Chapter 4

Interest Rates

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Remark 4.1

Types of rates are:

- Treasury rates (government, virtually risk free)
- LIBOR rates (1/3/6/12-month in all major currencies, not totally risk free)
- Repo-rates (very little credit risk)

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Definition 4.2

Let V(t) be the wealth at time t (years).

We talk about discrete or periodic compounding with frequency m times a year and interest rate r per annum provided

$V(t)=V(0)(1+r/m)^{mt}$ for all t≥0.

$(1+r/m)^{mt}$ is called the growth factor,

$(1+r/m)^{-mt}$ is called the discount factor.

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Example 4.3

- Let r=0.1. Find the value of $100 after 1 year with periodic compounding and m=1, 2, 4, 12, 52, 365.
- How long does it take to double a capital attracting interest at 6% daily?
- What is r if a deposit subject to annual compounding is doubled after 10 years?

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Definition 4.4

An annuity is a sequence of finitely many payments of a fixed amount due at equal time intervals.

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Proposition 4.5

For discrete annual compounding with rate r and payments of C every year, the present value of an annuity for n years is

$C(1-(1+r)^{-n})/r$.

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Example 4.6

Consider a loan of \$1000 to be paid back in 5 equal installments due at yearly intervals. The installments include both the interest payable each year calculated at 15% of the current outstanding loan and the repayment of a fraction of the loan (amortized loan).

Example 4.7

Suppose that you took a mortgage of \$100,000 on a house to be paid back in 10 equal annual payments (r=6%). If you decided to clear the mortgage after 8 years, how much would you need to pay on top of the 8^{th} installment?

Definition 4.8

A perpetuity is an infinite sequence of equal payments due at equal time intervals.

Proposition 4.9

For discrete annual compounding with rate r and payments of C every year, the present value of a perpetuity is

C/r.

Definition 4.10

We talk about continuous compounding at rate r provided

$V(t)=V(0)e^{rt}$ for all $t\geq 0$.

e^{rt} is called the growth factor,

e^{-rt} is called the discount factor.

Remark 4.11

Under continuous compounding, the rate of growth of the wealth is proportional to the wealth:

$V'(t)=rV(t)$.

Example 4.12

How long will it take to earn $1 if r=0.1 (c.c.) and V(0)=$1 million?

Definition 4.13

- Two compounding methods are called equivalent if the corresponding growth factors over a period of one year are the same.
- If one of the growth factors is bigger, then that method is called preferable.

Example 4.14

- What is the equivalent continuous rate for 10% semiannual compounding?
- What is the equivalent quarterly rate for 8% continuous compounding?

Definition 4.15

For a given compounding method, the effective rate r_e is the rate for annual compounding equivalent to that method.

Example 4.16

What is the effective rate for semiannual compounding with r=10%?

Definition 4.17

A zero-coupon bond involves a single payment, and the issuing institution promises to exchange the bond for its face value (principal value) at a given maturity date.

Example 4.18

- Suppose a bond has face value F=100 and matures in 1 year. If r=12% (a.c.), find the present value of the bond.
- Find the interest rates for annual, semiannual, and continuous compounding implied by a unit bond with maturity 1 and value 0.9455 after half a year.

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Definition 4.19

A coupon bond promises a sequence of payments, consisting of the face value paid at maturity and coupons paid regularly, the last coupon being due at maturity.

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Example 4.20

Consider a bond with F=100, T=5, C=10 paid annually, r=0.12 continuously compounded. Find the value of this bond at times 0, 1, and 4.

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Proposition 4.21

For coupons paid annually and continuous compounding with constant rate r, the price of a bond with coupon value C, face value F, and maturity T years is

$$C(1-e^{-rT})/(e^{r}-1)+Fe^{-rT}.$$

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Definition 4.22

- Assuming that coupons are paid annually, i=C/F is called the coupon rate.
- If the price of a bond is equal to its face value, we say the bond sells at par.
- The coupon rate that causes the bond to sell at par is called the par yield.

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Proposition 4.23

Assume that coupons are paid annually and interest rates are constant. Then the par yield is equal to the effective rate.

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Definition 4.24

The bond yield is the discount rate that, when applied to all cash flows, gives a bond price equal to its market price.

Example 4.25

Suppose a 2-year Treasury bond with F=100 provides coupons at rate of 6% p.a. semiannually.

Maturity (years)	Treasury zero rate (%) c.c.
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Example 4.26

In this example we discuss the most popular approach to calculate Treasury zero rates from the prices of Treasury bonds, the bootstrap method.

Example 4.26 (continued)

Bond principal ($)	Time to maturity (years)	Annual coupon ($)	Bond price ($)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

Definition 4.27

The forward rate is the future zero rate implied by today's term structure of zero interest rates.

Example 4.28

Find the forward rates for the nth year (% p.a.).

Year (n)	Zero rate for an n-year investment (% p.a.)
1	3.0
2	4.0
3	4.6
4	5.0
5	5.3

Proposition 4.29

Assume R_1 and R_2 are the zero rates for maturities T_1 and T_2. Then the forward rate R_F between T_1 and T_2 is given by

$$R_F=(R_2T_2-R_1T_1)/(T_2-T_1).$$

Definition 4.30

A forward rate agreement (FRA) is an agreement that a certain rate will apply to a certain principal during a certain future period.

Example 4.31

Suppose company X enters into an FRA with Y that specifies that it will receive a fixed rate of R_K=4% on a principal of L=1 million for a 3-month period starting in 3 years. The actual 3-month LIBOR proves to be R_M=4.5%. Find the cash flow to Y.

Example 4.32

Suppose rates are as Example 4.28. Consider an FRA where we will receive R_K=6% (annual compounding) on L=1 million between times 1 and 2. Note R_F=5% is the forward rate calculated today. Find the present value of the FRA.

Definition 4.33

The duration of a bond with price B and yield y that provides cash flow c_i at time t_i, $1\le i\le n$, is defined by

$$D=\sum_{i=1}^{n} t_i c_i e^{-yt_i}/B$$

Remark 4.34

- A zero-coupon bond has duration t_1=T
- Duration is a measure of how long on average the holder has to wait before receiving cash payments
- D is a convex combination of payment times
- Express ΔB in terms of D

Example 4.35

Consider a 3-year 10% coupon bond (paid semiannually) with F=100, y=0.12 cc. Find the new bond price if the yield increases by ten basis points.