Chapter 5
Brownian Motion

5.1 Stochastic Processes in Continuous Time

Definition 5.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

(i) A *filtration* is a nondecreasing family $\mathbb{F} = \{F(t)\}_{t \geq 0}$ of sub-σ-algebras of $\mathcal{F}$:

$$\mathcal{F}(s) \subset \mathcal{F}(t) \subset \mathcal{F} \quad \text{for all} \quad 0 \leq s < t < \infty.$$

(ii) A *stochastic process* is a family of random variables $X = \{X(t)\}_{t \geq 0}$ defined on the probability space.

(iii) The stochastic process X is *adapted* provided

$$X(t) \quad \text{is } \mathcal{F}(t)\text{-measurable} \qquad \text{for all} \quad t \geq 0.$$

Definition 5.2. Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space. A random variable $\tau : \Omega \to [0, \infty]$ is called a *stopping time* provided

$$\{\omega \in \Omega : \ \tau(\omega) \leq t\} \in \mathcal{F}(t) \quad \text{for all} \quad t \geq 0.$$

The stopping time σ-algebra $\mathcal{F}(\tau)$ is then defined by

$$\mathcal{F}(\tau) = \{A \in \mathcal{F} : \ A \cap \{\tau \leq t\} \in \mathcal{F}(t) \text{ for all } t \geq 0\} .$$

Definition 5.3. Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space. Let X be a stochastic process that is adapted to $\mathbb{F}$ such that $\mathbb{E}(|X(t)|) < \infty$ for all $t \geq 0$. Then X is called a

(i) *martingale* if

$$\mathbb{E}(X(t)|\mathcal{F}(s)) = X(s) \text{ a.s.} \quad \text{for all} \quad 0 \leq s \leq t < \infty;$$

(ii) *supermartingale* if

$$\mathbb{E}(X(t)|\mathcal{F}(s)) \leq X(s) \text{ a.s.} \quad \text{for all} \quad 0 \leq s \leq t < \infty;$$

(iii) *submartingale* if

$$\mathbb{E}(X(t)|\mathcal{F}(s)) \geq X(s) \text{ a.s.} \quad \text{for all} \quad 0 \leq s \leq t < \infty.$$

5.2 Definition and Properties of Brownian Motion

Definition 5.4. A stochastic process W is called a (standard, one-dimensional) *Brownian motion* on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ provided

(i) $W(0) = 0$ a.s.;

(ii) W has independent increments, i.e.,

$$W(t+u) - W(t) \quad \text{is independent of} \quad \sigma(\{W(s) : s \leq t\}) \quad \text{for} \quad u \geq 0;$$

(iii) W has stationary increments, i.e., $W(t+u) - W(t)$ depends only on u;

(iv) W has Gaussian increments, i.e.,

$$W(t+u) - W(t) \quad \text{is normally distributed with mean 0 and variance } u;$$

(v) W has continuous paths, i.e., $W(\cdot, \omega)$ is continuous for all $\omega \in \Omega$.

Theorem 5.5. *Brownian motion satisfies*

$$\mathbb{E}(W(t)) = 0 \quad \textit{and} \quad \mathbb{V}(W(t)) = t.$$

Theorem 5.6. *The covariance function for Brownian motion is given by*

$$\mathbb{C}\text{ov}(W(s), W(t)) = s \wedge t.$$

Theorem 5.7. *Brownian motion is a martingale.*

Theorem 5.8. *If W is Brownian motion, then the Doob decomposition of W^2 is*

$$W^2(t) = W^2(0) + (W^2(t) - t) + t.$$

5.3 Linear and Quadratic Variation

Definition 5.9. Let $f, g : [0, t] \to \mathbb{R}$. Consider partitions $\mathcal{P}$ of the form

$$0 = t_0 < t_1 < \ldots < t_n = t.$$

We define the

(i) *variation* of f by

$$\bigvee_t f = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=0}^{n-1} |\Delta f(t_j)|;$$

(ii) *quadratic variation* of f by

$$\langle f\rangle_t = \lim_{\|\mathcal{P}\|\to 0} \sum_{j=0}^{n-1} |\Delta f(t_j)|^2;$$

(iii) *covariation* of f and g by

$$\langle f, g\rangle_t = \lim_{\|\mathcal{P}\|\to 0} \sum_{j=0}^{n-1} \Delta f(t_j)\Delta g(t_j).$$

Theorem 5.10. *If $f : [0,t] \to \mathbb{R}$ has a continuous derivative, then*

$$\bigvee_t f = \int_0^t |f'(u)|\mathrm{d}u \quad \textit{and} \quad \langle f\rangle_t = 0.$$

Theorem 5.11. *If W is Brownian motion and $\mathrm{id}(t) = t$, then*

(i) $\langle W, \mathrm{id}\rangle_t = 0$*;*

(ii) $\langle \mathrm{id}\rangle_t = 0$.

Theorem 5.12 (Lévy). *If W is Brownian motion, then*

$$\langle W\rangle_t = t.$$

Remark 5.13. We capture the above results by writing

$$\mathrm{d}W(t)\mathrm{d}t = 0, \quad (\mathrm{d}t)^2 = 0, \quad (\mathrm{d}W(t))^2 = \mathrm{d}t.$$

Theorem 5.14. *The paths of Brownian motion are of unbounded variation.*

5.4 Geometric Brownian Motion

Definition 5.15. We define *geometric Brownian motion* by

$$S(t) = S(0)\exp\left\{\sigma W(t) + \left(\alpha - \frac{\sigma^2}{2}\right)t\right\},$$

where $\alpha \in \mathbb{R}$, $\sigma > 0$, and W is Brownian motion.

Theorem 5.16. *Geometric Brownian motion (with $\alpha = 0$) is a martingale.*

Theorem 5.17. *If S is geometric Brownian motion, then*

$$\langle \log \circ S\rangle_t = \sigma^2 t.$$

5.5 First Passage Time

Definition 5.18. The *first passage time* to level x is defined by

$$\tau_x = \min \{t \geq 0 : \; W(t) = x\} .$$

Theorem 5.19. *$\tau_x < \infty$ a.s. for all $x \in \mathbb{R}$.*

Theorem 5.20. *$\mathbb{E}(\exp\{-\alpha\tau_x\}) = \exp\{-|x|\sqrt{2\alpha}\}$ for all $\alpha > 0$.*

Theorem 5.21. *$\mathbb{E}(\tau_x) = \infty$ for all $x \in \mathbb{R} \setminus \{0\}$.*

5.6 Existence of Brownian Motion

Theorem 5.22 (Wiener). *Brownian motion exists.*

Definition 5.23. We consider the Hilbert space $L^2([0, 1])$, equipped with

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x, \quad \|f\| = \sqrt{\langle f, f\rangle}.$$

A complete orthonormal system $\{\phi_n\}$ in $L^2([0, 1])$ is abbreviated as a *cons*.

Theorem 5.24. *If $\{\phi_n\}$ is a cons in $L^2([0, 1])$, then*

$$\sum_{n=0}^{\infty} \int_0^s \phi_n(x)\mathrm{d}x \int_0^t \phi_n(x)\mathrm{d}x = s \wedge t.$$

Definition 5.25. Define $H(t) = 1$ for $t \in [0, 1/2)$, $H(t) = -1$ for $t \in [1/2, 1)$, and $H(t) = 0$ otherwise. Put $H_0(t) \equiv 1$ and for $n \in \mathbb{N}$, write $n = 2^j + k$ with unique $j \in \mathbb{N}_0$ and $0 \leq k \leq 2^j - 1$ and define $H_n(t) = 2^{j/2}H(2^j t - k)$ for $t \in \mathbb{R}$. Then $\{H_n\}$ is called the *Haar system*.

Theorem 5.26. *The Haar system is a cons in $L^2([0, 1])$.*

Definition 5.27. Define $s(t) = 2t$ for $t \in [0, 1/2)$, $s(t) = 2(1 - t)$ for $t \in [1/2, 1]$, and $s(t) = 0$ otherwise. Put $s_0(t) = t$ and for $n \in \mathbb{N}$, write $n = 2^j + k$ with unique $j \in \mathbb{N}_0$ and $0 \leq k \leq 2^j - 1$ and define $s_n(t) = s(2^j t - k)$ for $t \in \mathbb{R}$. Then $\{s_n\}$ is called the *Schauder system*.

Theorem 5.28. *We have*

$$\int_0^t H_n(u)\mathrm{d}u = \ell_n s_n(t), \quad \text{where} \quad \ell_n = \frac{1}{2} \cdot 2^{-j/2}.$$

Lemma 5.29. *Let Z_n be independent standard normally distributed. Then there exists a random variable C such that $C < \infty$ a.s. and*

$$|Z_n| \leq C\sqrt{\log(n)} \quad \textit{for all} \quad n \geq 2.$$

Theorem 5.30 (Lévy–Cieselski). *Let Z_n be independent standard normally distributed. Define*

$$W(t) = \sum_{n=0}^{\infty} \ell_n Z_n s_n(t).$$

Then the series converges uniformly and W is Brownian motion.