to compute the Laplace transform of the function given by

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1, \\ t-1 & \text{if } 1 \le t < \infty. \end{cases}$$

Solution 64. We use the definition of the Laplace transform by splitting the integral at the point where the function's definition changes:

$$\mathcal{L}\{f(t)\} = \int_0^1 0 \cdot e^{-st} dt + \int_1^\infty (t-1)e^{-st} dt.$$

The first integral is zero. Thus, we only need to evaluate the second integral:

$$\mathcal{L}\{f(t)\} = \int_1^\infty (t-1)e^{-st} dt.$$

We use integration by parts. Let

$$u = t - 1$$

and

$$dv = e^{-st} dt$$

. Then

$$du = dt$$

and

$$v = -\frac{1}{s}e^{-st}$$

$$\int_1^\infty (t-1)e^{-st} dt = \lim_{b\to\infty}\left(\left[-\frac{t-1}{s}e^{-st}\right]_1^b - \int_1^b -\frac{1}{s}e^{-st} dt\right)$$

$$= \lim_{b\to\infty}\left(\left[-\frac{b-1}{s}e^{-sb} - 0\right] + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\right]_1^b\right)$$

$$= \lim_{b\to\infty}\left(-\frac{b-1}{s}e^{-sb} - \frac{1}{s^2}e^{-sb} + \frac{1}{s^2}e^{-s}\right)$$

For the integral to converge, we require

$$s > 0$$

In this case,

$$e^{-sb} \to 0 \text{ as } b \to \infty$$

Then we have

$$\mathcal{L}\{f(t)\} = 0 - 0 + \frac{e^{-s}}{s^2}.$$

Therefore, the Laplace transform is

$$\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2},\ s > 0.$$

Problem 65. Use the definition of the Laplace transform,

$$\mathcal{L}\{f\}(s) = \int_0^\infty f\,(t) e^{-st} dt$$

for those values of

$$s$$

for which the improper integral converges, to find the Laplace transform of the function

$$f(t) = te^{at}$$

where a is a real constant. For which values of s is the Laplace transform of f defined?

Solution 65. We use the definition of the Laplace transform:

$$\mathcal{L}\{te^{at}\} = \int_0^\infty t\, e^{at} e^{-st} dt = \int_0^\infty t\, e^{-(s-a)t} dt.$$

To evaluate this improper integral, we use integration by parts. Let

$$u = t$$

and

$$dv = e^{-(s-a)t} dt$$

Then

$$du = dt$$

and

$$v = -\frac{1}{s-a} e^{-(s-a)t}$$

$$\int_0^\infty t\, e^{-(s-a)t} dt = \lim_{b\to\infty} \left(\left[-\frac{t}{s-a} e^{-(s-a)t} \right]_0^b - \int_0^b -\frac{1}{s-a} e^{-(s-a)t} dt \right)$$

$$= \lim_{b\to\infty}\left(\left[-\frac{b}{s-a}e^{-(s-a)b} - 0\right] + \frac{1}{s-a}\int_0^b e^{-(s-a)t}\,dt\right)$$

$$= \lim_{b\to\infty}\left(-\frac{b}{s-a}e^{-(s-a)b} + \frac{1}{s-a}\left[-\frac{1}{s-a}e^{-(s-a)t}\right]_0^b\right)$$

$$= \lim_{b\to\infty}\left(-\frac{b}{s-a}e^{-(s-a)b} - \frac{1}{(s-a)^2}e^{-(s-a)b} + \frac{1}{(s-a)^2}e^{0}\right)$$

For the integral to converge, we require the exponential terms to approach zero as

$$b \to \infty$$

This occurs when the exponent is negative, which means

$$s - a > 0 \Rightarrow s > a$$

Under this condition, the limits of the first two terms are zero, leaving:

$$\mathcal{L}\{te^{at}\} = 0 - 0 + \frac{1}{(s-a)^2}.$$

Therefore, the Laplace transform is

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

and it is defined for

$$s > a$$

.

Problem 66. Use the definition of the Laplace transform,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

for those values of s for which this improper integral converges, to find the Laplace transform of the function

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 2, \\ e^t & \text{if } t \ge 2. \end{cases}$$

For which values of s is the Laplace transform of f defined?

Solution 66. We use the definition of the Laplace transform and split the integral into two parts based on the piecewise definition of $f(t)$:

$$F(s) = \int_0^2 t\,e^{-st}dt + \int_2^\infty e^t\,e^{-st}dt.$$

For the first integral, $\int_0^2 t\, e^{-st} dt$, we use integration by parts. Let $u = t$ and $dv = e^{-st} dt$. Then $du = dt$ and $v = -\frac{1}{s} e^{-st}$:

$$\int_0^2 t\, e^{-st} dt = \left[-\frac{t}{s} e^{-st}\right]_0^2 - \int_0^2 -\frac{1}{s} e^{-st} dt$$

$$= \left(-\frac{2}{s} e^{-2s} - 0\right) + \frac{1}{s}\left[-\frac{1}{s} e^{-st}\right]_0^2$$

$$= -\frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s} + \frac{1}{s^2}.$$

For the second integral, $\int_2^\infty e^t\, e^{-st} dt$, we combine the exponents:

$$\int_2^\infty e^{-(s-1)t}\, dt = \lim_{b\to\infty} \left[-\frac{1}{s-1} e^{-(s-1)t}\right]_2^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{s-1} e^{-(s-1)b} + \frac{1}{s-1} e^{-2(s-1)}\right).$$

For the integral to converge, we require the exponential term to approach zero as $b \to \infty$. This occurs when:

$$s - 1 > 0 \Longrightarrow s > 1.$$

Under this condition, the limit is:

$$\int_2^\infty e^{-(s-1)t}\, dt = 0 + \frac{1}{s-1} e^{-2(s-1)} = \frac{e^{2-2s}}{s-1}.$$

Combining the two parts, the Laplace transform is:

$$F(s) = \frac{1}{s^2} - \frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s} + \frac{1}{s-1} e^{2-2s}.$$

Therefore, the Laplace transform is

$$F(s) = \frac{1 - (1 + 2s)e^{-2s}}{s^2} + \frac{e^{2-2s}}{s-1}$$

and it is defined for $s > 1$.

Problem 67. Use the definition of the Laplace transform,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f\,(t) e^{-st} dt$$

for those values of s for which this improper integral converges, to find the Laplace transform of the function

$$f(t) = \begin{cases} \pi & \text{if } 0 \leq t < \pi, \\ t & \text{if } \pi \leq t < \infty. \end{cases}$$

For which values of s is the Laplace transform of f defined?

Solution 67. We use the definition of the Laplace transform and split the integral into two parts based on the piecewise definition of $f(t)$:

$$F(s) = \int_0^{\pi} \pi\, e^{-st} dt + \int_{\pi}^{\infty} t\, e^{-st} dt.$$

For the first integral, $\int_0^{\pi} \pi\, e^{-st} dt$, we integrate directly:

$$\int_0^{\pi} \pi\, e^{-st} dt = \pi \left[-\frac{1}{s} e^{-st} \right]_0^{\pi} = \pi \left(-\frac{1}{s} e^{-\pi s} + \frac{1}{s} \right) = \frac{\pi}{s} (1 - e^{-\pi s}).$$

For the second integral, $\int_{\pi}^{\infty} t\, e^{-st} dt$, we use integration by parts. Let $u = t$ and $dv = e^{-st} dt$. Then $du = dt$ and $v = -\frac{1}{s} e^{-st}$:

$$\int_{\pi}^{\infty} t\, e^{-st} dt = \lim_{b \to \infty} \left(\left[-\frac{t}{s} e^{-st} \right]_{\pi}^{b} - \int_{\pi}^{b} -\frac{1}{s} e^{-st} dt \right)$$

$$= \lim_{b \to \infty} \left(\left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_{\pi}^{b} \right)$$

$$= \lim_{b \to \infty} \left(-\frac{b}{s} e^{-bs} - \frac{1}{s^2} e^{-bs} + \frac{\pi}{s} e^{-\pi s} + \frac{1}{s^2} e^{-\pi s} \right).$$

For the integral to converge, we require the exponential terms to approach zero as $b \to \infty$. This occurs when:

$$s > 0.$$

Under this condition, the limit is:

$$\int_{\pi}^{\infty} t\, e^{-st} dt = 0 - 0 + \frac{\pi}{s} e^{-\pi s} + \frac{1}{s^2} e^{-\pi s} = \left(\frac{\pi s + 1}{s^2} \right) e^{-\pi s}.$$

Combining the two parts, the Laplace transform is:

$$F(s) = \frac{\pi}{s} - \frac{\pi}{s} e^{-\pi s} + \frac{\pi}{s} e^{-\pi s} + \frac{1}{s^2} e^{-\pi s}.$$

Simplifying the expression, we get:

$$F(s) = \frac{\pi}{s} + \frac{1}{s^2} e^{-\pi s}.$$

and it is defined for positive s.

Problem 68. Recall the definition of the Laplace transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t) e^{-st}\, dt.$$

(a) Use the definition to find the Laplace transform of $f(t) = t$.

(b) For which values of s is the transform defined?

Solution 68. (a) Using the definition,

$$\mathcal{L}\{t\}(s) = \int_0^\infty t\, e^{-st}\, dt.$$

Integrate by parts with $u = t$, $dv = e^{-st} dt$. Then $du = dt$ and $v = -\frac{1}{s} e^{-st}$.

Thus

$$\begin{aligned}\mathcal{L}\{t\}(s) &= \lim_{M\to\infty}\left[-\frac{t}{s} e^{-st}|_0^M + \frac{1}{s}\int_0^M e^{-st}\, dt\right] \\ &= (0-0) + \frac{1}{s}(0 - \frac{-1}{s}) \\ &= \frac{1}{s^2}\end{aligned}$$

(b) The exponential terms vanish only when $s > 0$. Hence the Laplace transform is defined for

$$s > 0.$$

Problem 69. Using the definition of the Laplace transform, compute the Laplace transform of

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t \le 2 \\ 2, & t > 2 \end{cases}$$

and determine for which s the transform is defined.

Solution 69. Recall the definition of Laplace transform:

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Since we have a piecewise function, we split the integral. Then, we obtain:

$$\mathcal{L}\{f\}(s) = \int_0^1 0\ dt + \int_1^2 e^{-st}\, dt + 2\int_2^\infty e^{-st}\, dt.$$

Compute each part:

$$\int_0^1 0 \ dt = 0$$

$$\int_1^2 e^{-st}\, dt = -\frac{1}{s}(e^{-2s} - e^{-s}).$$

$$2\int_2^\infty e^{-st}\, dt = \frac{2}{s}e^{-2s}.$$

Thus,

$$\mathcal{L}\{f\}(s) = -\frac{1}{s}(e^{-2s} - e^{-s}) + \frac{2}{s}e^{-2s}.$$

Simplifying,

$$\mathcal{L}\{f\}(s) = \frac{e^{-2s} + e^{-s}}{s}.$$

The integral converges for $s > 0$.

Problem 70. Recall the definition of the Laplace transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

(a) Use the definition to find $\mathcal{L}\{f\}(s)$ for

$$f(t) = \begin{cases} 2, & 0 \le t \le 1, \\ e^t, & t > 1. \end{cases}$$

(b) For which values of s is the Laplace transform defined?

Solution 70. Using the definition of the Laplace transform we split up the integral into two parts.

$$\mathcal{L}\{f\}(s) = \int_0^1 2\, e^{-st} dt + \int_1^\infty e^{-st}\, e^t dt$$
$$= \left[\frac{2}{-s}(e^{-st})\right]_0^1 + \lim_{M\to\infty}\left[\frac{1}{1-s}e^{(1-s)t}\right]_1^M$$

The second integral is improper. Taking $M \to \infty$ gives

$$= \frac{2}{s}(1 - e^{-s}) + \frac{e^{1-s}}{s-1}$$

(b) The second integral converges only if $1 - s < 0$, so, $s > 1$.

Problem 71. Using the definition of the Laplace transform, find $\mathcal{L}\{f(t)\}(s)$ if

$$f(t) = \begin{cases} 7e^{4t} & 0 < t < 3 \\ 4 & t > 3 \end{cases}$$

You must also specify the domain of the transform.

Solution 71. We use the definition of the Laplace transform, $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$. We split the integral into two parts based on the piecewise definition of $f(t)$:

$$\mathcal{L}\{f(t)\} = \int_0^3 e^{-st} \cdot 7e^{4t} dt + \int_3^\infty e^{-st} \cdot 4dt.$$

For the first integral, we combine the exponents:

$$7\int_0^3 e^{-(s-4)t}\, dt = 7\left[-\frac{1}{s-4}e^{-(s-4)t}\right]_{t=0}^{t=3} = \frac{7}{s-4}\left(-e^{-3(s-4)} + e^0\right) = \frac{7}{s-4}(1 - e^{-3(s-4)}).$$

This integral converges for all values of s, but the term $\frac{1}{s-4}$ requires $s \neq 4$.

For the second integral, we evaluate the improper integral:

$$4\int_3^\infty e^{-st}\, dt = 4\lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_3^b = 4\lim_{b\to\infty}\left(-\frac{1}{s}e^{-bs} + \frac{1}{s}e^{-3s}\right).$$

For this limit to converge, we require $s > 0$. Under this condition, $e^{-bs} \to 0$ as $b \to \infty$:

$$4\int_3^\infty e^{-st}\, dt = \frac{4}{s}e^{-3s}.$$

Additionally, for the entire transform to be defined, we must consider the growth of the function $f(t)$. Since $f(t)$ behaves like e^{4t}, the integral $\int_0^\infty e^{-st} f(t)dt$ only converges if $s > 4$.

Combining the two parts, the Laplace transform is:

$$\mathcal{L}\{f(t)\} = \frac{7}{s-4}(1 - e^{-3(s-4)}) + \frac{4}{s}e^{-3s}.$$

The transform is defined for:

$$s > 4.$$

Inverse Laplace Transforms

Problem 72. Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s-2)(s-3)(s-6)}$$

Solution 72. To find the inverse Laplace transform, we first break the fraction into simpler parts using partial fraction decomposition.

$$\frac{s}{(s-2)(s-3)(s-6)} = \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-6}$$

To solve for these numbers, we multiply both sides by the denominator $(s-2)(s-3)(s-6)$ to get:

$$s = A(s-3)(s-6) + B(s-2)(s-6) + C(s-2)(s-3)$$

In this case, we can evaluate at specific values of s to find the coefficients.

$$s = 2 \Longrightarrow 2 = A(2-3)(2-6) + B(0) + C(0)$$

$$2 = A(-1)(-4)$$

$$2 = 4A \Longrightarrow A = \frac{1}{2}$$

$$s = 3 \Longrightarrow 3 = A(0) + B(3-2)(3-6) + C(0)$$

$$3 = B(1)(-3)$$

$$3 = -3B \Longrightarrow B = -1$$

$$s - 6 \Longrightarrow 6 = A(0) + B(0) + C(6-2)(6-3)$$

$$6 = C(4)(3)$$

$$6 = 12C \Longrightarrow C = \frac{1}{2}$$

Now we substitute these values back into our partial fraction form:

$$F(s) = \frac{1/2}{s-2} - \frac{1}{s-3} + \frac{1/2}{s-6}$$

Taking the inverse Laplace of both sides yields

$$f(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$$

Applying formula 2 from the table of Laplace transforms, in the inverse direction, we obtain

$$f(t) = \frac{1}{2}e^{2t} - e^{3t} + \frac{1}{2}e^{6t}$$

Problem 73. Find the inverse Laplace transform of

$$\frac{3s+5}{s^2-4s+5}.$$

Solution 73. To find the inverse Laplace transform, we first look at the denominator. Since

$$s^2 - 4s + 5$$

does not factor easily into real linear factors, we complete the square.

$$s^2 - 4s + 5 = (s^2 - 4s + 4) + 1 = (s - 2)^2 + 1.$$

Now we rewrite the original function using this new denominator:

$$F(s) = \frac{3s + 5}{(s - 2)^2 + 1}.$$

Next, we need to adjust the numerator to match the term

$$(s - 2)$$

in the denominator. We can write

$$3s$$

as

$$3(s - 2 + 2)$$

Then we can write

$$3s + 5 = 3(s - 2) + 6 + 5 = 3(s - 2) + 11.$$

Substituting, we have

$$F(s) = \frac{3(s - 2) + 11}{(s - 2)^2 + 1} = 3\frac{s - 2}{(s - 2)^2 + 1} + 11\frac{1}{(s - 2)^2 + 1}.$$

Now we take the inverse Laplace transform of each part. We use the formulas 9 and 10 from the table of Laplace transforms.

$$\mathcal{L}^{-1}\left\{\frac{s - a}{(s - a)^2 + b^2}\right\} = e^{at}\cos(bt)$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s - a)^2 + b^2}\right\} = e^{at}\sin(bt)$$

In our case,

$$a = 2 \text{ and } b = 1$$

Applying these formulas and the linear property of the inverse Laplace we have

$$f(t) = 3\mathcal{L}^{-1}\left\{\frac{s - 2}{(s - 2)^2 + 1^2}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s - 2)^2 + 1^2}\right\}.$$

$$f(t) = 3e^{2t}\cos(t) + 11e^{2t}\sin(t).$$

Problem 74. Find the inverse Laplace transform of

$$F(s) = \frac{s^2 + s + 2}{s^3 + s}.$$

Solution 74. To find the inverse Laplace transform, we first break the fraction into simpler parts using partial fraction decomposition. We start by factoring the denominator:

$$s^3 + s = s(s^2 + 1).$$

Since

$$s^2 + 1$$

is an irreducible quadratic factor, the decomposition takes the form

$$\frac{s^2 + s + 2}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

To solve for the coefficients, we multiply both sides by the denominator

$$s(s^2 + 1)$$

to get:

$$s^2 + s + 2 = A(s^2 + 1) + (Bs + C)s.$$

$$s^2 + s + 2 = (A + B)s^2 + Cs + A.$$

Matching the coefficients of like terms:

$$A = 2$$

$$C = 1$$

$$B = -1.$$

Now we substitute these values back into our partial fraction form:

$$F(s) = \frac{2}{s} + \frac{-s + 1}{s^2 + 1} = \frac{2}{s} - \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1}.$$

Taking the inverse Laplace transform of each side and using linearity:

$$f(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}.$$

Applying formulas 1,5, and 6 from the table of Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1, \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} = \cos(t), \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin(t).$$

Thus, the inverse Laplace transform is

$$f(t) = 2 - \cos(t) + \sin(t).$$

Problem 75. Find the inverse Laplace transform of

$$F(s) = \frac{s+1}{s^2 - s - 6}.$$

Solution 75. To find the inverse Laplace transform, we first break the fraction into simpler parts using partial fraction decomposition. We start by factoring the denominator:

$$s^2 - s - 6 = (s-3)(s+2).$$

The decomposition takes the form

$$\frac{s+1}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}.$$

To solve for the coefficients, we multiply both sides by the denominator

$$(s-3)(s+2)$$

to get:

$$s + 1 = A(s+2) + B(s-3).$$

We can solve for A and B by substituting convenient values for s:

- Let $s = 3$:

$$3 + 1 = A(3+2) + B(0) \Longrightarrow 4 = 5A \Longrightarrow A = \frac{4}{5}.$$

- Let $s = -2$:

$$-2 + 1 = A(0) + B(-2-3) \Longrightarrow -1 = -5B \Longrightarrow B = \frac{1}{5}.$$

Now we substitute these values back into our partial fraction form:

$$F(s) = \frac{4/5}{s-3} + \frac{1/5}{s+2}.$$

Taking the inverse Laplace transform of each side and using linearity:

$$f(t) = \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}.$$

Applying the formula $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ from row 2 of the table of Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}, \quad \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}.$$

Thus, the inverse Laplace transform is

$$f(t) = \frac{4}{5}e^{3t} + \frac{1}{5}e^{-2t}.$$

Problem 76. Find the inverse Laplace transform of

$$F(s) = \frac{5s+1}{s^2 - 2s + 3}.$$

Solution 76. Notice how the denominator does not factor. Thus,we complete the square on the denominator:

$$s^2 - 2s + 3 = (s-1)^2 + 2.$$

We can also rewrite the numerator in a way which has a $s - 1$ term and a constant term:

$$5s + 1 = 5(s-1) + 6.$$

Combining these yields:

$$\begin{aligned} F(s) &= 5\frac{s-1}{(s-1)^2+2} + 6\frac{1}{(s-1)^2+2} \\ &= 5\frac{s-1}{(s-1)^2+(\sqrt{2})^2} + 6\frac{1}{(s-1)^2+(\sqrt{2})^2} \\ &= 5\frac{s-1}{(s-1)^2+(\sqrt{2})^2} + \frac{6}{\sqrt{2}}\frac{\sqrt{2}}{(s-1)^2+(\sqrt{2})^2} \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= 5e^t\cos(\sqrt{2}t) + \frac{6}{\sqrt{2}}\,e^t\sin(\sqrt{2}t) \\ &= 5e^t\cos(\sqrt{2}t) + 3\sqrt{2}\,e^t\sin(\sqrt{2}t) \end{aligned}$$

Problem 77. Find the inverse Laplace transform of

$$F(s) = \frac{5s}{s^2 - s - 6}.$$

Solution 77. First, we factor the denominator:

$$s^2 - s - 6 = (s-3)(s+2).$$

Then we perform partial fraction decomposition.

$$\begin{aligned} \frac{5s}{(s-3)(s+2)} &= \frac{A}{s-3} + \frac{B}{s+2} \\ 5s &= A(s+2) + B(s-3) \\ (s=3)15 &= 5A \Rightarrow A = 3 \\ (s=-2) -10 &= -5B \Rightarrow B = 2 \end{aligned}$$

Hence,

$$F(s) = \frac{3}{s-3} + \frac{2}{s+2}.$$

By taking inverse Laplace transforms (using the table),

$$\mathcal{L}^{-1}\{F(s)\} = 3e^{3t} + 2e^{-2t}.$$

Problem 78. Find the inverse Laplace transform of

$$H(s) = \frac{1}{s(s^2+1)}$$

Solution 78. To find the inverse Laplace transform, we first decompose $H(s)$ into partial fractions:

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

To find the constants A, B, and C, we multiply both sides by the common denominator $s(s^2+1)$:

$$1 = A(s^2+1) + (Bs+C)s$$

Expanding the right-hand side:

$$1 = As^2 + A + Bs^2 + Cs$$

Grouping the like terms:

$$1 = (A+B)s^2 + Cs + A$$

Comparing the coefficients on both sides of the equation:

- Constant term: $A = 1$
- Coefficient of s: $C = 0$
- Coefficient of s^2: $A + B = 0 \Longrightarrow 1 + B = 0 \Longrightarrow B = -1$

Substituting these values back into our partial fraction decomposition, we get:

$$H(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

Now, we find the inverse Laplace transform of each term separately:

- $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$

Thus, the inverse Laplace transform is:

$$h(t) = 1 - \cos(t)$$

Problem 79. Find the inverse Laplace transform of

$$F(s) = \frac{s + 11}{(s - 1)(s + 3)}$$

Solution 79. To find the inverse Laplace transform, we first decompose $F(s)$ into partial fractions:

$$\frac{s + 11}{(s - 1)(s + 3)} = \frac{A}{s - 1} + \frac{B}{s + 3}$$

Multiplying both sides by the denominator $(s - 1)(s + 3)$ gives the basic equation:

$$s + 11 = A(s + 3) + B(s - 1)$$

We solve for the constants A and B:

- Let $s = -3$:

$$-3 + 11 = B(-3 - 1) \Longrightarrow 8 = -4B \Longrightarrow B = -2$$

- Let $s = 1$:

$$1 + 11 = A(1 + 3) \Longrightarrow 12 = 4A \Longrightarrow A = 3$$

Substituting these values back into the partial fraction decomposition, we have:

$$F(s) = \frac{3}{s - 1} - \frac{2}{s + 3}$$

Using the linearity of the inverse Laplace transform and the fact that $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$:

$$f(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s - 1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s + 3}\right\}$$

Thus, the inverse Laplace transform is:

$$f(t) = 3e^{t} - 2e^{-3t}$$

Problem 80. Determine the inverse Laplace transform of

$$F(s) = \frac{5}{(s - 1)(s^2 + 4)}.$$

Solution 80. To find the inverse Laplace transform, we first decompose $F(s)$ into partial fractions:

$$\frac{5}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

Multiplying by the denominator gives the equation:

$$5 = A(s^2+4) + (Bs+C)(s-1)$$

Expanding and grouping terms:

$$5 = (A+B)s^2 + (C-B)s + (4A-C)$$

By matching coefficients, we set up a system of equations:

- For s^2: $A + B = 0 \Longrightarrow A = -B$
- For s: $C - B = 0 \Longrightarrow C = B$
- Constants: $4A - C = 5 \Longrightarrow 4(-B) - B = 5 \Longrightarrow -5B = 5 \Longrightarrow B = -1$

Solving the system, we find $B = -1$, $A = 1$, and $C = -1$. Substituting these back into the decomposition:

$$F(s) = \frac{1}{s-1} + \frac{-s-1}{s^2+4} = \frac{1}{s-1} - \frac{s}{s^2+2^2} - \frac{1}{s^2+2^2}$$

To match the standard Laplace transform table, we rewrite the last term:

$$F(s) = \frac{1}{s-1} - \frac{s}{s^2+2^2} - \frac{1}{2} \cdot \frac{2}{s^2+2^2}$$

Applying the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ to each term:

- $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^t$
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = \cos(2t)$
- $\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{s^2+2^2}\right\} = \frac{1}{2}\sin(2t)$

Thus, the inverse Laplace transform is:

$$\mathcal{L}^{-1}\{F(s)\} = e^t - \cos(2t) - \frac{1}{2}\sin(2t).$$

Problem 81. Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+9)}$$

Solution 81. To find the inverse Laplace transform, we first decompose $F(s)$ into partial fractions:

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

Multiplying both sides by the denominator $s(s^2+9)$ gives:

$$1 = A(s^2+9) + (Bs+C)s$$

Expanding the right-hand side and grouping terms:

$$1 = (A+B)s^2 + Cs + 9A$$

By matching the coefficients on both sides of the equation, we find:

- Constant term: $9A = 1 \Longrightarrow A = \frac{1}{9}$
- Coefficient of s: $C = 0$
- Coefficient of s^2: $A + B = 0 \Longrightarrow \frac{1}{9} + B = 0 \Longrightarrow B = -\frac{1}{9}$

Substituting these values back into the partial fraction decomposition, we have:

$$F(s) = \frac{1}{9} \cdot \frac{1}{s} - \frac{1}{9} \cdot \frac{s}{s^2+9}$$

Now we take the inverse Laplace transform $\mathcal{L}^{-1}$ of each term:

- $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos(3t)$

Thus, the inverse Laplace transform is:

$$f(t) = \frac{1}{9} - \frac{1}{9}\cos(3t)$$

Problem 82. Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+9)}$$

Solution 82. To find the inverse Laplace transform, we first decompose $F(s)$ into partial fractions:

$$\frac{1}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}$$

Multiplying both sides by the denominator $s(s^2+9)$ gives:

$$1 = A(s^2+9) + (Bs+C)s$$

Expanding the right-hand side and grouping terms:

$$1 = (A + B)s^2 + Cs + 9A$$

By matching the coefficients on both sides of the equation, we find:

- Constant term: $9A = 1 \Longrightarrow A = \frac{1}{9}$
- Coefficient of s: $C = 0$
- Coefficient of s^2: $A + B = 0 \Longrightarrow \frac{1}{9} + B = 0 \Longrightarrow B = -\frac{1}{9}$

Substituting these values back into the partial fraction decomposition, we have:

$$F(s) = \frac{1}{9} \cdot \frac{1}{s} - \frac{1}{9} \cdot \frac{s}{s^2 + 9}$$

Now we take the inverse Laplace transform $\mathcal{L}^{-1}$ of each term:

- $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos(3t)$

Thus, the inverse Laplace transform is:

$$f(t) = \frac{1}{9} - \frac{1}{9}\cos(3t)$$