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A critical point approach for a second-order dynamic Sturm–Liouville boundary value problem with p-Laplacian

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ABSTRACT

In this paper, we give conditions guaranteeing the existence of at least three solutions for a second-order dynamic Sturm–Liouville boundary value problem involving two parameters. In the proofs of the results, we utilize critical point theory and variational methods. In addition, an example is given in order to illustrate our results.

1. Introduction

Let $\emptyset \neq \mathbb{T} \subseteq \mathbb{R}$, called a time scale, be given. For example, $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ are time scales that correspond to differential and difference equations, respectively. Let $S > 0$ be fixed and suppose $0, S \in \mathbb{T}$. Consider the second-order dynamic Sturm–Liouville boundary value problem

$$\begin{cases} -(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = \lambda f(t, x^{\sigma}(t)) + \zeta g(t, x^{\sigma}(t)), & t \in [0, S]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^{\Delta}(0) = 0, \quad \alpha_3 x(\sigma^2(S)) + \alpha_4 x^{\Delta}(\sigma(S)) = 0, \end{cases} \qquad (P_{\lambda,\zeta})$$

where $p \in C^1([0, \sigma(S)]_{\mathbb{T}}, (0, \infty))$, $q \in C([0, S]_{\mathbb{T}}, [0, \infty))$, $f, g \in C([0, S]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$, $\lambda > 0$ and $\zeta \geq 0$ are real parameters, $\alpha_i \geq 0$ for $i = 1, 2, 3, 4$ and $\alpha_1 + \alpha_2 \geq 0$, $\alpha_3 + \alpha_4 > 0$, $\alpha_1 + \alpha_3 > 0$.

Time scales theory was created by Hilger [24] in 1988, and it serves to unify continuous and discrete analysis. Moreover, basic elements of variational calculus on time scales were introduced in 2004 [4], and it has since then been further developed by many authors in several different directions, e.g., [16,25,29]. Many classical results of variational calculus such as sufficient and necessary conditions for optimality have been generalized to arbitrary time scales. For related results concerning economic models, we refer to [3,7,8,10–12,19]. Also, Sturm–Liouville equations on time scales have attracted substantial

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interest, see for example [2,15,17,18,26,27,30,33,34,36]. Zhang and Sun in [34], using critical point theory and variational methods, have established existence of solutions for $(P_{\lambda,0})$. First, they have ensured an existence interval for λ such that $(P_{\lambda,0})$ possesses one or two solutions. Then, under entirely different assumptions on f and by employing a three critical point theorem, they have derived some sufficient conditions for existence of at least three solutions for $(P_{\lambda,0})$ when λ is located in a certain interval. In [15], the authors established a connection between dynamic Sturm–Liouville equations and corresponding equations with measure-valued coefficients. Based on this, they generalized several known results for dynamic Sturm–Liouville equations. In [30], Ozkan considered a boundary value problem involving a dynamic Sturm–Liouville equation and boundary conditions depending on a spectral parameter, and he also introduced an operator formulation for the problem and gave several properties of eigenvalues and eigenfunctions. Finally, for finite time scales, he derived the exact number of eigenvalues of the problem. In [33], existence of at least one and at least two positive solutions for $(P_{1,0})$ was obtained. In [31], for a periodic time scale, Su and Feng have studied a dynamic second-order p-Laplacian equation together with certain boundary value conditions. They used the three critical point theorem, the least action principle, and the saddle point theorem in order to obtain existence of at least one or at least three distinct periodic solutions. Existence results on periodic time scales, by establishing a suitable variational setting, were also proved in [36] for a class of dynamic p-Laplacian systems.

In the present paper, see Theorem 5 below, we obtain the existence of at least three distinct nonnegative solutions for $(P_{\lambda,\zeta})$ by utilizing a critical point result due to Bonanno and Candito [13, Theorem 3.3]. Demanding an additional asymptotical behaviour of the data at zero, nontriviality of the solution can be achieved also under appropriate assumptions, see Remark 6. Moreover, existence of solutions for $\lambda \to 0^+$ is investigated, see Remark 7. Theorem 8 follows from Theorem 5. As two special cases of Theorem 8, we present Theorems 9 and 10. Next, we offer Example 11, in which the assumptions of Theorem 10 are satisfied. Finally, in Theorem 12, existence of at least four distinct nontrivial solutions of $(P_{\lambda,0})$ is discussed.

2. Preliminaries

The main tool to derive existence of at least three solutions for $(P_{\lambda,\zeta})$ is the following three critical point theorem due to Bonanno and Candito. For $X \neq 0$, $\mathcal{J}_1, \mathcal{J}_2 : X \to \mathbb{R}$, and $r, r_1, r_2 > \inf_X \mathcal{J}_1$, $r_2 > r_1$, $r_3 > 0$, define

$$\varphi(r) := \inf_{x\in\mathcal{J}_1^{-1}(-\infty,r)} \frac{\sup_{y\in\mathcal{J}_1^{-1}(-\infty,r)} \mathcal{J}_2(y) - \mathcal{J}_2(x)}{r - \mathcal{J}_1(x)},$$

$$\beta(r_1,r_2) := \inf_{x\in\mathcal{J}_1^{-1}(-\infty,r_1)} \sup_{y\in\mathcal{J}_1^{-1}[r_1,r_2)} \frac{\mathcal{J}_2(y) - \mathcal{J}_2(x)}{\mathcal{J}_1(y) - \mathcal{J}_1(x)},$$

$$\gamma(r_2,r_3) := \frac{\sup_{x\in\mathcal{J}_1^{-1}(-\infty,r_2+r_3)} \mathcal{J}_2(x)}{r_3},$$

and

$$\alpha(r_1,r_2,r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2,r_3)\}.$$

Theorem 1 (See [13, Theorem 3.3]). *Let be given a reflexive real Banach space X. Suppose $\mathcal{J}_1 : X \to \mathbb{R}$ is a convex, coercive, and continuously Gâteaux-differentiable functional such that its Gâteaux derivative admits a continuous inverse on X^*. Assume that $\mathcal{J}_2 : X \to \mathbb{R}$ is a continuously Gâteaux-differentiable functional such that its Gâteaux derivative is compact. Assume*

(a_1) $\inf_X \mathcal{J}_1 = \mathcal{J}_1(0) = \mathcal{J}_2(0) = 0$,
(a_2) for every $x_1, x_2 \in X$ such that $\mathcal{J}_2(x_1) \geq 0$ and $\mathcal{J}_2(x_2) \geq 0$, one has

$$\inf_{s\in[0,1]} \mathcal{J}_2(sx_1 + (1-s)x_2) \geq 0.$$

Suppose that there exist $r_1, r_2, r_3 > 0$ with $r_1 < r_2$ and

(a_3) $\varphi(r_1) < \beta(r_1, r_2)$,
(a_4) $\varphi(r_2) < \beta(r_1, r_2)$,
(a_5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for any $\lambda \in \left(\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}\right)$, $\mathcal{J}_1 - \lambda\mathcal{J}_2$ admits at least three distinct critical points x_1, x_2, x_3 such that

$$x_1 \in \mathcal{J}_1^{-1}((-\infty,r_1)), \quad x_2 \in \mathcal{J}_1^{-1}([r_1,r_2)), \quad x_3 \in \mathcal{J}_1^{-1}((-\infty,r_2+r_3)).$$

We refer the reader to [1,5,6,9,14,20–23,28] for situations of successful employments of results such as Theorem 1 in order to prove existence of three solutions for various boundary value problems.

For $f \in \mathrm{L}^1_\Delta([t_1,t_2)_\mathbb{T})$, we abbreviate

$$\int_{t_1}^{t_2} f(s)\Delta s = \int_{[t_1,t_2)_\mathbb{T}} f(s)\Delta s.$$

It is known [35] that

$$\mathcal{H} := H^1_\Delta([0,\sigma^2(S)]_{\mathbb{T}}) := \left\{x : [0,\sigma^2(S)]_{\mathbb{T}} \to \mathbb{R} : \ x \in \mathrm{AC}[0,\sigma^2(S)]_{\mathbb{T}} \text{ and } x\, L^2_\Delta([0,\sigma^2(S))_{\mathbb{T}})\right\}$$

is a Hilbert space when equipped with the inner product

$$(x,y)_{\mathcal{H}} = \int_0^{\sigma^2(S)} x(t)y(t)\Delta t + \int_0^{\sigma^2(S)} x^\Delta(t)y^\Delta(t)\Delta t.$$

For $x, y \in \mathcal{H}$, we define

$$(x,y)_0 = \int_0^{\sigma^2(S)} p(t)x^\Delta(t)y^\Delta(t)\Delta t + \int_0^{\sigma(S)} q(t)x^\sigma(t)y^\sigma(t)\Delta t$$
$$+ \beta_1 p(0)x(0)y(0) + \beta_2 p(\sigma(S))x(\sigma^2(S))y(\sigma^2(S)),$$

where

$$\beta_1 = \begin{cases} \dfrac{\alpha_1}{\alpha_2} & \text{if } \alpha_2 > 0, \\ 0 & \text{if } \alpha_2 = 0 \end{cases} \tag{1}$$

and

$$\beta_2 = \begin{cases} \dfrac{\alpha_3}{\alpha_4} & \text{if } \alpha_4 > 0, \\ 0 & \text{if } \alpha_4 = 0. \end{cases} \tag{2}$$

We let $\|\cdot\|_0$ be the norm induced by the inner product $(\cdot,\ \cdot)_0$.

Lemma 2 (See [34, Lemmas 2.1, 2.2 and 4.2]). *The immersion $\mathcal{H} \hookrightarrow C([0,\sigma^2(S)]_{\mathbb{T}})$ is compact. If $x \in \mathcal{H}$, then*

$$|x(t)| \le \sqrt{2}\max\{(\sigma^2(S))^{\frac{1}{2}}, (\sigma^2(S))^{-\frac{1}{2}}\}\|x\|_{\mathcal{H}} \text{ for all } t \in [0,\sigma^2(S)]_{\mathbb{T}}.$$

If $\alpha_2, \alpha_4 > 0$ or $q(t) > 0$ for $t \in [0,S]_{\mathbb{T}}$, then for $x \in \mathcal{H}$, $|x(t)| \le C\|x\|_0$ for every $t \in [0,\sigma^2(S)]_{\mathbb{T}}$, where $C = \min\{M_1, M_2, M_3\}$ and

$$M_1 = \sqrt{2}\max\left\{\frac{1}{\sqrt{\beta_1 p(0)}}, \frac{\sqrt{\sigma^2(S)}}{\min_{t\in[0,\sigma(S)]_{\mathbb{T}}} p(t)}\right\},$$

$$M_2 = \sqrt{2}\max\left\{\frac{1}{\sqrt{\beta_2 p(0)}}, \frac{\sqrt{\sigma^2(S)}}{\min_{t\in[0,\sigma(S)]_{\mathbb{T}}} p(t)}\right\},$$

$$M_3 = \sqrt{2}\max\left\{\frac{\sqrt{\sigma(S)}}{\min_{t\in[0,S]_{\mathbb{T}}} q(t)}, \frac{\sqrt{\sigma^2(S)}}{\min_{t\in[0,\sigma(S)]_{\mathbb{T}}} p(t)}\right\},$$

and where $\frac{1}{0} = \infty$.

For each $x \in \mathcal{H}$, define the functionals $\mathcal{J}_1$ and $\mathcal{J}_2$ by

$$\mathcal{J}_1(x) = \frac{1}{2}\|x\|_0^2 \tag{3}$$

and

$$\mathcal{J}_2(x) = \int_0^{\sigma(S)} F(t,x^\sigma(t))\Delta t + \frac{\zeta}{\lambda}\int_0^{\sigma(S)} G(t,x^\sigma(t))\Delta t, \tag{4}$$

where

$$F(t,\xi) = \int_0^\xi f(t,s)ds \quad \text{for} \quad (t,\xi) \in [0,S]_{\mathbb{T}} \times \mathbb{R}$$

and

$$G(t,\xi) = \int_0^\xi g(t,s)ds \quad \text{for} \quad (t,\xi) \in [0,S]_{\mathbb{T}} \times \mathbb{R}.$$

Define also

$$I_\lambda = \mathcal{J}_1(x) - \lambda\mathcal{J}_2(x).$$

The following auxiliary result is used later.

Lemma 3. *Let $T : \mathcal{H} \to \mathcal{H}^*$ be defined by $T(x)(y) = (x,y)_0$. Then T possesses a continuous inverse on E^*.*

Proof. Note that

$$\lim_{\|x\|_0\to\infty}\frac{T(x)(x)}{\|x\|_0}=\lim_{\|x\|_0\to\infty}\frac{(x,x)_0}{\|x\|_0}=\lim_{\|x\|_0\to\infty}\|x\|_0=\infty.$$

Thus, the map T is coercive. Now, we will prove that T is strictly monotone:

$$\begin{aligned}T(x)(x-y)-T(y)(x-y)&=(x,x-y)_0-(y,x-y)_0\\&=(x-y,x-y)_0=\|x-y\|_0^2.\end{aligned}$$

By [32, Theorem 26.A(d)], T^{-1} exists and is continuous on $\mathcal{H}^*$. □

Proposition 4. *$x\in X$ is a critical point of $\mathcal{J}_1-\lambda\mathcal{J}_2$ iff x solves $(P_{\lambda,\zeta})$.*

Proof. Suppose $x\in X$ is a critical point of $\mathcal{J}_1-\lambda\mathcal{J}_2$. Thus, for any $y\in X$,

$$\langle(\mathcal{J}_1-\lambda\mathcal{J}_2)'(x),y\rangle=0,$$

that is,

$$\begin{aligned}&\int_0^{\sigma^2(S)}p(t)x^\Delta(t)y^\Delta(t)\Delta t+\int_0^{\sigma(S)}q(t)x^\sigma(t)y^\sigma(t)\Delta t-\lambda\int_0^{\sigma(S)}f(t,x^\sigma(t))y^\sigma(t)\Delta t-\zeta\int_0^{\sigma(S)}g(t,x^\sigma(t))y^\sigma(t)\Delta t\\&+\beta_1p(0)x(0)y(0)+\beta_2p(\sigma(S))x(\sigma^2(S))y(\sigma^2(S))=0.\end{aligned}$$

Simple calculations show that

$$\begin{aligned}&-\int_0^{\sigma^2(S)}(px^\Delta)^\Delta(t)y^\sigma(t)\Delta t+\int_0^{\sigma(S)}[q(t)x^\sigma(t)-\lambda f(t,x^\sigma(t))-\zeta g(t,x^\sigma(t))]y^\sigma(t)\Delta t\\&+p(0)y(0)[\beta_1x(0)-x0)]+y(\sigma^2(S))\big[p(\sigma^2(S))x^\Delta(\sigma^2(S))+\beta_2p(\sigma(S))x(\sigma^2(S))\big]=0.\end{aligned}\tag{5}$$

Thus, by the fundamental lemma of variational calculus, x satisfies the dynamic equation in $(P_{\lambda,\zeta})$. Then (5) becomes

$$p(0)y(0)[\beta_1x(0)-x0)]+y(\sigma^2(S))\big[p(\sigma^2(S))x^\Delta(\sigma^2(S))+\beta_2p(\sigma(S))x(\sigma^2(S))\big]=0.$$

By using (1) and (2), we have

$$p(0)y(0)[\alpha_1x(0)-\alpha_2x0)]+y(\sigma^2(S))\big[p(\sigma^2(S))\alpha_4x^\Delta(\sigma^2(S))+\alpha_3p(\sigma(S))x(\sigma^2(S))\big]=0$$

for all $y\in X$. We now demonstrate that x satisfies the boundary conditions in $(P_{\lambda,\zeta})$. Without restricting generality, suppose

$$\alpha_1x(0)-\alpha_2x0)>0.$$

We let $y(t)=\sigma^2(S)-t$. Then

$$\begin{aligned}&p(0)y(0)[\alpha_1x(0)-\alpha_2x0)]+y(\sigma^2(S))\big[p(\sigma^2(S))\alpha_4x^\Delta(\sigma^2(S))+\alpha_3p(\sigma(S))x(\sigma^2(S))\big]\\&=p(0)\sigma^2(S)[\alpha_1x(0)-\alpha_2x0)]>0,\end{aligned}$$

a contradiction. So x is a solution of $(P_{\lambda,\zeta})$. Conversely, if x is a solution of $(P_{\lambda,\zeta})$, for any $y\in X$, multiplying $y(t)$ on both sides of the dynamic equation in $(P_{\lambda,\zeta})$ and integrating on $[0,\sigma(S)]_{\mathbb{T}}$, in view of the boundary conditions, we observe that x satisfies $\langle(\mathcal{J}_1-\lambda\mathcal{J}_2)'(x),y\rangle=0$ for all $y\in X$. □

Next, for our convenience, let

$$G^\theta:=\int_{[0,\sigma(S))_{\mathbb{T}}}G(t,\theta)\Delta t\quad\text{for}\quad\theta>0\tag{6}$$

and

$$G_\eta:=\sigma(S)\inf_{[0,\sigma(S))_{\mathbb{T}}\times[0,\eta]}G\quad\text{for}\quad\eta>0.\tag{7}$$

3. Main Results

For a positive constant d, set

$$K_d=\frac{d^2}{2}\left(\int_0^{\sigma(S)}q(t)\Delta t+\beta_1p(0)+\beta_2p(\sigma(S))\right).$$

We fix four positive constants θ_1, θ_2, θ_3, d and define the constant $\delta_{\lambda,g}$ by

$$\min\left\{\frac{1}{2C^2}\min\left\{\frac{\theta_1^2-2C^2\lambda\int_0^{\sigma(S)}F(t,\theta_1)\Delta t}{G^{\theta_1}},\frac{\theta_2^2-2C^2\lambda\int_0^{\sigma(S)}F(t,\theta_2)\Delta t}{G^{\theta_2}},\right.\right.$$
$$\left.\left.\frac{\theta_3^2-\theta_2^2-2C^2\lambda\int_0^{\sigma(S)}F(t,\theta_3)\Delta t}{G^{\theta_3}}\right\},\frac{K_d-\lambda\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t}{G_d-G^{\theta_1}}\right\}. \tag{8}$$

Theorem 5. *Suppose $f:[0,S]_{\mathbb{T}}\times[0,\infty)\to(0,\infty)$ is continuous. Assume the existence of θ_1, θ_2, θ_3, $d>0$ such that*

$$\theta_1<C\sqrt{2K_d}<\theta_2<\theta_3$$

and

$$(\mathrm{A}_1)\ \max\left\{\frac{\int_0^{\sigma(S)}F(t,\theta_1)\Delta t}{\theta_1^2},\frac{\int_0^{\sigma(S)}F(t,\theta_2)\Delta t}{\theta_2^2},\frac{\int_0^{\sigma(S)}F(t,\theta_3)\Delta t}{\theta_3^2-\theta_2^2}\right\}<\frac{1}{2C^2}\frac{\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t}{K_d}.$$

Then, for every

$$\lambda\in\left(\frac{K_d}{\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t},\frac{1}{2C^2}\min\left\{\frac{\theta_1^2}{\int_0^{\sigma(S)}F(t,\theta_1)\Delta t},\frac{\theta_2^2}{\int_0^{\sigma(S)}F(t,\theta_2)\Delta t},\frac{\theta_3^2-\theta_2^2}{\int_0^{\sigma(S)}F(t,\theta_3)\Delta t}\right\}\right),$$

for every nonnegative continuous function $g:[0,S]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$, there exists $\delta_{\lambda,g}>0$ given by (8) such that, for each $\zeta\in[0,\delta_{\lambda,g})$, the problem $(P_{\lambda,\zeta})$ possesses at least three nonnegative solutions $x_1,x_2,x_3\in\mathcal{H}$ such that

$$\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}}|x_1(t)|<\theta_1,\quad\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}}|x_2(t)|<\theta_2,\quad\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}}|x_3(t)|<\theta_3.$$

Proof. We may assume $f(t,x)=f(t,0)$ for all $(t,x)\in[0,S]_{\mathbb{T}}\times(-\infty,0)$. Let $X=\mathcal{H}$, and we consider $\mathcal{J}_1$ and $\mathcal{J}_2$ defined by (3) and (4), respectively. We now prove that $\mathcal{J}_1$ and $\mathcal{J}_2$ fulfill the assumptions of Theorem 1. It is clear that $\mathcal{J}_2$ is differentiable with

$$\mathcal{J}_2'(x)(y)=\int_0^{\sigma(S)}f(t,x^\sigma(t))y^\sigma(t)\Delta t+\frac{\zeta}{\lambda}\int_0^{\sigma(S)}g(t,x^\sigma(t))y^\sigma(t)\Delta t$$

for $x,y\in X$. Moreover, $\mathcal{J}_2':X\to X^*$ is compact. Also, $\mathcal{J}_1$ is continuously differentiable with

$$\mathcal{J}_1'(x)(y)=\int_0^{\sigma^2(S)}p(t)x^\Delta(t)y^\Delta(t)\Delta t+\int_0^{\sigma(S)}q(t)x^\sigma(t)y^\sigma(t)\Delta t$$
$$+\beta_1p(0)x(0)y(0)+\beta_2p(\sigma(S))x(\sigma^2(S))y(\sigma^2(S))$$

for $x,y\in X$, while Lemma 3 yields that $\mathcal{J}_1'$ has a continuous inverse on X^*. In addition, $\mathcal{J}_1$ is sequentially weakly lower semicontinuous. Denote

$$r_1:=\frac{\theta_1^2}{2C^2},\quad r_2:=\frac{\theta_2^2}{2C^2},\quad r_3:=\frac{\theta_3^2-\theta_2^2}{2C^2}$$

and $w(t)=d$ for $t\in[0,S]_{\mathbb{T}}$. Clearly, $w\in X$. From (3), we observe that $\mathcal{J}_1(w)=K_d$, and by the condition

$$\theta_1<C\sqrt{2K_d}<\theta_2<\theta_3,$$

we get $r_3>0$ and $r_1<\mathcal{J}_1(w)<r_2$. From the definition of r_1 and Lemma 2, we obtain

$$\mathcal{J}_1^{-1}(-\infty,r_1)\subseteq\left\{x\in X:\ \|x\|_0\le\sqrt{2r_1}\right\}$$
$$\subseteq\left\{x\in X:\ |x(t)|\le C\sqrt{2r_1}\text{ for all }t\in[0,\sigma^2(S)]_{\mathbb{T}}\right\}$$
$$=\left\{x\in X:\ |x(t)|\le\theta_1\text{ for all }t\in[0,\sigma^2(S)]_{\mathbb{T}}\right\},$$

and since we assumed that f and g are nonnegative, this ensures

$$\mathcal{J}_2(x)\le\sup_{y\in\mathcal{J}_1^{-1}(-\infty,r_1)}\left[\int_0^{\sigma(S)}F(t,y^\sigma(t))\Delta t+\frac{\zeta}{\lambda}\int_0^{\sigma(S)}G(t,y^\sigma(t))\Delta t.\right]$$
$$\le\int_0^{\sigma(S)}F(t,\theta_1)\Delta t+\frac{\zeta}{\lambda}\int_0^{\sigma(S)}G(t,\theta_1)\Delta t$$
$$=\int_0^{\sigma(S)}F(t,\theta_1)\Delta t+\frac{\zeta}{\lambda}G^{\theta_1}$$

for every $x \in X$ such that $\mathcal{J}_1(x) < r_1$. Thus,

$$\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_1)} \mathcal{J}_2(x) \le \int_0^{\sigma(S)} F(t, \theta_1)\Delta t + \frac{\zeta}{\lambda} G^{\theta_1}. \tag{9}$$

Similarly,

$$\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2)} \mathcal{J}_2(x) \le \int_0^{\sigma(S)} F(t, \theta_2)\Delta t + \frac{\zeta}{\lambda} G^{\theta_2} \tag{10}$$

and

$$\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2 + r_3)} \mathcal{J}_2(x) \le \int_0^{\sigma(S)} F(t, \theta_3)\Delta t + \frac{\zeta}{\lambda} G^{\theta_3}. \tag{11}$$

Therefore, since $0 \in \mathcal{J}_1^{-1}(-\infty, r_1)$ and $\mathcal{J}_1(0) = \mathcal{J}_2(0) = 0$, using (9), one has

$$\varphi(r_1) \le \frac{\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_1)} \mathcal{J}_2(x)}{r_1} \le 2C^2 \frac{\int_0^{\sigma(S)} F(t, \theta_1)\Delta t + \frac{\zeta}{\lambda} G^{\theta_1}}{\theta_1^2} < \frac{1}{\lambda}$$

because $\zeta < \frac{1}{2C^2} \frac{\theta_1^2 - 2C^2 \lambda \int_0^{\sigma(S)} F(t, \theta_1)\Delta t}{G^{\theta_1}}$ due to (8), and using (10), one has

$$\varphi(r_2) \le \frac{\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2)} \mathcal{J}_2(x)}{r_2} \le 2C^2 \frac{\int_0^{\sigma(S)} F(t, \theta_2)\Delta t + \frac{\zeta}{\lambda} G^{\theta_2}}{\theta_2^2} < \frac{1}{\lambda}$$

because $\zeta < \frac{1}{2C^2} \frac{\theta_2^2 - 2C^2 \lambda \int_0^{\sigma(S)} F(t, \theta_2)\Delta t}{G^{\theta_2}}$ due to (8), and using (11), one has

$$\gamma(r_2, r_3) \le \frac{\sup_{x \in \mathcal{J}_1^{-1}(-\infty, r_2 + r_3)} \mathcal{J}_2(x)}{r_3} \le 2C^2 \frac{\int_0^{\sigma(S)} F(t, \theta_3)\Delta t + \frac{\zeta}{\lambda} G^{\theta_3}}{\theta_3^2 - \theta_2^2} < \frac{1}{\lambda}$$

because $\zeta < \frac{\theta_3^2 - \theta_2^2 - 2C^2 \lambda \int_0^{\sigma(S)} F(t, \theta_3)\Delta t}{G^{\theta_3}}$ due to (8). Also, taking into account (7), we get

$$\begin{aligned}
\mathcal{J}_2(w) &= \int_0^{\sigma(S)} F(t, w(t))\Delta t + \frac{\zeta}{\lambda} \int_0^{\sigma(S)} G(t, w(t))\Delta t \\
&\ge \int_0^{\sigma(S)} F(t, w(t))\Delta t + \sigma(S) \frac{\zeta}{\lambda} \inf_{[0, \sigma(S)]_{\mathbb{T}} \times [0, d]} G \\
&= \int_0^{\sigma(S)} F(t, d)\Delta t + \frac{\zeta}{\lambda} G_d.
\end{aligned}$$

Hence, if $x \in \mathcal{J}_1^{-1}(-\infty, r_1)$, then, taking (9) into account,

$$\mathcal{J}_2(w) - \mathcal{J}_2(x) \ge \int_0^{\sigma(S)} (F(t, d) - F(t, \theta_1))\Delta t + \frac{\zeta}{\lambda}(G_d - G^{\theta_1})$$

and

$$0 < \mathcal{J}_1(w) - \mathcal{J}_1(x) \le \mathcal{J}_1(w) = K_d,$$

so that

$$\beta(r_1, r_2) \ge \frac{\int_0^{\sigma(S)} (F(t, d) - F(t, \theta_1))\Delta t + \frac{\zeta}{\lambda}(G_d - G^{\theta_1})}{K_d} > \frac{1}{\lambda}$$

because $\zeta < \frac{K_d - \lambda \int_0^{\sigma(S)} (F(t,d) - F(t, \theta_1))\Delta t}{G_d - G^{\theta_1}}$ due to (8). Altogether, we get

$$\alpha(r_1, r_2, r_3) < \beta(r_1, r_2).$$

Next, we illustrate that $\mathcal{J}_2$ fulfills (a_2) of Theorem 1. Suppose x_1, x_2 are local minima for I_λ. Then x_1, x_2 are critical points for I_λ and hence nonnegative solutions of $(P_{\lambda, \zeta})$. Thus, it follows that

$$(\lambda f + \zeta g)(t, sx_1 + (1 - s)x_2) \ge 0$$

for all $s \in [0, 1]$, and therefore, $\mathcal{J}_2(sx_1 + (1 - s)x_2) \ge 0$ for all $s \in [0, 1]$. By utilizing Theorem 1, we get that for every

$$\lambda \in \left(\frac{K_d}{\int_0^{\sigma(S)} (F(t, d) - F(t, \theta_1))\Delta t}, \frac{1}{2C^2} \min \left\{ \frac{\theta_1^2}{\int_0^{\sigma(S)} F(t, \theta_1)\Delta t}, \frac{\theta_2^2}{\int_0^{\sigma(S)} F(t, \theta_2)\Delta t}, \frac{\theta_3^2 - \theta_2^2}{\int_0^{\sigma(S)} F(t, \theta_3)\Delta t} \right\} \right)$$

and $\zeta \in [0, \delta_{\lambda,g})$, I_λ has three critical points x_i, $i = 1, 2, 3$, in X such that $\mathcal{J}_1(x_1) < r_1$, $\mathcal{J}_1(x_2) < r_2$, and $\mathcal{J}_1(x_3) < r_2 + r_3$, that is,

$$\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_1(t)| < \theta_1, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_2(t)| < \theta_2, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_3(t)| < \theta_3.$$

The proof is complete. □

Remark 6. If $f(t, 0) \neq 0$ or $g(t, 0) \neq 0$ for some $t \in [0, S]_{\mathbb{T}}$, then the solutions obtained in Theorem 5 are nontrivial. Moreover, nontriviality can be demonstrated also if $f(t, 0) = 0$ for some $t \in [0, S]_{\mathbb{T}}$, requiring an extra condition at zero, namely the existence of $\emptyset \neq D \subseteq [0, S]_{\mathbb{T}}$ (D open) and $B \subset D$ such that

$$\limsup_{\xi\to 0^+} \frac{\inf_{t\in B} F(t,\xi)}{|\xi|^2} = \infty \quad \text{and} \quad \liminf_{\xi\to 0^+} \frac{\inf_{t\in D} F(t,\xi)}{|\xi|^2} > -\infty. \tag{12}$$

To see this, let $0 < \overline{\lambda} < \lambda^*$, where

$$\lambda^* = \frac{1}{2C^2} \min\left\{ \frac{\theta_1^2}{\int_0^{\sigma(S)} F(t,\theta_1)\Delta t}, \frac{\theta_2^2}{\int_0^{\sigma(S)} F(t,\theta_2)\Delta t}, \frac{\theta_3^2-\theta_2^2}{\int_0^{\sigma(S)} F(t,\theta_3)\Delta t} \right\}.$$

Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be as given in (3) and (4), respectively. Because of Theorem 1, for all $\lambda \in \left(\frac{K_d}{\int_0^{\sigma(S)} (F(t,d)-F(t,\theta_1))\Delta t}, \overline{\lambda} \right)$, there exist three critical points $x_{1\lambda}$, $x_{2\lambda}$ and $x_{3\lambda}$ of the functional $I_\lambda = \mathcal{J}_1(x) - \lambda \mathcal{J}_2(x)$ such that

$$x_{1\lambda} \in \mathcal{J}_1^{-1}(-\infty, r_{1\lambda}), \ x_{2\lambda} \in \mathcal{J}_1^{-1}(-\infty, r_{2\lambda}), \ x_{3\lambda} \in \mathcal{J}_1^{-1}(-\infty, r_{3\lambda}),$$

where

$$r_{1\lambda} = \frac{\theta_{1\lambda}^2}{2C^2}, \ r_{2\lambda} = \frac{\theta_{2\lambda}^2}{2C^2}, \ r_{3\lambda} = \frac{\theta_{3\lambda}^2 - \theta_{2\lambda}^2}{2C^2}.$$

In particular, $x_{i\lambda}$ for $i = 1, 2, 3$ is a global minimum of the restriction of I_λ to $\mathcal{J}_1^{-1}(-\infty, r_{i\lambda})$ for $i = 1, 2, 3$. We will prove that $x_{1\lambda}$ cannot be trivial. Let us show that

$$\limsup_{\|x\|\to 0^+} \frac{\mathcal{J}_2(x)}{\mathcal{J}_1(x)} = \infty. \tag{13}$$

According to (12), we may pick $\tau > 0$ and κ and a sequence $\{\xi_n\} \subset \mathbb{R}^+$ that converges to zero such that, for all $\xi \in [0, \tau]$,

$$\lim_{\xi\to 0^+} \frac{\inf_{t\in B} F(t,\xi_n)}{|\xi_n|^2} = \infty \quad \text{and} \quad \inf_{t\in D} F(t,\xi) > \kappa|\xi|^2.$$

We consider $E \subset B$ of positive measure and $y \in X$ satisfying

(k_1) $y(t) \in [0, 1]$ for all $t \in [0, S]_{\mathbb{T}}$,
(k_2) $y(t) = 1$ for all $t \in E$,
(k_3) $y(t) = 0$ for all $t \in [0, S]_{\mathbb{T}} \setminus D$.

Finally, let $M > 0$ and let $\eta > 0$ such that

$$M < 2\frac{\operatorname{meas}(E) + mp\kappa \int_{D\setminus E} |y(t)|\Delta t}{K_y},$$

where

$$K_y := \int_0^{\sigma^2(S)} p(t)|y^\Delta(t)|^2\Delta t + \int_E q(t)\Delta t + \int_{D\setminus E} q(t)|y^\sigma(t)|^2\Delta t + \beta_1 p(0)y^2(0) + \beta_2 p(\sigma(S))y^2(\sigma^2(S)).$$

Then, there is $n_0 \in \mathbb{N}$ such that

$$\xi_n < \tau \quad \text{and} \quad \inf_{t\in B} F(t,\xi_n) \geq \kappa|\xi_n|^p$$

for all $n > n_0$. Now, for every $n > n_0$, by taking into account $0 \leq \xi_n y(t) < \tau$ for sufficiently large n, since g is nonnegative, one has

$$\begin{aligned}
\frac{\mathcal{J}_2(\xi_n y)}{\mathcal{J}_1(\xi_n y)} &= \frac{\int_E F(t,\xi_n)\Delta t + \int_{D\setminus E} F(t,\xi_n y(t))\Delta t + \frac{\zeta}{\lambda}\int_0^{\sigma(S)} G(t,\xi_n y(t))\Delta t}{\mathcal{J}_1(\xi_n y)} \\
&\geq \frac{\int_E F(t,\xi_n)\Delta t + \int_{D\setminus E} F(t,\xi_n y(t))\Delta t + \frac{\zeta}{\lambda}G_\tau}{\mathcal{J}_1(\xi_n y)} \\
&\geq \frac{\eta \operatorname{meas}(E) + \kappa \int_{D\setminus E} |y(t)|\Delta t}{K_y} > M.
\end{aligned}$$

Hence, by the arbitrariness of M, we obtain

$$\lim_{n\to\infty}\frac{\mathcal{J}_2(\xi_n y)}{\mathcal{J}_1(\xi_n y)}=\infty,$$

from which (13) follows. Thus, there exists $\{\omega_n\}\subset X$ that converges strongly to zero such that $\omega_n\in\mathcal{J}_1^{-1}(-\infty,r_{1\lambda})$ and

$$I_\lambda(\omega_n)=\mathcal{J}_1(\omega_n)-\lambda\mathcal{J}_2(\omega_n)<0.$$

Since $x_{1\lambda}$ is a global minimum of the restriction of I_λ to $\mathcal{J}_1^{-1}(-\infty,r_{1\lambda})$, we get

$$I_\lambda(x_{1\lambda})<0, \tag{14}$$

which means $x_{1\lambda}$ is nontrivial. By the same arguments, we see that $x_{2\lambda}$ and $x_{3\lambda}$ are nontrivial. If we assume that there exist $\emptyset\neq D\subseteq[0,S]_{\mathbb{T}}$ (D open) and $B\subset D$ such that

$$\limsup_{\xi\to 0^+}\frac{\inf_{t\in B}G(t,\xi)}{|\xi|^2}=\infty \text{ and } \liminf_{\xi\to 0^+}\frac{\inf_{t\in D}G(t,\xi)}{|\xi|^2}>-\infty$$

instead of (12), respectively, then the solutions are again nontrivial.

Remark 7. Using (14), we obtain negativity of the map

$$\Lambda:=\left(\frac{K_d}{\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t},\lambda^*\right)\ni\lambda\mapsto I_\lambda(x_{i\lambda}),\quad i=1,2,3. \tag{15}$$

Also, one has

$$\lim_{\lambda\to 0^+}\|x_{i\lambda}\|=0,\quad i=1,2,3.$$

Indeed, recalling that $\mathcal{J}_2$ is coercive and for all $\lambda\in\Lambda$, for the solution $x_{i\lambda}\in\mathcal{J}_1^{-1}(-\infty,r_{i\lambda})$, $i=1,2,3,$ we get the existence of $L>0$ satisfying $\|x_{i\lambda}\|\le L$, $i=1,2,3$ for all $\lambda\in\Lambda$. Then, there exists $N>0$ with

$$\left|\int_0^{\sigma(S)}f(t,x_{i\lambda}^\sigma(t))x_{i\lambda}^\sigma(t)\Delta t+\frac{\zeta}{\lambda}\int_0^{\sigma(S)}g(t,x_{i\lambda}^\sigma(t))x_{i\lambda}^\sigma(t)\Delta t\right|\le N\|x_{i\lambda}\|\le NL \tag{16}$$

for $i=1,2,3,$ for every $\lambda\in\Lambda$. Since $x_{i\lambda}$, $i=1,2,3$ is a critical point of I_λ, we have $I'_\lambda(x_{i\lambda})(y)=0,$ $i=1,2,3,$ for all $y\in X$ and all $\lambda\in\Lambda$. Hence, $I'_\lambda(x_{i\lambda})(x_{i\lambda})=0,$ $i=1,2,3,$ that is,

$$\mathcal{J}_1'(x_{i\lambda})(x_{i\lambda})=\lambda\int_0^{\sigma(S)}f(t,x_{i\lambda}^\sigma(t))x_{i\lambda}^\sigma(t)\Delta t+\zeta\int_0^{\sigma(S)}g(t,x_{i\lambda}^\sigma(t))x_{i\lambda}^\sigma(t)\Delta t$$

for all $\lambda\in\Lambda$. Then, it follows

$$0\le\mathcal{J}_1'(x_{i\lambda})(x_{i\lambda})=\lambda\int_0^{\sigma(S)}f(t,x_{i\lambda}^\sigma(t))x_{i\lambda}^\sigma(t)\Delta t+\zeta\int_0^{\sigma(S)}g(t,x_{i\lambda}^\sigma(t))x_{i\lambda}^\sigma(t)\Delta t$$

for all $\lambda\in\Lambda$. Letting $\lambda\to 0^+$ by (16), we obtain

$$\lim_{\lambda\to 0^+}\|x_{i\lambda}\|=0,\quad i=1,2,3.$$

This gives the desired conclusion. Finally, we show that the map

$$\lambda\mapsto I_\lambda(x_{i\lambda}),\quad i=1,2,3$$

strictly decreases in $\lambda\in\Lambda$. For any $x\in X$, we have

$$I_\lambda(x)=\lambda\left(\frac{\mathcal{J}_1(x)}{\lambda}-\mathcal{J}_2(x)\right). \tag{17}$$

Take $0<\lambda_1<\lambda_2<\lambda^*$ and let $x_{i\lambda_j}$ be the global minimum of I_{λ_j} restricted to $\mathcal{J}_1(-\infty,r_{i\lambda_j})$ for $i=1,2,3,$ $j=1,2$. Also, put

$$m_{i\lambda_j}=\frac{\mathcal{J}_1(x_{i\lambda_j})}{\lambda_i}-\mathcal{J}_2(x_{i\lambda_j})=\inf_{y\in\mathcal{J}_1^{-1}(-\infty,r_{i\lambda_j})}\left(\frac{\mathcal{J}_1(y)}{\lambda_j}-\mathcal{J}_2(y)\right)$$

for every $i=1,2,3,$ $j=1,2$. Then, (15) in conjunction with (17) and $\lambda>0$ yields

$$m_{i\lambda_j}<0\quad\text{for}\quad i=1,2,3,\quad j=1,2. \tag{18}$$

Moreover,

$$m_{i\lambda_2}<m_{i\lambda_1},\quad i=1,2,3 \tag{19}$$

due to the fact that $0<\lambda_1<\lambda_2$. Then, by (17)–(19) and again by the fact that $0<\lambda_1<\lambda_2$, we get

$$I_{\lambda_2}(x_{i\lambda_2})=\lambda_2 m_{i\lambda_2}\le\lambda_2 m_{i\lambda_1}<\lambda_1 m_{i\lambda_1},\quad i=1,2,3$$

so that the map $\lambda\mapsto I_\lambda(x_{i\lambda})$, $i=1,2,3$, decreases strictly in $\lambda\in\Lambda$. Since $\lambda<\lambda^*$ is arbitrary, we obtain that $\lambda\mapsto I_\lambda(x_{i\lambda})$ decreases strictly in $\lambda\in\Lambda$.

For positive constants θ_1, θ_4, and d, set

$$\delta'_{\lambda,g}:=\min\left\{\frac{1}{2C^2}\min\left\{\frac{\theta_1^2-\lambda\int_a^{\sigma(S)}F(t,\theta_1)\Delta t}{G^{\theta_1}},\frac{\theta_4^2-2\lambda\int_0^{\sigma(S)}F(t,\frac{1}{\sqrt{2}}\theta_4)\Delta t}{2G^{\frac{1}{\sqrt{2}}\theta_4}},\frac{\theta_4^2-2\lambda\int_0^{\sigma(S)}F(t,\theta_4)\Delta t}{2G^{\theta_4}}\right\},\right.$$
$$\left.\frac{K_d-\lambda\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t}{G_d-G^{\theta_1}}\right\}. \tag{20}$$

Theorem 8. *Let $f:[0,S]_{\mathbb{T}}\times[0,\infty)\to[0,\infty)$ be continuous. Assume the existence of θ_1, θ_4, $d>0$ such that*

$$\theta_1<C\sqrt{2K_d}<\frac{\theta_4}{\sqrt{2}}$$

and

$$(\mathrm{A}_2)\ \max\left\{\frac{\int_0^{\sigma(S)}F(t,\theta_1)\Delta t}{\theta_1^2},\ \frac{2\int_0^{\sigma(S)}F(t,\theta_4)\Delta t}{\theta_4^2}\right\}<\frac{1}{4C^2}\frac{\int_0^{\sigma(S)}F(t,d)\Delta t}{K_d}.$$

Then, for every

$$\lambda\in\left(\frac{K_d}{\int_0^{\sigma(S)}F(t,d)\Delta t},\frac{1}{4C^2}\min\left\{\frac{\theta_1^2}{\int_0^{\sigma(S)}F(t,\theta_1)\Delta t},\frac{\theta_4^2}{2\int_0^{\sigma(S)}F(t,\theta_4)\Delta t}\right\}\right)$$

and every continuous $g:[0,S]_{\mathbb{T}}\times\mathbb{R}\to[0,\infty)$, there is $\delta'_{\lambda,g}>0$ defined by (20) such that, for all $\zeta\in[0,\delta'_{\lambda,g})$, $(P_{\lambda,\zeta})$ admits at least three nonnegative solutions $x_1,x_2,x_3\in\mathcal{H}$ satisfying

$$\max_{t\in[0,S]_{\mathbb{T}}}|x_1(t)|<\theta_1,\ \max_{t\in[0,S]_{\mathbb{T}}}|x_2(t)|<\frac{\theta_4}{\sqrt{2}},\ \max_{t\in[0,S]_{\mathbb{T}}}|x_3(t)|<\theta_4.$$

Proof. Choose $\theta_2=\frac{\theta_4}{\sqrt{2}}$ and $\theta_3=\theta_4$. So, from (A_2) one has

$$\frac{\int_0^{\sigma(S)}F(t,\theta_2)\Delta t}{\theta_2^2}=\frac{2\int_0^{\sigma(S)}F(t,\frac{\theta_4}{\sqrt{2}})\Delta t}{\theta_4^2}\le\frac{2\int_a^{\sigma(S)}F(t,\theta_4)\Delta t}{\theta_4^2}<\frac{1}{4C^2}\frac{\int_0^{\sigma(S)}F(t,d)\Delta t}{K_d} \tag{21}$$

and

$$\frac{\int_0^{\sigma(S)}F(t,\theta_3)\Delta t}{\theta_3^2-\theta_2^2}=\frac{2\int_0^{\sigma(S)}F(t,\theta_4)\Delta t}{\theta_4^2}<\frac{1}{4C^2}\frac{\int_0^{\sigma(S)}F(t,d)\Delta t}{K_d}. \tag{22}$$

Moreover, taking into account that $\theta_1<C\sqrt{2K_d}<\frac{\theta_4}{\sqrt{2}}$, by using (A_2), we have

$$\frac{1}{2C^2}\frac{\int_0^{\sigma(S)}(F(t,d)-F(t,\theta_1))\Delta t}{K_d}>\frac{1}{2C^2}\frac{\int_0^{\sigma(S)}F(t,d)\Delta t}{K_d}-\frac{\int_0^{\sigma(S)}F(t,\theta_1)\Delta t}{\theta_1^2}>\frac{1}{2C^2}\left(\frac{\int_0^{\sigma(S)}F(t,d)\Delta t}{K_d}-\frac{2C^2}{4C^2}\frac{\int_0^{\sigma(S)}F(t,d)\Delta t}{K_d}\right)$$
$$=\frac{1}{4C^2}\frac{\int_0^{\sigma(S)}F(t,d)\Delta t}{K_d}.$$

Hence, from (A_2), (21), and (22), we observe that (A_1) of Theorem 5 is fulfilled, completing the proof. □

The following two results are special cases of Theorem 8.

Theorem 9. *Let $f_1\in\mathrm{L}^1([0,S]_{\mathbb{T}})$ and $f_2\in\mathrm{C}(\mathbb{R})$. Put $\tilde{F}(\xi)=\int_0^{\xi}f_2(s)ds$, $\xi\in\mathbb{R}$, and assume the existence of θ_1, θ_4, $d>0$ with*

$$\theta_1<C\sqrt{2K_d}<\frac{\theta_4}{\sqrt{2}}$$

and

(A_3) *$f_1(t)\ge 0$ for all $t\in[0,S]_{\mathbb{T}}$ and $f_2(\xi)\ge 0$ for each $\xi\in[0,\infty)$,*

$$(\mathrm{A}_4)\ \max\left\{\frac{\tilde{F}(\theta_1)}{\theta_1^2},\ \frac{2\tilde{F}(\theta_4)}{\theta_4^2}\right\}<\frac{1}{4C^2}\frac{\tilde{F}(d)}{K_d}.$$

Fig. 1. $\mathbb{T}$ in Example 11.

Then, for every

$$\lambda \in \left(\frac{K_d}{\tilde{F}(d) \int_0^{\sigma(S)} f_1(t)\Delta t}, \frac{1}{4C^2 \int_0^{\sigma(S)} f_1(t)\Delta t} \min\left\{ \frac{\theta_1^2}{\tilde{F}(\theta_1)}, \frac{\theta_4^2}{2\tilde{F}(\theta_4)} \right\} \right)$$

and every continuous $g : [0, S]_{\mathbb{T}} \times \mathbb{R} \to [0, \infty)$, *whenever*

$$\zeta \in \left[0, \min\left\{ \frac{1}{2C} \min\left\{ \frac{\theta_1^2 - \lambda\tilde{F}(\theta_1)\int_0^{\sigma(S)} f_1(t)\Delta t}{G^{\theta_1}}, \frac{\theta_4^2 - 2\lambda\tilde{F}(\frac{\theta_4}{\sqrt{2}})\int_0^{\sigma(S)} f_1(t)\Delta t}{2G^{\frac{\theta_4}{\sqrt{2}}}}, \frac{\theta_4^2 - 2\lambda\tilde{F}(\theta_4)\int_0^{\sigma(S)} f_1(t)\Delta t}{2G^{\theta_4}} \right\}, \frac{K_d - \lambda\int_0^{\sigma(S)} f_1(t)\Delta t - \left(\tilde{F}(d) - \tilde{F}(\theta_1)\right)}{G_d - G^{\theta_1}} \right\} \right],$$

the problem

$$\begin{cases} -(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = \lambda f_1(t) f_2(x^{\sigma}(t)) + \zeta g(t, x^{\sigma}(t)), & t \in [0, S]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^{\Delta}(0) = 0, \quad \alpha_3 x(\sigma^2(S)) + \alpha_4 x^{\Delta}(\sigma(S)) = 0 \end{cases} \qquad (P_\lambda)$$

admits at least three nonnegative solutions $x_1, x_2, x_3 \in \mathcal{H}$ *satisfying*

$$\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_1(t)| < \theta_1, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_2(t)| < \frac{\theta_4}{\sqrt{2}}, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_3(t)| < \theta_4.$$

Proof. Put $f(t,x) = f_1(t)f_2(x)$ for $(t,x) \in [0,S]_{\mathbb{T}} \times \mathbb{R}$. Since $F(t,x) = f_1(t)\tilde{F}(x)$ for all $(t,x) \in [0,S]_{\mathbb{T}} \times \mathbb{R}$, from (A_4), we obtain (A_2). □

Theorem 10. *Assume the existence of* θ_1, θ_4, $d > 0$ *with*

$$\theta_1 < C\sqrt{2K_d} < \frac{\theta_4}{\sqrt{2}}$$

and

(A_5) *$f(\xi) \geq 0$ for all $\xi \in [0, \infty)$,*
(A_6) $\max\left\{ \frac{F(\theta_1)}{\theta_1^2}, \frac{2F(\theta_4)}{\theta_4^2} \right\} < \frac{1}{4C^2}\frac{F(d)}{K_d}$.

Then, for every

$$\lambda \in \left(\frac{K_d}{\sigma(S)F(d)}, \frac{1}{4C^2\sigma(S)} \min\left\{ \frac{\theta_1^2}{F(\theta_1)}, \frac{\theta_4^2}{2F(\theta_4)} \right\} \right)$$

and every continuous $g : [0, S]_{\mathbb{T}} \times \mathbb{R} \to [0, \infty)$, *whenever*

$$\zeta \in \left[0, \min\left\{ \frac{1}{2C^2} \min\left\{ \frac{\theta_1^2 - \lambda\sigma(S)F(\theta_1)}{G^{\theta_1}}, \frac{\theta_4^2 - 2\lambda\sigma(S)F(\frac{1}{\sqrt{2}}\theta_4)}{2G^{\frac{1}{\sqrt{2}}\theta_4}} \right\}, \frac{K_d - \lambda\sigma(S)(F(d) - F(\theta_1))}{G_d - G^{\theta_1}} \right\} \right],$$

the problem

$$\begin{cases} -(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = \lambda f(x^{\sigma}(t)) + \zeta g(t, x^{\sigma}(t)), & t \in [0, S]_{\mathbb{T}}, \\ \alpha_1 x(0) - \alpha_2 x^{\Delta}(0) = 0, \quad \alpha_3 x(\sigma^2(S)) + \alpha_4 x^{\Delta}(\sigma(S)) = 0 \end{cases} \qquad (23)$$

admits at least three nonnegative solutions $x_1, x_2, x_3 \in \mathcal{H}$ *satisfying*

$$\max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_1(t)| < \theta_1, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_2(t)| < \frac{1}{\sqrt{2}}\theta_4, \quad \max_{t\in[0,\sigma^2(S)]_{\mathbb{T}}} |x_3(t)| < \theta_4.$$

In the following example, all assumptions of Theorem 10 are fulfilled.

Example 11. Consider the nontrivial time scale (see Figure 1)

$$\mathbb{T} = \left\{ 1 - \frac{1}{2^n} : n \in \mathbb{N}_0 \right\} \cup [1, 2] \cup \{3\}.$$

Let $p(t)\equiv 1$ and $q(t)\equiv 1$ on $\mathbb{T}$. Let

$$S=1,\quad \alpha_1=1,\quad \alpha_2=2,\quad \alpha_3=1,\quad \alpha_4=4,$$

and

$$f(\xi)=\begin{cases}13\xi^{12}, & \xi\le 1,\\ \dfrac{13}{\xi}, & \xi>1.\end{cases}$$

Hence, we have

$$\sigma(S)=\sigma^2(S)=1,\quad \beta_1=\frac{1}{2},\quad \beta_2=\frac{1}{4},\quad C=\min\left\{2,2\sqrt{2},\sqrt{2}\right\}=\sqrt{2},$$

and

$$F(\xi)=\begin{cases}\xi^{13}, & \xi\le 1,\\ 1+13\ln(\xi), & \xi>1.\end{cases}$$

We now choose

$$\theta_1=10^{-8},\quad \theta_4=10^3,\quad \text{and}\quad d=10.$$

Now, it is easy to check that all assumptions of Theorem 10 are fulfilled. Thus, for every

$$\lambda\in\left(\frac{87.5}{1+13\ln 10},\frac{10^6}{16(1+39\ln 10)}\right)$$

and every continuous $g:[0,1]_{\mathbb{T}}\times\mathbb{R}\to[0,\infty)$, whenever

$$\zeta\in\left[0,\min\left\{\frac{1}{4}\min\left\{\frac{10^{-16}-\lambda 10^{-104}}{G^{10^{-8}}},\frac{10^6-2\lambda\left(1+13\ln\left(\frac{1000}{\sqrt{2}}\right)\right)}{2G^{\frac{1000}{\sqrt{2}}}}\right\},\frac{87.5-\lambda\left(1+13\ln 10-10^{-104}\right)}{G_{10}-G^{10^{-8}}}\right\}\right],$$

the problem

$$\begin{cases}-x^{\Delta\Delta}(t)+x^{\sigma}(t)=\lambda f(x^{\sigma}(t))+\zeta g(t,x^{\sigma}(t)), & t\in[0,1]_{\mathbb{T}},\\ x\left(\dfrac{1}{2}\right)=2x^{\Delta}(0),\quad x(1)+4x^{\Delta}(1)=0\end{cases}\tag{24}$$

admits at least three nonnegative solutions x_1, x_2, and x_3 satisfying

$$\max_{t\in[0,1]_{\mathbb{T}}}|x_1(t)|<10^{-8},\ \max_{t\in[0,1]_{\mathbb{T}}}|x_2(t)|<\frac{1000}{\sqrt{2}},\ \max_{t\in[0,1]_{\mathbb{T}}}|x_3(t)|<10^3.$$

Our final result is concerned with the case $\zeta=0$.

Theorem 12. *Let $f:[0,S]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$ be continuous such that $\xi f(t,\xi)>0$ for all $(t,\xi)\in[0,S]_{\mathbb{T}}\times(\mathbb{R}\setminus\{0\})$. Suppose*

(A_7) $\displaystyle\lim_{\xi\to 0}\frac{f(t,\xi)}{|\xi|}=\lim_{|\xi|\to\infty}\frac{f(t,\xi)}{|\xi|}=0.$

Then, for all

$$\lambda>\lambda^{**}:=\max\left\{\inf_{d>0}\frac{K_d}{\int_0^{\sigma(S)}F(t,d)\Delta t},\ \inf_{d<0}\frac{K_d}{\int_0^{\sigma(S)}F(t,d)\Delta t}\right\},$$

the problem $(P_{\lambda,0})$ admits at least four distinct nontrivial solutions.

Proof. Put

$$f_1(t,\xi)=\begin{cases}f(t,\xi) & \text{if } (t,\xi)\in[0,S]_{\mathbb{T}}\times[0,\infty),\\ 0 & \text{otherwise}\end{cases}$$

and

$$f_2(t,\xi)=\begin{cases}-f(t,-\xi) & \text{if } (t,\xi)\in[0,S]_{\mathbb{T}}\times[0,\infty),\\ 0 & \text{otherwise}\end{cases}$$

and set $F_1(t,\xi):=\int_0^t f_1(t,s)ds$ for every $(t,\xi)\in[0,S]_{\mathbb{T}}\times\mathbb{R}$. Take $\lambda>\lambda^{**}$ and $d>0$ with $\lambda>\frac{K_d}{\int_0^{\sigma(S)}F_1(t,d)\Delta t}$. From

$$\lim_{t\to 0^+}\frac{f_1(t,\xi)}{|\xi|}=\lim_{t\to\infty}\frac{f_1(t,\xi)}{|\xi|}=0,$$

there exist θ_1 and θ_4 such that

$$\theta_1 < C\sqrt{2K_d} < \frac{\theta_4}{\sqrt{2}}, \quad \frac{\int_a^{\sigma(S)} F_1(t,\theta_1)\Delta t}{\theta_1^2} < \frac{1}{4C^2\lambda}, \quad \frac{\int_0^{\sigma(S)} F_1(t,\theta_4)\Delta t}{\theta_4^2} < \frac{1}{8C^2\lambda}.$$

Then, (A_2) in Theorem 8 is fulfilled, and

$$\lambda \in \left(\frac{K_d}{\int_0^{\sigma(S)} F_1(t,d)\Delta t}, \frac{1}{4C^2} \min\left\{ \frac{\theta_1^2}{\int_0^{\sigma(S)} F_1(t,\theta_1)\Delta t}, \frac{\theta_4^2}{2\int_0^{\sigma(S)} F_1(t,\theta_4)\Delta t} \right\} \right).$$

Hence, the problem $(P^{f_1}_{\lambda,0})$ admits two positive solutions x_1 and x_2, and they are positive solutions of $(P_{\lambda,0})$. Next, by the same arguments, from

$$\lim_{t\to 0^+} \frac{f_2(t,\xi)}{|\xi|} = \lim_{\xi\to\infty} \frac{f_2(t,\xi)}{|\xi|} = 0,$$

we guarantee existence of two positive solutions x_3 and x_4 for $(P^{f_2}_{\lambda,0})$. Clearly, $-x_3$ and $-x_4$ are negative solutions of $(P_{\lambda,0})$, and the proof is complete. □

Remark 13. We remark that in Theorem 12, f is not assumed to be symmetric. But, if $f \not\equiv 0$ is odd and continuous satisfying $f(t,\xi) \geq 0$ for all $(t,\xi) \in [0,S]_{\mathbb{T}} \times [0,\infty)$, then (A_7) may be substituted by

$$(\mathrm{A}_8) \lim_{\xi\to 0^+} \frac{f(t,\xi)}{\xi} = \lim_{\xi\to\infty} \frac{f(t,\xi)}{\xi} = 0,$$

guaranteeing existence of at least four distinct nontrivial solutions of $(P_{\lambda,0})$ for every $\lambda > \inf_{d>0} \frac{K_d}{\int_0^{\sigma(S)} F(t,d)\Delta t}$.

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