Oscillation Theory for Second Order Dynamic Equations. *By Ravi P. Agarwal, Said R. Grace, and Donal O'Regan.* Taylor & Francis, Ltd., London, 2003. $80.00. viii+404 pp., hardcover. ISBN 0-415-30074-6.

There are numerous books on oscillation theory for differential equations such as [1, 2, 5, 6], to name but a few. The monograph by Agarwal, Grace, and O'Regan is an excellent addition to the existing literature. It covers topics related to oscillation theory for differential equations with deviating arguments, neutral functional differential equations, second order ordinary differential equations, and impulsive differential equations.

This book is very well organized; being in the classical mathematical style, the presented material is divided into definitions, theorems, proofs, lemmas, remarks, and examples. It is self-contained and consistent, and it is easy to follow the presentation. Only some background in calculus and differential equations is required. One definitely does not have to be a differential equations specialist in order to follow this well-written compact but thorough treatment.

In an effort to make this book available to a wide audience of scientists, a first chapter collects some basic results for second order linear ordinary differential equations such as Sturm's comparison theorem and Picone's identity, and several fixed point theorems are presented that are used in the remainder of the book, e.g., Brower's, Schauder's, Banach's, Krasnosel′skiĭ's, and Knaster's fixed point theorem. The remaining four chapters deal with criteria for oscillation, nonoscillation, and various other asymptotic behavior for solutions of the studied second order differential equations.

This book can easily be used as a textbook for a special topics course at the graduate level, and it can also serve as an encyclopedia and reference book on oscillation theory for researchers in various fields. Finally, it can be an inspiring source for advisors and graduate students who are seeking topics for their theses. Even though some of the material has been studied for a long time, there are still many open problems, and research in the area is continuing and even rapidly growing. A search on MathSciNet using the word "oscillation" in the title produces 8707 hits. The three authors of this monograph, Professors Agarwal, Grace, and O'Regan, are well-known and very active experts in this and many other areas.

The title of the book can be interpreted in several ways. The authors use the expression "dynamic equation" in the sense of a collection of differential equations such as delay, neutral, functional, ordinary, and impulsive differential equations. On the other hand, the study of "dynamic equations" is indeed a mathematical subject, not very well known among the mathematical community due to its recent origin, but this subject is also connected to the material presented in the monograph under review. In fact, dynamic equations on so-called "time scales" have been introduced in order to unify the theories of differential and difference equations and in order to extend them to cases "in between".

A time scale $\mathbb{T}$ is an arbitrary closed subset of the real numbers $\mathbb{R}$, and for functions $f : \mathbb{T} \to \mathbb{R}$ a delta derivative f^Δ may be defined which has the following properties: When $\mathbb{T} = \mathbb{R}$, f^Δ equals f', the ordinary derivative. When $\mathbb{T} = \mathbb{Z}$ (the set of integers), f^Δ equals Δf, the ordinary forward difference. Two important objects in the study of time scales that are unbounded above are the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the graininess operator $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$. Now one can study equations that involve unknown functions and their delta derivatives, i.e., so-called dynamic equations. For example, one of the main equations that is studied in the book under review, in particular in Chapter 4, namely

$$(a(t)x')' + b(t)x = 0, \tag{1}$$

could be studied in the form of a dynamic equation as

$$(a(t)x^\Delta)^\Delta + b(t)x = 0. \tag{2}$$

However, with such an equation one would be unable to prove classical analoga for the differential equation (1) in the dynamic equations setting. It turns out that even the well-studied discrete version of (1),

$$a_{k+1}x_{k+2} - (a_k + a_{k+1} - b_k)\, x_{k+1} + a_k x_k = 0, \tag{3}$$

does not fit into the form of (2). Instead one should consider the equation

$$(a(t)x^{\Delta})^{\Delta} + b(t)x^{\sigma} = 0, \tag{4}$$

where x^σ is defined as $x \circ \sigma$. For dynamic equations of the form (4), it is in fact possible to obtain standard results and oscillation criteria that have results for differential equations (such as those presented in the book under review) and also results for corresponding difference equations as special cases. Furthermore, such dynamic equations contain not only differential and difference equations as special cases but also much different kinds of equations because a time scale is allowed to be any closed subset of the real numbers. For example, so-called q-difference equations are included, and when $\mathbb{T}$ is the collection of nonnegative integer powers of a number $q > 1$, (4) takes the form

$$a(qt)x(q^2t) - \big(qa(t) + a(qt) - q(q-1)^2t^2b(t)\big)\, x(qt) + qa(t)x(t) = 0, \tag{5}$$

where $t \in \mathbb{T}$.

Oscillation of the dynamic equation (4) has been studied intensively in the recent literature (see [3, 4]), but the current book under review does not contain any of those results; in particular, it does not deal with difference equations. For difference equations, see the monograph [1] by the same three authors, Agarwal, Grace, and O'Regan. The book under review deals with differential equations solely, and hence the connection to dynamic equations is only a remote one in the sense that differential equations are special cases of dynamic equations for $\mathbb{T} = \mathbb{R}$.

Now, returning to the actual topic of the monograph by Agarwal, Grace, and O'Regan, the book covers oscillation results for differential equations of the following forms. In Chapter 2, the second order functional differential equation

$$(a(t)x')' + q(t)f(x(g(t))) = 0 \tag{6}$$

is considered, where the occuring functions are assumed to satisfy certain conditions; f may be a nonlinear function, a is positive, and q is either positive or negative. In later sections of this chapter, damping and forcing terms are added so that the most general form appears as

$$(a(t)x')' + p(t)x' + F(t, x(g(t))) = e(t). \tag{7}$$

Special cases such as the Emden–Fowler or the Klein–Gordon equation are treated as well. Chapter 3 covers nonlinear neutral differential equations such as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t) + p(t)x(\tau(t))\right)\right) + q(t)F(x(g(t))) = 0, \tag{8}$$

and again, in later sections of this chapter, damping and forcing terms are added to the equation (8). Equations with mixed type, i.e., both advanced and retarded arguments occur, are also studied. Chapter 4 is concerned with the classical self-adjoint differential equation (1). Standard results of this equation are already contained in Chapter 1, so Chapter 4 offers some results connected to conjugacy (rather than disconugacy) that are not easily accesible in the literature. The Emden–Fowler equation in both the sublinear and superlinear cases is studied as well. Finally, Chapter 5 offers a treatment of impulsive delay differential equations in the form

$$x'(t) + q(t)x(t-\tau) = 0,\ t \neq t_k;\ x(t_k^+) - x(t_k) = p_k x(t_k), \tag{9}$$

where the numbers t_k are the impulse points.

Each chapter concludes with a very nice section on "Notes and General Discussions", followed by the bibliography for that particular chapter (i.e., this book has five different lists of references). An extremely important and nice feature is the inclusion of 82 examples throughout the book that serve well to illustrate the presented theory. This monograph opens research directions in various ways: One could try to find analogues of the results presented for difference equations or q-difference equations, or in more generality, for dynamic equations on time scales. The discrete case has already been well studied, but although first steps and progress have been made for dynamic equations, it will take quite a while until time scale results corresponding to the continuous theory presented in this book are really investigated and an "oscillation theory for second order dynamic equations" is fully developed.

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