56. Define the Hermite polynomials h_n by

$$h_n(t,x) = \frac{(-t)^n}{n!} e^{x^2/2t} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2/2t}\right).$$

(a) Find the first four Hermite polynomials.

(b) Show that for $n \in \mathbb{N}_0$, we have

$$\int_0^T h_n(t, W(t))\mathrm{d}W(t) = h_{n+1}(T, W(T)).$$

57. Show that $e^{iW(t)} = X_1(t) + iX_2(t)$ is a process on the unit circle satisfying

$$\mathrm{d}X_1 = -\frac{1}{2}X_1\mathrm{d}t - X_2\mathrm{d}W, \quad \mathrm{d}X_2 = -\frac{1}{2}X_2\mathrm{d}t + X_1\mathrm{d}W.$$

58. Compute $\mathrm{d}(S(t))^p$ and $\mathbb{E}((S(t))^p)$, where S is geometric Brownian motion.

59. Write down the stochastic differential equation obtained via Itô's formula for the process $Y(t) = (W(t))^4$ and use it to calculate $\mathbb{E}((W(t))^4)$.

60. Write down the stochastic differential equation obtained via Itô's formula for the process $Y(t) = (W(t))^6$ and use it to calculate $\mathbb{E}((W(t))^6)$.

61. Let W be a Brownian motion and define $B(t) = \int_0^t \mathrm{sgn}(W(s))\mathrm{d}W(s)$ with $\mathrm{sgn}(x) = 1$ for $x \geq 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$.

(a) Show that B is a Brownian motion.

(b) Use Itô's product rule to compute $\mathrm{d}(B(t)W(t))$.

(c) Show that B and W are uncorrelated.

(d) Use Itô's product rule to compute $\mathrm{d}W^2(t)$ and $\mathrm{d}(B(t)W^2(t))$.

(e) Show that B and W are not independent.