Chapter 7
Stochastic Differential Equations

7.1 Some Equations and their Solutions

Definition 7.1. Let W be Brownian motion. A *stochastic differential equation* (SDE) is an equation of the form

$$\mathrm{d}X(t) = \mu(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t),$$

also written as

$$\mathrm{d}X = \mu(t, X)\mathrm{d}t + \sigma(t, X)\mathrm{d}W,$$

where the *drift* μ and the *diffusion* σ are given. A process X is called a (strong) *solution* of the SDE if

$$X(t) = X(0) + \int_0^t \mu(u, X(u))\mathrm{d}u + \int_0^t \sigma(u, X(u))\mathrm{d}W(u)$$

for all $t \geq 0$, where both occurring integrals are assumed to exist.

Theorem 7.2. *Let g be nonrandom. The solution of the problem*

$$\mathrm{d}X = g(t)X\mathrm{d}W, \quad X(0) = 1$$

is given by

$$X(t) = \exp\left\{\int_0^t g(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^t g^2(s)\mathrm{d}s\right\}.$$

Theorem 7.3. *Let f and g be nonrandom. The solution of the problem*

$$\mathrm{d}X = f(t)X\mathrm{d}t + g(t)X\mathrm{d}W, \quad X(0) = 1$$

is given by

$$X(t) = \exp\left\{\int_0^t g(s)\mathrm{d}W(s) + \int_0^t \left(f(s) - \frac{1}{2}g^2(s)\right)\mathrm{d}s\right\}.$$

Theorem 7.4 (Stock Price). *Let $\mu > 0$ be the* drift*, $\sigma > 0$ the volatility. A model for the* relative change of price $\mathrm{d}S/S$ *is*

$$\frac{\mathrm{d}S}{S} = \mu\mathrm{d}t + \sigma\mathrm{d}W.$$

Its solution is geometric Brownian motion

$$S(t) = S(0)\exp\left\{\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right)t\right\}.$$

We also have

$$\mathbb{E}(S(t)) = S(0)e^{\mu t} \quad \textit{and} \quad \mathbb{V}(S(t)) = (S(0))^2 e^{2\mu t}(e^{\sigma^2 t} - 1).$$

Example 7.5 (Generalized Geometric Brownian Motion). Let W be Brownian motion with associated filtration $\mathbb{F}$ and α, σ adapted to $\mathbb{F}$. Another model for the relative change of the stock price is

$$\frac{\mathrm{d}S}{S} = \alpha(t)\mathrm{d}t + \sigma(t)\mathrm{d}W.$$

Its solution is given by

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(s)\mathrm{d}W(s) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)\mathrm{d}s\right\},$$

and this example includes all possible models of an asset price which is always positive, has no jumps, and is driven by a single Brownian motion.

7.2 Interest Rate Models

Theorem 7.6 (Itô Integral of Deterministic Integrand). *Let W be Brownian motion, $g : \mathbb{R} \to \mathbb{R}$ be nonrandom, and*

$$I(t) = \int_0^t g(s)\mathrm{d}W(s).$$

Then

$$I(t) \sim N\left(0, \int_0^t g^2(s)\mathrm{d}s\right).$$

Example 7.7 (Langevin Equation). Let $\alpha, \sigma > 0$ and consider the *Langevin equation*

$$\mathrm{d}X(t) = -\alpha X(t)\mathrm{d}t + \sigma \mathrm{d}W(t).$$

Its solution is called an *Ornstein–Uhlenbeck process* and is given by

$$X(t) = e^{-\alpha t}X(0) + \sigma\int_0^t e^{-\alpha(t-u)}\mathrm{d}W(u)$$

and satisfies

$$X(t) \sim N\left(e^{-\alpha t}X(0), \frac{\sigma^2(1 - e^{-2\alpha t})}{2\alpha}\right),$$

in particular

$$\mathbb{E}(X(t)) \to 0 \quad \text{and} \quad \mathbb{V}(X(t)) \to \frac{\sigma^2}{2\alpha} \quad \text{as} \quad t \to \infty.$$

Example 7.8 (Vasicek Interest Rate Model). Let $\alpha, \beta, \sigma > 0$. A model for the interest rate process R is

$$\mathrm{d}R = (\alpha - \beta R)\mathrm{d}t + \sigma \mathrm{d}W.$$

Its solution is

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma e^{-\beta t} \int_0^t e^{\beta\tau} \mathrm{d}W(\tau)$$

and it satisfies

$$R(t) \sim N\left(e^{-\beta t}R(0) + \frac{\alpha(1 - e^{-\beta t})}{\beta}, \frac{\sigma^2(1 - e^{-2\beta t})}{2\beta}\right).$$

Example 7.9 (Cox–Ingersoll–Ross Interest Rate Model). Let $\alpha, \beta, \sigma > 0$. A model for the interest rate process R is

$$\mathrm{d}R = (\alpha - \beta R)\mathrm{d}t + \sigma\sqrt{R}\mathrm{d}W.$$

We have

$$\mathbb{E}(R(t)) = e^{-\beta t}R(0) + \frac{\alpha(1 - e^{-\beta t})}{\beta}$$

and

$$\mathbb{V}(R(t)) = \frac{\sigma^2}{\beta}R(0)(e^{-\beta t} - e^{-2\beta t}) + \frac{\alpha\sigma^2}{2\beta^2}(1 - e^{-\beta t})^2.$$

7.3 Black–Scholes–Merton Equation

Example 7.10 (Black–Scholes–Merton Equation). The price of a European call option $c(t, x)$ at time t when the stock price is x satisfies the problem

$$c_t + rxc_x + \frac{1}{2}\sigma^2 x^2 c_{xx} = rc \quad \text{with} \quad c(T, x) = (x - K)^+,$$

where r is the risk-free interest rate, σ is the volatility, T is the expiration time of the call, and K is the strike price.

Theorem 7.11. *Define*

$$c(t, x) = xN(d_+(T - t, x)) - Ke^{-r(T-t)}N(d_-(T - t, x)),$$

where

$$d_{\pm}(\tau, x) = \frac{\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

We have

$$d_+(\tau, x) - d_-(\tau, x) = \sigma\sqrt{\tau}; \tag{a}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}; \tag{b}$$

$$xN'(d_+(\tau, x)) - Ke^{-r\tau} N'(d_-(\tau, x)) = 0; \tag{c}$$

$$\frac{\partial d_+(T-t, x)}{\partial t} = \frac{\partial d_-(T-t, x)}{\partial t} - \frac{\sigma}{2\sqrt{T-t}}; \tag{d}$$

$$\frac{\partial c(t, x)}{\partial t} = -rKe^{-r(T-t)} N(d_-(T-t, x)) - \frac{\sigma x}{2\sqrt{T-t}} N'(d_+(T-t, x)); \tag{e}$$

$$\frac{\partial d_+(\tau, x)}{\partial x} = \frac{\partial d_-(\tau, x)}{\partial x} = \frac{1}{\sigma x\sqrt{\tau}}; \tag{f}$$

$$\frac{\partial c(t, x)}{\partial x} = N(d_+(T-t, x)); \tag{g}$$

$$\frac{\partial^2 c(t, x)}{\partial x^2} = \frac{1}{\sigma x\sqrt{T-t}} N'(d_+(T-t, x)); \tag{h}$$

$$\frac{\partial c(t, x)}{\partial t} + rx\frac{\partial c(t, x)}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 c(t, x)}{\partial x^2} = rc(t, x); \tag{i}$$

$$\lim_{t \to T^-} c(t, x) = (x - K)^+. \tag{j}$$

Theorem 7.12. *The price of a European call at time zero with expiration time T and strike price K is given by*

$$S(0)N(d_+(T, S(0))) - Ke^{-rT} N(d_-(T, S(0))).$$

7.4 The Greeks

Definition 7.13. The *Greeks* of a European call are defined as follows:

$$\Delta = \frac{\partial c(t,x)}{\partial x} \quad \text{(the } \textit{Delta}\text{);}$$
$$\Theta = \frac{\partial c(t,x)}{\partial t} \quad \text{(the } \textit{Theta}\text{);}$$
$$\Gamma = \frac{\partial^2 c(t,x)}{\partial x^2} \quad \text{(the } \textit{Gamma}\text{);}$$
$$\mathcal{V} = \frac{\partial c(t,x)}{\partial \sigma} \quad \text{(the } \textit{Vega}\text{);}$$
$$\rho = \frac{\partial c(t,x)}{\partial r} \quad \text{(the } \textit{Rho}\text{).}$$

Theorem 7.14. *The Greeks for a European call are given by*

$$\Delta = N(d_+(T-t,x)),$$
$$\Theta = -rKe^{-r(T-t)}N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{2\pi(T-t)}}e^{-\frac{(d_+(T-t,x))^2}{2}},$$
$$\Gamma = \frac{1}{\sigma x\sqrt{2\pi(T-t)}}e^{-\frac{(d_+(T-t,x))^2}{2}},$$
$$\mathcal{V} = x\sqrt{\frac{T-t}{2\pi}}e^{-\frac{(d_+(T-t,x))^2}{2}},$$
$$\rho = K(T-t)e^{-r(T-t)}N(d_-(T-t,x)).$$