Chapter 6
Stochastic Calculus

6.1 Itô Integral for Simple Processes

Definition 6.1. For a *simple* process, i.e., an adapted process of the form

$$X = \sum_{i=0}^{n-1} C_i \chi_{[t_i, t_{i+1})},$$

where

$$0 = t_0 < t_1 < \ldots < t_n = T \quad \text{is a partition of} \quad [0, T],$$

we define the *Itô integral* of X with respect to Brownian motion by

$$I(t) := \int_0^t X(u)\mathrm{d}W(u) = \sum_{j=0}^{k-1} C_j \Delta W(t_j) + C_k[W(t) - W(t_k)],$$

where $t_k \leq t < t_{k+1}$.

Theorem 6.2. *The Itô integral for simple processes is linear.*

Theorem 6.3. *The Itô integral for simple processes is adapted.*

Theorem 6.4. *The Itô integral for simple processes is a martingale.*

Theorem 6.5 (Zero Mean Property). *The Itô integral for simple processes satisfies*

$$\mathbb{E}(I(t)) = 0 \quad \textit{for all} \quad t \geq 0.$$

Theorem 6.6 (Itô Isometry). *The Itô integral for simple processes satisfies*

$$\mathbb{V}(I(t)) = \mathbb{E}\left(\int_0^t X^2(u)\mathrm{d}u\right) \quad \textit{for all} \quad t \geq 0.$$

Theorem 6.7. *The quadratic variation of the Itô integral for simple processes is*

$$\langle I \rangle_t = \int_0^t X^2(u)\mathrm{d}u.$$

6.2 Properties of the General Itô Integral

Definition 6.8. Let X be any adapted process with

$$\mathbb{E}\left(\int_0^T X^2(u)\mathrm{d}u\right) < \infty.$$

Then there exist simple processes X_n with

$$\mathbb{E}\left(\int_0^T |X_n(u) - X(u)|^2\mathrm{d}u\right) \to 0, \quad n \to \infty,$$

and we define

$$I(t) := \int_0^t X(u)\mathrm{d}W(u) = \lim_{n\to\infty}\int_0^t X_n(u)\mathrm{d}W(u).$$

Theorem 6.9. *The Itô integral I is continuous, adapted, linear, a martingale, the Itô isometry*

$$\mathbb{E}(I^2(t)) = \mathbb{E}\left(\int_0^t X^2(u)\mathrm{d}u\right)$$

holds, and its quadratic variation is

$$\langle I\rangle_t = \int_0^t X^2(u)\mathrm{d}u.$$

Example 6.10. If W is Brownian motion, then

$$\int_0^T W(u)\mathrm{d}W(u) = \frac{W^2(T)}{2} - \frac{T}{2}.$$

Remark 6.11. If g is differentiable with $g(0) = 0$, then

$$\int_0^T g(u)\mathrm{d}g(u) = \frac{g^2(T)}{2}$$

without the extra term $-T/2$.

Theorem 6.12 (Itô–Doeblin Formula for Brownian Motion). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that f_t, f_x, and f_{xx} are defined and continuous, and let W be Brownian motion. Then, for $T \geq 0$,*

$$\begin{aligned} f(T, W(T)) \;=\; & f(0, W(0)) + \int_0^T f_t(t, W(t))\mathrm{d}t \\ & + \int_0^T f_x(t, W(t))\mathrm{d}W(t) + \frac{1}{2}\int_0^T f_{xx}(t, W(t))\mathrm{d}t. \end{aligned}$$

Remark 6.13. We capture this result by writing

$$\mathrm{d}f(t,W(t)) = f_t(t,W(t))\mathrm{d}t + f_x(t,W(t))\mathrm{d}W(t) + \frac{1}{2}f_{xx}(t,W(t))\mathrm{d}t.$$

Example 6.14. Use Itô's formula to find $\int_0^T W(t)\mathrm{d}W(t)$ and $\int_0^T W^2(t)\mathrm{d}W(t)$.

6.3 Itô Processes

Definition 6.15. Let W be Brownian motion. An *Itô process* Y is defined by

$$Y(t) = Y(0) + \int_0^t X(u)\mathrm{d}W(u) + \int_0^t \Theta(u)\mathrm{d}u,$$

where $Y(0)$ is nonrandom, X and Θ are adapted, and both

$$\mathbb{E}\left(\int_0^t X^2(u)\mathrm{d}u\right) \quad \text{and} \quad \int_0^t |\Theta(u)|\mathrm{d}u$$

are finite for all $t \geq 0$.

Theorem 6.16. *The quadratic variation of an Itô process is given by*

$$\langle Y\rangle_t = \int_0^t X^2(u)\mathrm{d}u.$$

Definition 6.17. Let Y be an Itô process as in Definition 6.15 and let Γ be adapted. We define the integral of Γ with respect to Y by

$$\int_0^t \Gamma(u)\mathrm{d}Y(u) = \int_0^t \Gamma(u)X(u)\mathrm{d}W(u) + \int_0^t \Gamma(u)\Theta(u)\mathrm{d}u.$$

Theorem 6.18 (Itô–Doeblin Formula for Itô Processes). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that f_t, f_x, and f_{xx} are defined and continuous, and let Y be an Itô process as in Definition 6.15 and W Brownian motion. Then, for $T \geq 0$,*

$$\begin{aligned} f(T,Y(T)) &= f(0,Y(0)) + \int_0^T f_t(t,Y(t))\mathrm{d}t + \int_0^T f_x(t,Y(t))X(t)\mathrm{d}W(t) \\ &\quad + \int_0^T f_x(t,Y(t))\Theta(t)\mathrm{d}t + \frac{1}{2}\int_0^T f_{xx}(t,Y(t))X^2(t)\mathrm{d}t. \end{aligned}$$

Remark 6.19. We capture this result by writing

$$\mathrm{d}Y(t) = X(t)\mathrm{d}W(t) + \Theta(t)\mathrm{d}t$$

implies

$$\mathrm{d}f(t,Y(t)) = f_t(t,Y(t))\mathrm{d}t + f_x(t,Y(t))\mathrm{d}Y(t) + \frac{1}{2}f_{xx}(t,Y(t))X^2(t)\mathrm{d}t.$$

Corollary 6.20. *Under the assumptions of Theorem 6.18, we have*

$$\begin{aligned}\mathbb{E}(f(T,Y(T))) &= f(0,Y(0))\\ &\quad + \int_0^T \mathbb{E}\left(f_t(t,Y(t)) + f_x(t,Y(t))\Theta(t) + \frac{1}{2}f_{xx}(t,Y(t))X^2(t)\right)\mathrm{d}t.\end{aligned}$$

6.4 Multivariate Stochastic Calculus

Definition 6.21. A d-dimensional Brownian motion is a process

$$W(t) = (W_1(t),\ldots,W_d(t))$$

such that

(i) each W_j is a (one-dimensional) Brownian motion;

(ii) if $i \neq j$, then W_i and W_j are independent.

Associated with a d-dimensional Brownian motion we have a filtration $\mathcal{F}(t)$ with

(iii) $\mathcal{F}(s) \subset \mathcal{F}(t)$ for all $0 \leq s \leq t$;

(iv) $W(t)$ is $\mathcal{F}(t)$-measurable for all $t \geq 0$;

(v) $W(t+h) - W(t)$ is independent of $\mathcal{F}(t)$ for all $h > 0$ and all $t \geq 0$.

Theorem 6.22. *If $W = (W_1,\ldots,W_d)$ is d-dimensional Brownian motion, then*

$$\mathrm{d}W_i(t)\mathrm{d}W_j(t) = 0 \quad \textit{for all} \quad 1 \leq i < j \leq d.$$

Theorem 6.23. *For Itô processes*

$$\begin{aligned}X(t) &= X(0) + \int_0^t \Theta_1(u)\mathrm{d}u + \int_0^t \sigma_{11}(u)\mathrm{d}W_1(u) + \int_0^t \sigma_{12}(u)\mathrm{d}W_2(u),\\ Y(t) &= Y(0) + \int_0^t \Theta_2(u)\mathrm{d}u + \int_0^t \sigma_{21}(u)\mathrm{d}W_1(u) + \int_0^t \sigma_{22}(u)\mathrm{d}W_2(u),\end{aligned}$$

we have

$$\begin{aligned}\mathrm{d}X(t)\mathrm{d}X(t) &= \left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right)\mathrm{d}t,\\ \mathrm{d}Y(t)\mathrm{d}Y(t) &= \left(\sigma_{21}^2(t) + \sigma_{22}^2(t)\right)\mathrm{d}t,\\ \mathrm{d}X(t)\mathrm{d}Y(t) &= (\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t))\,\mathrm{d}t.\end{aligned}$$

Theorem 6.24 (Two-Dimensional Itô–Doeblin Formula). *Let $f(t,x,y)$ be such that f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy} are defined and continuous. Let X and Y be Itô processes as in Theorem 6.23. Then*

$$\begin{aligned} \mathrm{d}f(t,X(t),Y(t)) \;=\; & f_t(t,X(t),Y(t))\mathrm{d}t \\ & + f_x(t,X(t),Y(t))\mathrm{d}X(t) + f_y(t,X(t),Y(t))\mathrm{d}Y(t) \\ & + \frac{1}{2} f_{xx}(t,X(t),Y(t))\mathrm{d}X(t)\mathrm{d}X(t) \\ & + f_{xy}(t,X(t),Y(t))\mathrm{d}X(t)\mathrm{d}Y(t) \\ & + \frac{1}{2} f_{yy}(t,X(t),Y(t))\mathrm{d}Y(t)\mathrm{d}Y(t). \end{aligned}$$

Theorem 6.25 (Itô Product Rule). *For Itô processes X and Y we have*

$$\mathrm{d}(X(t)Y(t)) = X(t)\mathrm{d}Y(t) + Y(t)\mathrm{d}X(t) + \mathrm{d}X(t)\mathrm{d}Y(t).$$

Theorem 6.26 (Lévy Characterization of Brownian Motion). *Any martingale M that starts at zero, has continuous paths, and satisfies $\langle M\rangle_t = t$ for all $t \geq 0$ is a Brownian motion.*

Theorem 6.27 (Lévy, Two Dimensions). *Let M_1 and M_2 be martingales with continuous paths such that $M_1(0) = M_2(0) = 0$ and $\langle M_1\rangle_t = \langle M_2\rangle_t = t$ for all $t \geq 0$. If, in addition, $\langle M_1, M_2\rangle_t = 0$ for all $t \geq 0$, then M_1 and M_2 are independent Brownian motions.*

Example 6.28 (Correlated Brownian Motions). Let W_1 and W_2 be independent Brownian motions. Let $|\rho| < 1$ and define

$$W_3(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_2(t).$$

Then W_3 is a Brownian motion and we have $\rho(W_1(t), W_3(t)) = \rho$.