Section 7.8

Convolution

Goals for Today

We need to consider some very specific inverse Laplace transform cases.

We also need to consider how to solve integral equations and integrodifferential equations.

The Convolution Product

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$. The convolution of $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

Note that

$$(f * g)(t) = (g * f)(t) = \int_0^t g(t-\tau)f(\tau)d\tau$$

Example 1

Compute the convolution product of $f(t) = t$ and $g(t) = t^2$.

The Convolution Theorem

Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$ and of exponential order α. If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ both exist for $s > \alpha \geq 0$, then

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$$

and, equivalently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Example 2

Use the Convolution Theorem to compute $\mathcal{L}\{(f * g)(t)\}$ given $f(t) = t$ and $g(t) = t^2$.

Then, use the result of Example 1 to verify your answer.

Example 3

Find $\mathcal{L}^{-1}\{H(s)\}$ where $H(s) = \frac{k^2}{(s^2+k^2)^2}$

Example 4

Use the Convolution Theorem to compute

$$\mathcal{L}\left\{\int_0^t \tau e^{t-\tau} d\tau\right\}$$

Example 5

Solve the integral equation

$$y(t) + \int_0^t (t-\tau) y(\tau) d\tau = 1$$

Example 6

Solve the integrodifferential equation

$$y'(t) + 3\int_0^t y(t-\tau)e^{-4\tau}d\tau = \sin t, \qquad y(0) = 0$$

Example 7

Use the convolution theorem to obtain a formula for the solution to the initial value problem

$$y''' + 4y' = f(t), \qquad y(0) = 1, y'(0) = y''(0) = 0$$

Assume $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order.