Chapter 4
American Derivative Securities in Discrete Time

4.1 Stopping Times

Definition 4.1. Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space. A random variable $\tau : \Omega \to \overline{\mathbb{N}}_0$ is called a *stopping time* provided

$$\{\omega \in \Omega : \ \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for all} \quad k \in \overline{\mathbb{N}}_0.$$

Example 4.2. Consider an American put with strike price 5 in the BAPM with $N = 2$, $S_0 = 4$, $u = 2$, $d = 1/2$, $r = 1/4$. Let

(i) $\tau(\omega) = 1$ if $\omega \in A_D$ and $\tau(\omega) = 2$ if $\omega \in A_U$;

(ii) $\rho(\omega) = \min\{k \in \mathbb{N}_0 : S_k(\omega) = m_2(\omega)\}$, where $m_2 = \min_{0 \le j \le 2} S_j$.

Lemma 4.3. *τ is a stopping time iff $\{\tau \le k\} \in \mathcal{F}_k$ for all $k \in \overline{\mathbb{N}}_0$.*

Definition 4.4. Let τ be a stopping time. We say that a set $A \subset \Omega$ is *determined by time τ* provided

$$A \cap \{\omega \in \Omega : \ \tau(\omega) = k\} \in \mathcal{F}_k \quad \text{for all} \quad k \in \overline{\mathbb{N}}_0.$$

The collection of all sets determined by τ is denoted by $\mathcal{F}_\tau$.

Example 4.5. Find $\mathcal{F}_\tau$ for τ given in Example 4.2.

Theorem 4.6. *If τ is a stopping time, then $\mathcal{F}_\tau$ is a σ-algebra.*

Lemma 4.7. $\mathcal{F}_\tau = \{A \subset \Omega : \ A \cap \{\tau \le k\} \in \mathcal{F}_k$ *for all* $k \in \overline{\mathbb{N}}_0\}$.

Theorem 4.8. *Let σ, τ be stopping times with $\sigma \le \tau$. Then $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$.*

Definition 4.9. If X is adapted to the filtration $\mathbb{F}$ and τ is a stopping time, we define

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega) \quad \text{for all} \quad \omega \in \Omega.$$

Example 4.10. Find S_τ for τ given in Example 4.2.

Theorem 4.11. *If τ is a stopping time, then X_τ is $\mathcal{F}_\tau$-measurable.*

Lemma 4.12. *We have* $X_\tau = \sum_{n=0}^{\infty} X_n \chi_{\{\tau=n\}}$.

Theorem 4.13 (Doob's Stopping-time Principle). *Let τ be a bounded stopping time and X a martingale. Then X_τ is integrable and*

$$\mathbb{E}(X_\tau) = \mathbb{E}(X_0).$$

Example 4.14. Show $\tilde{\mathbb{E}}((\beta S)_\tau) = (\beta S)_0$ in Example 4.2.

Example 4.15. If τ is not bounded, then STP does not hold in general.

Theorem 4.16 (Doob's Optional Sampling Theorem). *Let X be a martingale and let σ, τ be bounded stopping times with $\sigma \leq \tau$. Then*

$$\mathbb{E}(X_\tau | \mathcal{F}_\sigma) = X_\sigma.$$

Example 4.17. Continuation of Example 4.2 with $\rho(\omega) \equiv 2$.

Theorem 4.18. *Let $\{X_n\}$ be adapted integrable random variables with $\mathbb{E}(X_\tau) = 0$ for all bounded stopping times τ. Then X is a martingale.*

Definition 4.19. For an adapted process X and a stopping time τ we define the *stopped process* X^τ by

$$X_n^\tau(\omega) := X_{\tau(\omega)\wedge n}(\omega).$$

Example 4.20. Find S_1^τ and S_2^τ, where τ is given in Example 4.2.

Lemma 4.21. *We have* $X_n^\tau = X_0 + \sum_{k=0}^{n-1} \chi_{\{\tau \leq k\}^c} \Delta X_k$.

Theorem 4.22. *Let X be a discrete stochastic process and let τ be a stopping time.*

(i) If X is adapted, then so is X^τ.

(ii) If X is a martingale, then so is X^τ.

(iii) If X is a supermartingale, then so is X^τ.

(iv) If X is a submartingale, then so is X^τ.

4.2 The Snell Envelope and Optimal Stopping

Definition 4.23. Let X be an adapted sequence of integrable random variables. The sequence Z defined by

$$Z_N = X_N, \quad Z_n = \max\{X_n, \mathbb{E}(Z_{n+1}|\mathcal{F}_n)\},\ 0 \leq n \leq N-1$$

is called the *Snell envelope* of X.

Theorem 4.24. *The Snell envelope of X is the smallest supermartingale dominating X.*

Theorem 4.25. $\tau^* = \inf\{n \in \mathbb{N}_0 :\ Z_n = X_n\}$ *is a stopping time, and the stopped process Z^{τ^*} is a martingale.*

Definition 4.26. We define the set $\mathcal{S}_n$ by

$$\mathcal{S}_n = \{\tau :\ n \leq \tau \leq N \text{ a.s. and } \tau \text{ is a stopping time}\}.$$

A stopping time $\sigma \in \mathcal{S}_n$ is called *optimal* for X if

$$\mathbb{E}(X_\sigma|\mathcal{F}_n) = \sup_{\tau\in\mathcal{S}_n} \mathbb{E}(X_\tau|\mathcal{F}_n).$$

Theorem 4.27. τ^* *solves the optimal stopping problem for X:*

$$Z_0 = \mathbb{E}(X_{\tau^*}) = \sup_{\tau\in\mathcal{S}_0} \mathbb{E}(X_\tau).$$

Theorem 4.28. *If $\tau_n^* = \inf\{k \geq n :\ Z_k = X_k\}$, then*

$$Z_n = \mathbb{E}(X_{\tau_n^*}|\mathcal{F}_n) = \sup_{\tau\in\mathcal{S}_n} \mathbb{E}(X_\tau|\mathcal{F}_n).$$

Theorem 4.29. *The stopping time $\sigma \in \mathcal{S}_0$ is optimal for X iff*

$$Z_\sigma = X_\sigma \quad \textit{and} \quad Z^\sigma \textit{ is a martingale.}$$

Theorem 4.30 (Doob Decomposition). *Let X be an adapted process such that each X_n is integrable. Then X has a unique Doob decomposition*

$$X_n = X_0 + M_n + A_{n-1} \quad \textit{for all} \quad n \in \mathbb{N}_0,$$

where M is a martingale with $M_0 = 0$ and A is adapted with $A_{-1} = 0$. If X is a supermartingale, then A is decreasing.

Theorem 4.31. *If Z has a Doob decomposition with M and A, then*

$$\nu(\omega) = \begin{cases} N & \textit{if} \quad A_{N-1}(\omega) = 0 \\ \min\{n \in \mathbb{N}_0 :\ A_n(\omega) < 0\} & \textit{if} \quad A_{N-1}(\omega) < 0 \end{cases}$$

is an optimal stopping time for X, and it is the largest optimal stopping time for X.

4.3 Properties of American Derivatives

Definition 4.32. An *American derivative security* (Ads) is a discrete stochastic process $G \geq 0$ that is adapted to a filtration $\mathbb{F}$. We also define the *value* of an Ads by

$$V_k = \beta_k^{-1} \max_{\tau \in \mathcal{S}_k} \tilde{\mathbb{E}}(\beta_\tau G_\tau | \mathcal{F}_k), \quad 0 \leq k \leq N-1,$$

in particular

$$V_0 = \max_{\tau \in \mathcal{S}_0} \tilde{\mathbb{E}}(\beta_\tau G_\tau).$$

Theorem 4.33. *Let G be an Ads and V its value. Then we have*

(i) $V \geq G$;

(ii) βV *is a supermartingale.*

Theorem 4.34. *Let G be an Ads and V its value. Then V is the smallest process satisfying the properties given in Theorem 4.33.*

Theorem 4.35. *βV is the Snell envelope of βG, i.e.,*

$$V_N = G_N, \quad V_n = \max\{G_n, \beta_1 \tilde{\mathbb{E}}(V_{n+1}|\mathcal{F}_n)\}, \; 0 \leq n \leq N-1.$$

Theorem 4.36. *Let G be an Ads and V its value. Any stopping time τ which satisfies*

$$V_0 = \tilde{\mathbb{E}}(\beta_\tau G_\tau)$$

is an optimal stopping time. In particular,

$$\tau^* = \min\{k \in \mathbb{N}_0 : \; V_k = G_k\}$$

is an optimal stopping time.

Definition 4.37. An Ads G is said to be *hedgeable* (or *attainable*) if there exists a constant X_0 and a portfolio process φ such that the wealth process X given by

$$X_{k+1} = \varphi_k S_{k+1} + (1+r)(X_k - C_k - \varphi_k S_k),$$

where

$$C_k = V_k - \beta_1 \tilde{\mathbb{E}}(V_{k+1}|\mathcal{F}_k),$$

satisfies

$$X_k = V_k \quad \text{for all} \quad 0 \leq k \leq N.$$

Theorem 4.38. *Under $\tilde{\mathbb{P}}$, the discounted wealth process βX is a supermartingale.*

Theorem 4.39. *In the BAPM, any Ads is hedgeable.*

4.4 Comparison of American and European Derivative Securities

Remark 4.40. Let G be an Ads. Then it is also an sEds, and we denote the value process of the Ads G by V^A and the value process of the sEds G by V^E.

Theorem 4.41. *(i)* $V^A \geq V^E$*;*

(ii) $V^E \geq G$ *implies* $V^A = V^E$.

Theorem 4.42. *Suppose* g *is convex and satisfies* $g(0) = 0$*. If* $G = g(S)$*, then* $V^A = V^E$.

Corollary 4.43. *The American call option is equivalent to its European counterpart.*