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# DISCRETE, CONTINUOUS, DELTA, NABLA, AND DIAMOND–ALPHA OPIAL INEQUALITIES

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*Abstract.* In this paper, we prove some new diamond-alpha dynamic inequalities of Opial type with one and with two weight functions on time scales. These results contain as special cases improvements of results given in the literature, and these improvements are new even in the important discrete case.

## 1. Introduction

In 1960, Opial [13] published an inequality, involving integrals of functions and their derivatives, of the form

$$\int_0^h |f(t)f'(t)|\mathrm{d}t \leqslant \frac{h}{4}\int_0^h |f'(t)|^2\mathrm{d}t, \tag{1.1}$$

where $f\in \mathrm{C}^1[0,h]$, $f(0)=f(h)=0$ and $f>0$ on $(0,h)$, and the constant $h/4$ is the best possible. Olech [12] extended the inequality (1.1) and proved that

$$\int_0^h |f(t)f'(t)|\mathrm{d}t \leqslant \frac{h}{2}\int_0^h |f'(t)|^2\mathrm{d}t,$$

where $f$ is absolutely continuous on $[0,h]$ and $f(0)=0$. The discrete analogues of Opial-type inequalities started in the papers by Beesack [4], Lasota [10], and Wong [24]. The Lasota inequality is given by

$$\sum_{i=1}^{N-1}|x_i\Delta x_i| \leqslant \frac{1}{2}\left[\frac{N+1}{2}\right]\sum_{i=1}^{N-1}|\Delta x_i|^2,$$

where $\{x_i\}_{i=0}^N$ is a sequence of numbers satisfying $x_0=x_N=0$, $\Delta$ is the forward difference operator, and $[\cdot]$ is the greatest integer function. Opial-type inequalities have various significant applications in the study of differential equations, difference equations, approximations, and probability. We refer the reader to the monograph [2] and the papers [14–16, 20–22] for further details.

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The study of dynamic inequalities on time scales has recently received a lot of attention and is becoming a major field in pure and applied mathematics. The general idea is to prove an inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$. For example, in [6] (see also [7, Theorem 6.23]), the authors proved the dynamic Opial-type inequality

$$\int_0^h |(f^2)^\Delta(t)|\Delta t \leqslant h\int_0^h (f^\Delta)^2(t)\Delta t, \tag{1.2}$$

where $f\in \mathrm{C}^1_{\mathrm{rd}}([0,h]_{\mathbb{T}},\mathbb{R})$ satisfies $f(0)=0$. This inequality contains the continuous and the discrete inequalities as special cases and has been extended in [6, 9]. For applications of dynamic Opial-type inequalities on time scales, we refer to the papers [17–19]. For $f\in \mathrm{C}^1_{\mathrm{ld}}([0,h]_{\mathbb{T}},\mathbb{R})$ satisfying $f(0)=0$, we have

$$\int_0^h |(f^2)^\nabla(t)|\nabla t \leqslant h\int_0^h (f^\nabla)^2(t)\nabla t. \tag{1.3}$$

Note that equality in (1.2) and (1.3) holds when $f(t)=ct$, since if $f(t)=ct$ for some $c\in\mathbb{R}$, then $f^\Delta(t)=f^\nabla(t)=c$.

One question arising is the following: Is it possible to prove a new inequality which as special cases contains both inequalities (1.2) and (1.3)? The answer of this question has been partially solved in [3, 5] by using the diamond-alpha calculus on time scales that has been developed in [23]. This calculus gives a combination between the delta and the nabla calculus and will be discussed in Section 2. In [5, Theorem 3.2], the authors proved: If $f\in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}},\mathbb{R})$ with $f(0)=0$ and $\alpha(1-\alpha)f^\Delta f^\nabla \geqslant 0$, then

$$\alpha^3\int_0^h |(f^2)^\Delta(t)|\Delta t + (1-\alpha)^3\int_0^h |(f^2)^\nabla(t)|\nabla t \leqslant h\int_0^h (f^{\diamondsuit_\alpha})^2(t)\diamondsuit_\alpha t, \tag{1.4}$$

where $\alpha\in[0,1]$ and $h\in\mathbb{T}$ with $h>0$. This inequality (1.4) has been extended to an inequality with a weight function $g$ in the form [5, Theorem 3.4]

$$\alpha^3\int_0^h g(\sigma(t))|(f^2)^\Delta(t)|\Delta t + (1-\alpha)^3\int_0^h g(\rho(t))|(f^2)^\nabla(t)|\nabla t \leqslant h\int_0^h g(t)(f^{\diamondsuit_\alpha})^2(t)\diamondsuit_\alpha t,$$

where in addition to the assumptions for (1.4), $g\in \mathrm{C}([0,h]_{\mathbb{T}},(0,\infty))$ is assumed to be nonincreasing. In [3], related results are given.

Following this trend, in this paper we will prove some new diamond-alpha dynamic inequalities of Opial type on time scales via the concepts of diamond-alpha differentiability and diamond-alpha integrability. Our results contain as special cases some improvements of results proved in [9, 18, 25]. The paper is organized as follows: In Section 2, we give some basics of diamond-alpha calculus on time scales which will be used throughout the paper. In Sections 3–5, we state and prove our main results and discuss some special cases. The main results may help in studying the distribution of zeros of solutions of diamond-alpha dynamic equations of the form (see [19])

$$x^{\diamondsuit_\alpha\diamondsuit_\alpha}(t)+p(t)x(t)=0.$$

## 2. Diamond-alpha calculus on time scales

In this section, we briefly introduce the diamond-alpha dynamic derivative and diamond-alpha dynamic integration, which are combinations of the delta and the nabla derivatives and integrations. The development of the calculus of the $\diamondsuit_\alpha$-derivative and $\diamondsuit_\alpha$-integration is given in [23] (see also [11]). The foundations of calculus on time scales can be found in [7, 8].

The definition and some properties related to $\diamondsuit_\alpha$-derivatives are given next.

DEFINITION 2.1. Let $\mathbb{T}$ be a time scale and $f:\mathbb{T}\to\mathbb{R}$ be delta-differentiable and nabla-differentiable. Then we define the $\diamondsuit_\alpha$-derivative $f^{\diamondsuit_\alpha}$ by

$$f^{\diamondsuit_\alpha} = \alpha f^{\Delta} + (1-\alpha) f^{\nabla}, \quad \text{where} \quad 0 \leqslant \alpha \leqslant 1.$$

The function $f$ is said to be in the class $\mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}},\mathbb{R})$ if $f$ is $\diamondsuit_\alpha$-differentiable such that $\alpha f^{\Delta}$ is rd-continuous, $(1-\alpha)f^{\nabla}$ is ld-continuous, and $\alpha(1-\alpha)f^{\diamondsuit_\alpha}$ is continuous.

REMARK 2.2. The $\diamondsuit_\alpha$-derivative reduces to the standard $\Delta$-derivative for $\alpha=1$ while it reduces to the standard $\nabla$-derivative for $\alpha=0$. On the other hand, the $\diamondsuit_\alpha$-derivative represents a weighted dynamic derivative for $\alpha\in(0,1)$. Furthermore, the combined dynamic derivative offers a centralized derivative for any discrete time scale $\mathbb{T}$ when $\alpha=\frac{1}{2}$.

THEOREM 2.3. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be $\diamondsuit_\alpha$-differentiable. Then*

(*i*) *$f+g$ is $\diamondsuit_\alpha$-differentiable with*

$$(f+g)^{\diamondsuit_\alpha} = f^{\diamondsuit_\alpha} + g^{\diamondsuit_\alpha};$$

(*ii*) *$fg$ is $\diamondsuit_\alpha$-differentiable with*

$$(fg)^{\diamondsuit_\alpha} = f^{\diamondsuit_\alpha} g + \alpha f^{\sigma} g^{\Delta} + (1-\alpha) f^{\rho} g^{\nabla};$$

(*iii*) *$f/g$ is $\diamondsuit_\alpha$-differentiable with*

$$\left(\frac{f}{g}\right)^{\diamondsuit_\alpha} = \frac{f^{\diamondsuit_\alpha} g^{\sigma} g^{\rho} - \alpha f^{\sigma} g^{\rho} g^{\Delta} - (1-\alpha) f^{\rho} g^{\sigma} g^{\nabla}}{g g^{\sigma} g^{\rho}}$$

*provided $g(t)g^{\sigma}(t)g^{\rho}(t)\neq 0$ for all $t\in\mathbb{T}$.*

The definition and some properties related to $\diamondsuit_\alpha$-integrals are given next.

DEFINITION 2.4. Let $a,b\in\mathbb{T}$ and suppose that $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is delta-integrable and nabla-integrable on $[a,b]_{\mathbb{T}}$. Then the $\diamondsuit_\alpha$-integral of $f$ on $[a,b]_{\mathbb{T}}$ is defined by

$$\int_a^b f(t)\diamondsuit_\alpha t = \alpha\int_a^b f(t)\Delta t + (1-\alpha)\int_a^b f(t)\nabla t, \quad \text{where} \quad 0\leqslant\alpha\leqslant 1.$$

REMARK 2.5. In general, we do not have

$$F^{\diamondsuit_\alpha} = f, \quad \text{where} \quad F(t) = \int_a^t f(s)\diamondsuit_\alpha s.$$

THEOREM 2.6. *Let $a,b,c \in \mathbb{T}$, $\beta \in \mathbb{R}$ and $f,g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be continuous. Then*

(i) $\int_a^b (f+g)(t)\diamondsuit_\alpha t = \int_a^b f(t)\diamondsuit_\alpha t + \int_a^b g(t)\diamondsuit_\alpha t$;

(ii) $\int_a^b (\beta f)(t)\diamondsuit_\alpha t = \beta \int_a^b f(t)\diamondsuit_\alpha t$;

(iii) $\int_a^b f(t)\diamondsuit_\alpha t = -\int_b^a f(t)\diamondsuit_\alpha t$;

(iv) $\int_a^b f(t)\diamondsuit_\alpha t = \int_a^c f(t)\diamondsuit_\alpha t + \int_c^b f(t)\diamondsuit_\alpha t$ *for* $c \in [a,b]_{\mathbb{T}}$;

(v) $f(t) \geqslant g(t)$ *for all* $t \in [a,b]_{\mathbb{T}}$ *implies* $\int_a^b f(t)\diamondsuit_\alpha t \geqslant \int_a^b g(t)\diamondsuit_\alpha t$.

EXAMPLE 2.7. If we let $\mathbb{T} = \mathbb{R}$ and $a,b \in \mathbb{R}$ with $a \leqslant b$, then we obtain

$$\int_a^b f(t)\diamondsuit_\alpha t = \int_a^b f(t)\mathrm{d}t,$$

and if we let $\mathbb{T} = \mathbb{Z}$ and $m,n \in \mathbb{N}_0$ with $m < n$, then we obtain

$$\int_m^n f(t)\diamondsuit_\alpha t = \sum_{i=m}^{n-1} [\alpha f(i) + (1-\alpha)f(i+1)]. \tag{2.1}$$

EXAMPLE 2.8. If we let $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$ and $m,n \in \mathbb{N}_0$ with $m < n$, then we obtain

$$\int_{q^m}^{q^n} f(t)\diamondsuit_\alpha t = (q-1)\sum_{i=m}^{n-1} q^i \left[\alpha f(q^i) + (1-\alpha)f(q^{i+1})\right], \tag{2.2}$$

and if we let $\mathbb{T} = \{t_i : i \in \mathbb{N}_0\}$ such that $t_i < t_{i+1}$ and $m,n \in \mathbb{N}_0$ with $m < n$, then we obtain the generalization of both (2.1) and (2.2)

$$\int_{t_m}^{t_n} f(t)\diamondsuit_\alpha t = \sum_{i=m}^{n-1} (t_{i+1} - t_i)\left[\alpha f(t_i) + (1-\alpha)f(t_{i+1})\right].$$

## 3. Generalizations of results by Zhao et al

In this section, we generalize some delta Opial inequalities obtained by Zhao, Xu, and Li [25] to the diamond-alpha case. The main result utilized here is [25, Theorem 4.1], see (3.2) at the beginning of the proof of Theorem 3.1 below.

### 3.1. Zero initial condition

We first consider the case when $f(0)=0$.

THEOREM 3.1. *Assume that* $p>1$, $q=p/(p-1)$, $\alpha\in[0,1]$, $h\in(0,\infty)_{\mathbb{T}}$,

$$\omega\in \mathrm{C}([0,h]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f\in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha f^{\Delta}\geqslant 0$, $(1-\alpha)f^{\nabla}\geqslant 0$, *and* $f(0)=0$, *then*

$$\alpha^4\int_0^h |(f^2)^{\Delta}(t)|\Delta t+(1-\alpha)^4\int_0^h |(f^2)^{\nabla}(t)|\nabla t \leqslant \left(\int_0^h \omega^{1-q}(t)\diamondsuit_\alpha t\right)^{\frac{2}{q}}\left(\int_0^h \omega(t)|f^{\diamondsuit_\alpha}(t)|^p\diamondsuit_\alpha t\right)^{\frac{2}{p}}. \tag{3.1}$$

*Proof.* By [25, Theorem 4.1], we see that

$$\int_0^h |(f^2)^{\Delta}(t)|\Delta t\leqslant \left(\int_0^h \omega^{1-q}(t)\Delta t\right)^{\frac{2}{q}}\left(\int_0^h \omega(t)|f^{\Delta}(t)|^p\Delta t\right)^{\frac{2}{p}}. \tag{3.2}$$

For the nabla case, we obtain similarly

$$\int_0^h |(f^2)^{\nabla}(t)|\nabla t\leqslant \left(\int_0^h \omega^{1-q}(t)\nabla t\right)^{\frac{2}{q}}\left(\int_0^h \omega(t)|f^{\nabla}(t)|^p\nabla t\right)^{\frac{2}{p}}. \tag{3.3}$$

From the definition of $f^{\diamondsuit_\alpha}$ and since $\alpha f^{\Delta}\geqslant 0$ and $(1-\alpha)f^{\nabla}\geqslant 0$, we have

$$|\alpha f^{\Delta}|^p\leqslant |f^{\diamondsuit_\alpha}|^p \quad \text{and} \quad |(1-\alpha)f^{\nabla}|^p\leqslant |f^{\diamondsuit_\alpha}|^p. \tag{3.4}$$

To simplify notation, we introduce

$$a=\alpha\int_0^h \omega^{1-q}(t)\Delta t, \quad b=(1-\alpha)\int_0^h \omega^{1-q}(t)\nabla t,$$
$$c=\alpha\int_0^h \omega(t)|f^{\diamondsuit_\alpha}(t)|^p\Delta t, \quad d=(1-\alpha)\int_0^h \omega(t)|f^{\diamondsuit_\alpha}(t)|^p\nabla t.$$

Now, using (3.2), (3.3), (3.4), Hölder's inequality [7, Theorem 6.13], and

$$4=\frac{2}{q}+(1+p)\frac{2}{p},$$

we get

$$\alpha^4\int_0^h |(f^2)^{\Delta}(t)|\Delta t+(1-\alpha)^4\int_0^h |(f^2)^{\nabla}(t)|\nabla t\leqslant a^{\frac{2}{q}}c^{\frac{2}{p}}+b^{\frac{2}{q}}d^{\frac{2}{p}}$$
$$\leqslant (a^2+b^2)^{\frac{1}{q}}(c^2+d^2)^{\frac{1}{p}}\leqslant \left[(a+b)^2\right]^{\frac{1}{q}}\left[(c+d)^2\right]^{\frac{1}{p}}=(a+b)^{\frac{2}{q}}(c+d)^{\frac{2}{p}},$$

and so we obtain the desired inequality (3.1). □

REMARK 3.2. If $p \in \mathbb{N}$ is even, then the two conditions

$$\alpha f^{\Delta} \geqslant 0 \quad \text{and} \quad (1-\alpha) f^{\nabla} \geqslant 0$$

from Theorem 3.1 may be replaced by the single condition $\alpha(1-\alpha) f^{\Delta} f^{\nabla} \geqslant 0$ as in this case (3.4) holds as well.

If $p=q=2$, then Theorem 3.1 reduces (observe Remark 3.2) as follows.

COROLLARY 3.3. *Assume that* $\alpha \in [0,1]$, $h \in (0,\infty)_{\mathbb{T}}$,

$$\omega \in \mathrm{C}([0,h]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha(1-\alpha) f^{\Delta} f^{\nabla} \geqslant 0$ *and* $f(0)=0$, *then*

$$\alpha^4 \int_0^h |(f^2)^{\Delta}(t)| \Delta t + (1-\alpha)^4 \int_0^h |(f^2)^{\nabla}(t)| \nabla t \leqslant \left( \int_0^h \frac{\diamondsuit_\alpha t}{\omega(t)} \right) \left( \int_0^h \omega(t) |f^{\diamondsuit_\alpha}(t)|^2 \diamondsuit_\alpha t \right).$$

If $\omega(t) \equiv 1$, then Theorem 3.1 reduces to the following result.

COROLLARY 3.4. *Assume that* $p>1$, $q=p/(p-1)$, $\alpha \in [0,1]$, $h \in (0,\infty)_{\mathbb{T}}$, *and* $f \in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}},\mathbb{R})$. *If* $\alpha f^{\Delta} \geqslant 0$, $(1-\alpha) f^{\nabla} \geqslant 0$, *and* $f(0)=0$, *then*

$$\alpha^4 \int_0^h |(f^2)^{\Delta}(t)| \Delta t + (1-\alpha)^4 \int_0^h |(f^2)^{\nabla}(t)| \nabla t \leqslant h^{\frac{2}{q}} \left( \int_0^h |f^{\diamondsuit_\alpha}(t)|^p \diamondsuit_\alpha t \right)^{\frac{2}{p}}.$$

If $p=q=2$, then Corollary 3.4 reduces (observe Remark 3.2) to the following consequence of [5, Theorem 3.1].

COROLLARY 3.5. *Assume that* $\alpha \in [0,1]$, $h \in (0,\infty)_{\mathbb{T}}$, *and* $f \in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}},\mathbb{R})$. *If* $\alpha(1-\alpha) f^{\Delta} f^{\nabla} \geqslant 0$ *and* $f(0)=0$, *then*

$$\alpha^4 \int_0^h |(f^2)^{\Delta}(t)| \Delta t + (1-\alpha)^4 \int_0^h |(f^2)^{\nabla}(t)| \nabla t \leqslant h \int_0^h |f^{\diamondsuit_\alpha}(t)|^2 \diamondsuit_\alpha t.$$

If $\alpha=1$, then Corollary 3.5 reduces to the following result [1, Theorem 6.1].

COROLLARY 3.6. *Assume that* $h \in (0,\infty)_{\mathbb{T}}$ *and* $f \in \mathrm{C}^1_{\mathrm{rd}}([0,h]_{\mathbb{T}},\mathbb{R})$. *If* $f(0)=0$, *then*

$$\int_0^h |(f^2)^{\Delta}(t)| \Delta t \leqslant h \int_0^h |f^{\Delta}(t)|^2 \Delta t.$$

### 3.2. Arbitrary boundary conditions

Now we consider the case when $f(0)$ and $f(h)$ are allowed to be arbitrary.

THEOREM 3.7. *Assume that* $1 < p \leqslant 2,\ q = p/(p-1),\ \alpha \in [0,1],\ h \in (0,\infty)_{\mathbb{T}}$,

$$\omega \in \mathrm{C}([0,h]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha f^{\Delta} \geqslant 0$ *and* $(1-\alpha)f^{\nabla} \geqslant 0$, *then*

$$\alpha^4 \int_0^h |(f^2)^{\Delta}(t)|\Delta t + (1-\alpha)^4 \int_0^h |(f^2)^{\nabla}(t)|\nabla t \\ \leqslant \beta^{\frac{2}{q}} \left( \int_0^h \omega(t)|f^{\diamondsuit_\alpha}(t)|^p \diamondsuit_\alpha t \right)^{\frac{2}{p}} + 2\gamma(\alpha^4 + (1-\alpha)^4)(f(h)-f(0)), \quad (3.5)$$

*where*

$$\beta := \min_{u \in [0,h]_{\mathbb{T}}} \upsilon(u) \quad \textit{with} \quad \upsilon(u) := \max\left\{ \int_0^u \omega^{1-q}(t)\diamondsuit_\alpha t, \int_u^h \omega^{1-q}(t)\diamondsuit_\alpha t \right\}$$

*and*

$$\gamma := \max\{|f(0)|,|f(h)|\}.$$

*Proof.* We let $u \in [0,h]_{\mathbb{T}}$ be arbitrary. First, we apply Theorem 3.1 to the function $g := f - f(0)$ to obtain

$$\begin{aligned}
&\alpha^4 \int_0^u |(f^2)^{\Delta}(t)|\Delta t + (1-\alpha)^4 \int_0^u |(f^2)^{\nabla}(t)|\nabla t \\
&= \alpha^4 \int_0^u |(g^2)^{\Delta}(t) + 2f(0)f^{\Delta}(t)|\Delta t + (1-\alpha)^4 \int_0^u |(g^2)^{\nabla}(t) + 2f(0)f^{\nabla}(t)|\nabla t \\
&\leqslant \alpha^4 \int_0^u |(g^2)^{\Delta}(t)|\Delta t + (1-\alpha)^4 \int_0^u |(g^2)^{\nabla}(t)|\nabla t \\
&\quad + 2\alpha^4|f(0)| \int_0^u |f^{\Delta}(t)|\Delta t + 2(1-\alpha)^4|f(0)| \int_0^u |f^{\nabla}(t)|\nabla t \\
&\leqslant \left( \int_0^u \omega^{1-q}(t)\diamondsuit_\alpha t \right)^{\frac{2}{q}} \left( \int_0^u \omega(t)|g^{\diamondsuit_\alpha}(t)|^p \diamondsuit_\alpha t \right)^{\frac{2}{p}} \\
&\quad + 2\gamma(\alpha^4 + (1-\alpha)^4)(f(u)-f(0)) \\
&\leqslant (\upsilon(u))^{\frac{2}{q}} \left( \int_0^u \omega(t)|f^{\diamondsuit_\alpha}(t)|^p \diamondsuit_\alpha t \right)^{\frac{2}{p}} + 2\gamma(\alpha^4 + (1-\alpha)^4)(f(u)-f(0))
\end{aligned}$$

and similarly

$$\alpha^4 \int_u^h |(f^2)^{\Delta}(t)|\Delta t + (1-\alpha)^4 \int_u^h |(f^2)^{\nabla}(t)|\nabla t \\ \leqslant (\upsilon(u))^{\frac{2}{q}} \left( \int_u^h \omega(t)|f^{\diamondsuit_\alpha}(t)|^p \diamondsuit_\alpha t \right)^{\frac{2}{p}} + 2\gamma(\alpha^4 + (1-\alpha)^4)(f(h)-f(u)).$$

Adding these two inequalities and observing that $a^r + b^r \leqslant (a+b)^r$ holds for $a,b \geqslant 0$ and $r \geqslant 1$ yields the desired inequality (3.5). $\square$

REMARK 3.8. Note that when $p > 2$, then we may use that $a^r + b^r \leqslant 2^{1-r}(a+b)^r$ holds for $a,b \geqslant 0$ and $0 < r < 1$ in the last part of the proof of Theorem 3.7, and thus (3.5) holds with an additional factor $2^{1-2/p}$ of the first term on the right-hand side of (3.5). The same situation occurs also for corresponding delta or nabla results when $p > 2$. This remark applies to the subsequent corollaries as well.

If $\alpha = 1$, then Theorem 3.7 reduces to the following, see [25, Theorem 4.3].

COROLLARY 3.9. *Assume that* $1 < p \leqslant 2$, $q = p/(p-1)$, $h \in (0,\infty)_{\mathbb{T}}$,

$$\omega \in \mathrm{C}([0,h]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([0,h]_{\mathbb{T}}, \mathbb{R}).$$

*If* $f^{\Delta} \geqslant 0$, *then*

$$\int_0^h |(f^2)^{\Delta}(t)|\Delta t \leqslant \beta^{\frac{2}{q}} \left( \int_0^h \omega(t)|f^{\Delta}(t)|^p \Delta t \right)^{\frac{2}{p}} + 2\gamma(f(h) - f(0)),$$

*where*

$$\beta := \min_{u \in [0,h]_{\mathbb{T}}} \max \left\{ \int_0^u \omega^{1-q}(t)\Delta t, \int_u^h \omega^{1-q}(t)\Delta t \right\}$$

*and*

$$\gamma := \max\{|f(0)|, |f(h)|\}.$$

If $f(0) = f(h) = 0$, then Theorem 3.7 reduces to the following result.

COROLLARY 3.10. *Assume that* $1 < p \leqslant 2$, $q = p/(p-1)$, $\alpha \in [0,1]$, $h \in (0,\infty)_{\mathbb{T}}$,

$$\omega \in \mathrm{C}([0,h]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}}, \mathbb{R}).$$

*If* $\alpha f^{\Delta} \geqslant 0$, $(1-\alpha) f^{\nabla} \geqslant 0$, *and* $f(0) = f(h) = 0$, *then*

$$\alpha^4 \int_0^h |(f^2)^{\Delta}(t)|\Delta t + (1-\alpha)^4 \int_0^h |(f^2)^{\nabla}(t)|\nabla t \leqslant \beta^{\frac{2}{q}} \left( \int_0^h \omega(t)|f^{\diamondsuit_\alpha}(t)|^p \diamondsuit_\alpha t \right)^{\frac{2}{p}},$$

*where*

$$\beta := \min_{u \in [0,h]_{\mathbb{T}}} \max \left\{ \int_0^u \omega^{1-q}(t)\diamondsuit_\alpha t, \int_u^h \omega^{1-q}(t)\diamondsuit_\alpha t \right\}.$$

If $\omega(t) \equiv 1$, then Theorem 3.7 reduces to the following result.

COROLLARY 3.11. *Assume that* $1 < p \leqslant 2$, $q = p/(p-1)$, $\alpha \in [0,1]$, $h \in (0,\infty)_{\mathbb{T}}$, *and* $f \in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_{\mathbb{T}}, \mathbb{R})$. *If* $\alpha f^{\Delta} \geqslant 0$ *and* $(1-\alpha) f^{\nabla} \geqslant 0$, *then*

$$\alpha^4 \int_0^h |(f^2)^\Delta(t)|\Delta t + (1-\alpha)^4 \int_0^h |(f^2)^\nabla(t)|\nabla t \leqslant \beta^{\frac{2}{q}} \left( \int_0^h |f^{\diamondsuit_\alpha}(t)|^p \diamondsuit_\alpha t \right)^{\frac{2}{p}} + 2\gamma(\alpha^4 + (1-\alpha)^4)(f(h) - f(0)),$$

*where*

$$\beta := \min_{u \in [0,h]_\mathbb{T}} \max\{u, h-u\} \quad \text{and} \quad \gamma := \max\{|f(0)|, |f(h)|\}.$$

If $p = q = 2$, then Corollary 3.11 reduces to the following result related to [5, Theorem 3.2].

COROLLARY 3.12. *Assume that* $\alpha \in [0,1]$, $h \in (0,\infty)_\mathbb{T}$, *and* $f \in \mathrm{C}^1_{\diamondsuit_\alpha}([0,h]_\mathbb{T}, \mathbb{R})$. *If* $\alpha f^\Delta \geqslant 0$ *and* $(1-\alpha) f^\nabla \geqslant 0$, *then*

$$\alpha^4 \int_0^h |(f^2)^\Delta(t)|\Delta t + (1-\alpha)^4 \int_0^h |(f^2)^\nabla(t)|\nabla t \leqslant \beta \int_0^h |f^{\diamondsuit_\alpha}(t)|^2 \diamondsuit_\alpha t + 2\gamma(\alpha^4 + (1-\alpha)^4)(f(h) - f(0)),$$

*where*

$$\beta := \min_{u \in [0,h]_\mathbb{T}} \max\{u, h-u\} \quad \text{and} \quad \gamma := \max\{|f(0)|, |f(h)|\}.$$

If $\alpha = 1$, then Corollary 3.12 reduces to the following result [7, Theorem 6.27].

COROLLARY 3.13. *Assume that* $h \in (0,\infty)_\mathbb{T}$ *and* $f \in \mathrm{C}^1_{\mathrm{rd}}([0,h]_\mathbb{T}, \mathbb{R})$. *If* $f^\Delta \geqslant 0$, *then*

$$\int_0^h |(f^2)^\Delta(t)|\Delta t \leqslant \beta \int_0^h |f^\Delta(t)|^2 \Delta t + 2\gamma(f(h) - f(0)),$$

*where*

$$\beta := \min_{u \in [0,h]_\mathbb{T}} \max\{u, h-u\} \quad \text{and} \quad \gamma := \max\{|f(0)|, |f(h)|\}.$$

## 4. Generalizations of results by Karpuz et al

In this section, we deal with one weight function and generalize some delta Opial inequalities obtained by Karpuz, Kaymakçalan, and Öcalan [9] to the diamond-alpha case. The starting point here is the following result given in [9].

THEOREM 4.1. (See [9, Theorem 3.1]) *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_\mathbb{T}$,

$$\omega \in \mathrm{C}_{\mathrm{rd}}([a,b]_\mathbb{T}, (0,\infty)), \quad \text{and} \quad f \in \mathrm{C}^1_{\mathrm{rd}}([a,b]_\mathbb{T}, \mathbb{R}).$$

*If* $f(a) = 0$, *then*

$$\int_a^b \omega(t)|(f^2)^\Delta(t)|\Delta t \leqslant K \int_a^b |f^\Delta(t)|^2 \Delta t,$$

*where*

$$K=\sqrt{2\int_a^b \omega^2(t)(\sigma(t)-a)\Delta t}.$$

For our purposes, we offer the following slight but essential improvement of Theorem 4.1.

THEOREM 4.2. *Assume that* $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,

$$\omega\in \mathrm{C_{rd}}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f\in \mathrm{C^1_{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $f(a)=0$, *then*

$$\int_a^b \omega(t)|(f^2)^\Delta(t)|\Delta t\leqslant K\int_a^b |f^\Delta(t)|^2\Delta t,$$

*where*

$$K=\sqrt{\int_a^b \omega^2(t)h^\Delta(t)\Delta t} \quad \textit{with} \quad h(t)=(t-a)^2.$$

*Proof.* Define $g(t):=\int_a^t|f^\Delta(s)|^2\Delta s$. Then $g(a)=0$, $g^\Delta(t)=|f^\Delta(t)|^2$, and

$$\begin{aligned}|f(t)|&=|f(t)-f(a)|=\left|\int_a^t f^\Delta(s)\Delta s\right|\leqslant\int_a^t|f^\Delta(s)|\Delta s\\&\leqslant\sqrt{t-a}\sqrt{\int_a^t|f^\Delta(s)|^2\Delta s}=\sqrt{(t-a)g(t)},\end{aligned}$$

where we have used the time scales Cauchy–Schwarz inequality [7, Theorem 6.15]. Thus

$$\begin{aligned}|(f^2)^\Delta(t)|&=|(f(t)+f(\sigma(t)))f^\Delta(t)|\\&\leqslant(|f(t)|+|f(\sigma(t))|)\,|f^\Delta(t)|\\&\leqslant\left(\sqrt{t-a}\sqrt{g(t)}+\sqrt{\sigma(t)-a}\sqrt{g(\sigma(t))}\right)\sqrt{g^\Delta(t)}\\&\leqslant\sqrt{t-a+\sigma(t)-a}\sqrt{g(t)+g(\sigma(t))}\sqrt{g^\Delta(t)}\\&=\sqrt{h^\Delta(t)}\sqrt{(g^2)^\Delta(t)},\end{aligned}$$

where we have used the classical Cauchy–Schwarz inequality, and hence

$$\begin{aligned}\int_a^b\omega(t)|(f^2)^\Delta(t)|\Delta t&\leqslant\int_a^b\omega(t)\sqrt{h^\Delta(t)}\sqrt{(g^2)^\Delta(t)}\Delta t\\&\leqslant\sqrt{\int_a^b\omega^2(t)h^\Delta(t)\Delta t}\sqrt{\int_a^b(g^2)^\Delta(t)\Delta t}\\&=K\sqrt{g^2(b)}=Kg(b)=K\int_a^b|f^\Delta(t)|^2\Delta t,\end{aligned}$$

where we have used the time scales Cauchy–Schwarz inequality one last time. □

Similarly, we may prove the following nabla result.

THEOREM 4.3. *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$\omega \in \mathrm{C}_{\mathrm{ld}}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{ld}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $f(a)=0$, *then*

$$\int_a^b \omega(t)|(f^2)^{\nabla}(t)|\nabla t \leqslant L\int_a^b |f^{\nabla}(t)|^2\nabla t,$$

*where*

$$L=\sqrt{\int_a^b \omega^2(t)h^{\nabla}(t)\nabla t} \quad \textit{with} \quad h(t)=(t-a)^2.$$

Now we are ready to prove the corresponding diamond-alpha inequality.

THEOREM 4.4. *Assume that* $\alpha \in [0,1]$, $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$\omega \in \mathrm{C}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\diamondsuit_\alpha}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha(1-\alpha)f^{\Delta}f^{\nabla} \geqslant 0$ *and* $f(a)=0$, *then*

$$\alpha^4\int_a^b \omega(t)|(f^2)^{\Delta}(t)|\Delta t+(1-\alpha)^4\int_a^b \omega(t)|(f^2)^{\nabla}(t)|\nabla t \leqslant \Lambda\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\diamondsuit_\alpha t,$$

*where*

$$\Lambda=\sqrt{\int_a^b \omega^2(t)h^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha t} \quad \textit{with} \quad h(t)=(t-a)^2.$$

*Proof.* By Theorem 4.2 and Theorem 4.3, we have

$$\begin{aligned}
&\alpha^4\int_a^b \omega(t)|(f^2)^{\Delta}(t)|\Delta t+(1-\alpha)^4\int_a^b \omega(t)|(f^2)^{\nabla}(t)|\nabla t\\
&\leqslant \alpha^4K\int_a^b |f^{\Delta}(t)|^2\Delta t+(1-\alpha)^4L\int_a^b |f^{\nabla}(t)|^2\nabla t\\
&= \alpha^2K\int_a^b |\alpha f^{\Delta}(t)|^2\Delta t+(1-\alpha)^2L\int_a^b |(1-\alpha)f^{\nabla}(t)|^2\nabla t\\
&\leqslant \alpha^2K\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\Delta t+(1-\alpha)^2L\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\nabla t\\
&= (\alpha K)\left(\alpha\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\Delta t\right)+((1-\alpha)L)\left((1-\alpha)\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\nabla t\right)\\
&\leqslant \sqrt{\alpha^2K^2+(1-\alpha)^2L^2}\sqrt{\left(\alpha\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\Delta t\right)^2+\left((1-\alpha)\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\nabla t\right)^2}
\end{aligned}$$

$$\leqslant \sqrt{\alpha^2K^2+(1-\alpha)^2L^2}\sqrt{\left(\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\diamondsuit_\alpha t\right)^2}$$
$$= \tilde{\Lambda}\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\diamondsuit_\alpha t,$$

where we have used the classical Cauchy–Schwarz inequality and

$$\begin{aligned}\tilde{\Lambda} &= \sqrt{\alpha^2K^2+(1-\alpha)^2L^2}\\ &= \sqrt{\alpha^2\int_a^b \omega^2(t)h^{\Delta}(t)\Delta t+(1-\alpha)^2\int_a^b \omega^2(t)h^{\nabla}(t)\nabla t}\\ &\leqslant \sqrt{\alpha\int_a^b \omega^2(t)h^{\diamondsuit_\alpha}(t)\Delta t+(1-\alpha)\int_a^b \omega^2(t)h^{\diamondsuit_\alpha}(t)\nabla t}\\ &= \sqrt{\int_a^b \omega^2(t)h^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha t}=\Lambda,\end{aligned}$$

which completes the proof. □

Following the same steps as in the proofs of all previous results in this section, we can establish the following result.

THEOREM 4.5. *Assume that* $\alpha\in[0,1]$, $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,

$$\omega\in \mathrm{C}([a,b]_{\mathbb{T}},(0,\infty)),\quad \textit{and}\quad f\in \mathrm{C}^1_{\diamondsuit_\alpha}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha(1-\alpha)f^{\Delta}f^{\nabla}\geqslant 0$ *and* $f(b)=0$, *then*

$$\alpha^4\int_a^b \omega(t)|(f^2)^{\Delta}(t)|\Delta t+(1-\alpha)^4\int_a^b \omega(t)|(f^2)^{\nabla}(t)|\nabla t\leqslant \Omega\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\diamondsuit_\alpha t,$$

*where*

$$\Omega=\sqrt{\int_a^b \omega^2(t)h^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha t}\quad \textit{with}\quad h(t)=(b-t)^2.$$

The next result combines Theorem 4.4 and Theorem 4.5.

THEOREM 4.6. *Assume that* $\alpha\in[0,1]$, $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,

$$\omega\in \mathrm{C}([a,b]_{\mathbb{T}},(0,\infty)),\quad \textit{and}\quad f\in \mathrm{C}^1_{\diamondsuit_\alpha}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha(1-\alpha)f^{\Delta}f^{\nabla}\geqslant 0$ *and* $f(a)=f(b)=0$, *then*

$$\alpha^4\int_a^b \omega(t)|(f^2)^{\Delta}(t)|\Delta t+(1-\alpha)^4\int_a^b \omega(t)|(f^2)^{\nabla}(t)|\nabla t\leqslant \beta\int_a^b |f^{\diamondsuit_\alpha}(t)|^2\diamondsuit_\alpha t,$$

*where*

$$\beta := \min_{u\in[a,b]_{\mathbb{T}}} \upsilon(u)$$

*with*

$$\upsilon(u) := \max\left\{\sqrt{\int_a^u \omega^2(t)h_a^{\Diamond_\alpha}(t)\Diamond_\alpha t}, \sqrt{\int_u^b \omega^2(t)h_b^{\Diamond_\alpha}(t)\Diamond_\alpha t}\right\},$$

*and* $h_c$ *for* $c\in\mathbb{T}$ *is defined by* $h_c(t) = (t-c)^2$.

*Proof.* We let $u\in[a,b]_{\mathbb{T}}$ be arbitrary. By Theorem 4.4, we have

$$\alpha^4\int_a^u \omega(t)|(f^2)^\Delta(t)|\Delta t + (1-\alpha)^4\int_a^u \omega(t)|(f^2)^\nabla(t)|\nabla t \leqslant \upsilon(u)\int_a^u |f^{\Diamond_\alpha}(t)|^2\Diamond_\alpha t,$$

and by Theorem 4.5, we have

$$\alpha^4\int_u^b \omega(t)|(f^2)^\Delta(t)|\Delta t + (1-\alpha)^4\int_u^b \omega(t)|(f^2)^\nabla(t)|\nabla t \leqslant \upsilon(u)\int_u^b |f^{\Diamond_\alpha}(t)|^2\Diamond_\alpha t.$$

Adding these two inequalities yields

$$\alpha^4\int_a^b \omega(t)|(f^2)^\Delta(t)|\Delta t + (1-\alpha)^4\int_a^b \omega(t)|(f^2)^\nabla(t)|\nabla t \leqslant \upsilon(u)\int_a^b |f^{\Diamond_\alpha}(t)|^2\Diamond_\alpha t.$$

Now passing to the minimum over $u\in[a,b]_{\mathbb{T}}$ completes the proof. □

## 5. Generalizations of results by Saker

In this section, we deal with two weight functions and generalize some delta Opial inequalities obtained by Saker [18] to the diamond-alpha case. The starting point here is the following result given in [18].

THEOREM 5.1. (See [18, Theorem 1]) *Assume that* $a\in\mathbb{T}$, $b\in(a,\infty)_{\mathbb{T}}$,

$$r,s\in \mathrm{C_{rd}}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f\in \mathrm{C}^1_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $f(a) = 0$, *then*

$$\int_a^b s(t)|(f^2)^\Delta(t)|\Delta t \leqslant K\int_a^b r(t)|f^\Delta(t)|^2\Delta t,$$

*where*

$$K = \sqrt{2\int_a^b \frac{s^2(t)}{r(t)}\left(\int_a^t \frac{\Delta s}{r(s)}\right)\Delta t} + \sup_{a\leqslant t\leqslant b}\frac{\mu(t)s(t)}{r(t)},$$

*where* $\mu$ *is the graininess of the time scale* $\mathbb{T}$.

For our purposes, we offer the following slight but essential improvement of Theorem 5.1.

THEOREM 5.2. *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r, s \in \mathrm{C_{rd}}([a,b]_{\mathbb{T}}, (0,\infty)), \quad \textit{and} \quad f \in \mathrm{C^1_{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}).$$

*If* $f(a) = 0$, *then*

$$\int_a^b s(t)|(f^2)^{\Delta}(t)|\Delta t \leqslant K \int_a^b r(t)|f^{\Delta}(t)|^2 \Delta t,$$

*where*

$$K = \sqrt{\int_a^b s^2(t)(R^2)^{\Delta}(t)\Delta t} \quad \textit{with} \quad R(t) = \int_a^t \frac{\Delta s}{r(s)}.$$

*Proof.* Define $g(t) := \int_a^t r(s)|f^{\Delta}(s)|^2 \Delta s$. Then $g(a) = 0$,

$$g^{\Delta}(t) = r(t)|f^{\Delta}(t)|^2 \quad \text{so that} \quad |f^{\Delta}(t)| = \sqrt{\frac{g^{\Delta}(t)}{r(t)}} = \sqrt{R^{\Delta}(t)g^{\Delta}(t)},$$

and

$$\begin{aligned} |f(t)| &= |f(t) - f(a)| = \left| \int_a^t f^{\Delta}(s)\Delta s \right| \leqslant \int_a^t |f^{\Delta}(s)|\Delta s \\ &= \int_a^t \frac{1}{\sqrt{r(s)}} \left( \sqrt{r(s)}|f^{\Delta}(s)| \right) \Delta s \\ &\leqslant \sqrt{\int_a^t \frac{\Delta s}{r(s)}} \sqrt{\int_a^t r(s)|f^{\Delta}(s)|^2 \Delta s} = \sqrt{R(t)}\sqrt{g(t)}, \end{aligned}$$

where we have used the time scales Cauchy–Schwarz inequality. Thus

$$\begin{aligned} |(f^2)^{\Delta}(t)| &= |(f(t) + f(\sigma(t)))f^{\Delta}(t)| \\ &\leqslant (|f(t)| + |f(\sigma(t))|)\,|f^{\Delta}(t)| \\ &\leqslant \left( \sqrt{R(t)}\sqrt{g(t)} + \sqrt{R(\sigma(t))}\sqrt{g(\sigma(t))} \right) \sqrt{R^{\Delta}(t)g^{\Delta}(t)} \\ &\leqslant \sqrt{R(t) + R(\sigma(t))}\sqrt{g(t) + g(\sigma(t))}\sqrt{R^{\Delta}(t)g^{\Delta}(t)} \\ &= \sqrt{(R^2)^{\Delta}(t)}\sqrt{(g^2)^{\Delta}(t)}, \end{aligned}$$

where we have used the classical Cauchy–Schwarz inequality, and hence

$$\begin{aligned} \int_a^b s(t)|(f^2)^{\Delta}(t)|\Delta t &\leqslant \int_a^b s(t)\sqrt{(R^2)^{\Delta}(t)}\sqrt{(g^2)^{\Delta}(t)}\Delta t \\ &\leqslant \sqrt{\int_a^b s^2(t)(R^2)^{\Delta}(t)\Delta t}\sqrt{\int_a^b (g^2)^{\Delta}(t)\Delta t} \\ &= K\sqrt{g^2(b)} = Kg(b) = K\int_a^b r(t)|f^{\Delta}(t)|^2 \Delta t, \end{aligned}$$

where we have used the time scales Cauchy–Schwarz inequality one last time. □

Similarly, we may prove the following nabla result.

THEOREM 5.3. *Assume that* $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r,s \in \mathrm{C}_{\mathrm{ld}}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\mathrm{ld}}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $f(a) = 0$, *then*

$$\int_a^b s(t)|(f^2)^\nabla(t)|\nabla t \leqslant L \int_a^b r(t)|f^\nabla(t)|^2 \nabla t,$$

*where*

$$L = \sqrt{\int_a^b s^2(t)(S^2)^\nabla(t)\nabla t} \quad \textit{with} \quad S(t) = \int_a^t \frac{\nabla s}{r(s)}.$$

Now we are ready to prove the corresponding diamond-alpha inequality.

THEOREM 5.4. *Assume that* $\alpha \in [0,1]$, $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r,s \in \mathrm{C}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\diamondsuit_\alpha}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha(1-\alpha)f^\Delta f^\nabla \geqslant 0$ *and* $f(a) = 0$, *then*

$$\alpha^5 \int_a^b s(t)|(f^2)^\Delta(t)|\Delta t + (1-\alpha)^5 \int_a^b s(t)|(f^2)^\nabla(t)|\nabla t \leqslant \Lambda \int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2 \diamondsuit_\alpha t,$$

*where*

$$\Lambda = \sqrt{\int_a^b s^2(t)(T^2)^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha t} \quad \textit{with} \quad T(t) = \int_a^t \frac{\diamondsuit_\alpha s}{r(s)}.$$

*Proof.* By Theorem 5.2 and Theorem 5.3, we have

$$\begin{aligned}
&\alpha^5 \int_a^b s(t)|(f^2)^\Delta(t)|\Delta t + (1-\alpha)^5 \int_a^b s(t)|(f^2)^\nabla(t)|\nabla t \\
&\leqslant \alpha^5 K \int_a^b r(t)|f^\Delta(t)|^2 \Delta t + (1-\alpha)^5 L \int_a^b r(t)|f^\nabla(t)|^2 \nabla t \\
&= \alpha^3 K \int_a^b r(t)|\alpha f^\Delta(t)|^2 \Delta t + (1-\alpha)^3 L \int_a^b r(t)|(1-\alpha) f^\nabla(t)|^2 \nabla t \\
&\leqslant \alpha^3 K \int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2 \Delta t + (1-\alpha)^3 L \int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2 \nabla t \\
&= (\alpha^2 K)\left(\alpha \int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2 \Delta t\right) + ((1-\alpha)^2 L)\left((1-\alpha)\int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2 \nabla t\right) \\
&\leqslant \tilde{\Lambda}\sqrt{\left(\alpha \int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2 \Delta t\right)^2 + \left((1-\alpha)\int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2 \nabla t\right)^2}
\end{aligned}$$

$$\leqslant \tilde{\Lambda}\sqrt{\left(\int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2\diamondsuit_\alpha t\right)^2} = \tilde{\Lambda}\int_a^b r(t)|f^{\diamondsuit_\alpha}(t)|^2\diamondsuit_\alpha t,$$

where we have used the classical Cauchy–Schwarz inequality and

$$\begin{aligned}
\tilde{\Lambda} &= \sqrt{\alpha^4K^2+(1-\alpha)^4L^2}\\
&= \sqrt{\alpha^4\int_a^b s^2(t)(R^2)^\Delta(t)\Delta t+(1-\alpha)^4\int_a^b s^2(t)(S^2)^\nabla(t)\nabla t}\\
&\leqslant \sqrt{\alpha\int_a^b s^2(t)(T^2)^{\diamondsuit_\alpha}(t)\Delta t+(1-\alpha)\int_a^b s^2(t)(T^2)^{\diamondsuit_\alpha}(t)\nabla t}\\
&= \sqrt{\int_a^b s^2(t)(T^2)^{\diamondsuit_\alpha}(t)\diamondsuit_\alpha t} = \Lambda,
\end{aligned}$$

where we have used the inequalities

$$\alpha^3(R^2)^\Delta \leqslant (T^2)^{\diamondsuit_\alpha} \quad \text{and} \quad (1-\alpha)^3(S^2)^\Delta \leqslant (T^2)^{\diamondsuit_\alpha}. \tag{5.1}$$

Now we show (5.1) in order to complete the proof. Note first that by [8, Theorem 5.37], we have

$$R^\Delta = S^\nabla = \frac{1}{r}, \quad R^\nabla = \frac{1}{r^\rho}, \quad \text{and} \quad S^\Delta = \frac{1}{r^\sigma},$$

and all of these derivatives are positive. Using these relations and the time scales product rules, we have

$$\begin{gathered}
(R^2)^\nabla = \frac{R+R^\rho}{r^\rho}, \quad (S^2)^\Delta = \frac{S+S^\sigma}{r^\sigma}, \quad (R^2)^\Delta = \frac{R+R^\sigma}{r},\\
(S^2)^\nabla = \frac{S+S^\rho}{r}, \quad (RS)^\Delta = \frac{S}{r}+\frac{R^\sigma}{r^\sigma}, \quad (RS)^\nabla = \frac{S}{r^\rho}+\frac{R^\rho}{r},
\end{gathered}$$

and again all of these derivatives are positive. Since $T = \alpha R+(1-\alpha)S$, the calculation

$$\begin{aligned}
(T^2)^{\diamondsuit_\alpha} &= \alpha(T^2)^\Delta+(1-\alpha)(T^2)^\nabla\\
&= \alpha\left(\alpha^2(R^2)^\Delta+2\alpha(1-\alpha)(RS)^\Delta+(1-\alpha)^2(S^2)^\Delta\right)\\
&\quad +(1-\alpha)\left(\alpha^2(R^2)^\nabla+2\alpha(1-\alpha)(RS)^\nabla+(1-\alpha)^2(S^2)^\nabla\right)\\
&= \alpha^3(R^2)^\Delta+(1-\alpha)^3(S^2)^\nabla+2\alpha^2(1-\alpha)(RS)^\Delta\\
&\quad +2\alpha(1-\alpha)^2(RS)^\nabla+\alpha(1-\alpha)^2(S^2)^\Delta+\alpha^2(1-\alpha)(R^2)^\nabla
\end{aligned}$$

confirms the validity of the inequalities (5.1). □

Following the same steps as in the proofs of all previous results in this section, we can establish the following result.

THEOREM 5.5. *Assume that* $\alpha \in [0,1]$, $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r,s \in \mathrm{C}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\Diamond_\alpha}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha(1-\alpha)f^{\Delta}f^{\nabla} \geqslant 0$ *and* $f(b)=0$, *then*

$$\alpha^5 \int_a^b s(t)|(f^2)^{\Delta}(t)|\Delta t + (1-\alpha)^5 \int_a^b s(t)|(f^2)^{\nabla}(t)|\nabla t \leqslant \Omega \int_a^b r(t)|f^{\Diamond_\alpha}(t)|^2 \Diamond_\alpha t,$$

*where*

$$\Omega = \sqrt{\int_a^b s^2(t)(T^2)^{\Diamond_\alpha}(t)\Diamond_\alpha t} \quad \textit{with} \quad T(t) = \int_t^b \frac{\Diamond_\alpha s}{r(s)}.$$

The last result combines Theorem 5.4 and Theorem 5.5. Its proof is omitted as it is the same as the proof of Theorem 4.6.

THEOREM 5.6. *Assume that* $\alpha \in [0,1]$, $a \in \mathbb{T}$, $b \in (a,\infty)_{\mathbb{T}}$,

$$r,s \in \mathrm{C}([a,b]_{\mathbb{T}},(0,\infty)), \quad \textit{and} \quad f \in \mathrm{C}^1_{\Diamond_\alpha}([a,b]_{\mathbb{T}},\mathbb{R}).$$

*If* $\alpha(1-\alpha)f^{\Delta}f^{\nabla} \geqslant 0$ *and* $f(a)=f(b)=0$, *then*

$$\alpha^5 \int_a^b s(t)|(f^2)^{\Delta}(t)|\Delta t + (1-\alpha)^5 \int_a^b s(t)|(f^2)^{\nabla}(t)|\nabla t \leqslant \beta \int_a^b r(t)|f^{\Diamond_\alpha}(t)|^2 \Diamond_\alpha t,$$

*where*

$$\beta := \min_{u \in [a,b]_{\mathbb{T}}} \upsilon(u)$$

*with*

$$\upsilon(u) := \max\left\{ \sqrt{\int_a^u s^2(t)(T_a^2)^{\Diamond_\alpha}(t)\Diamond_\alpha t}, \sqrt{\int_u^b s^2(t)(T_b^2)^{\Diamond_\alpha}(t)\Diamond_\alpha t} \right\},$$

*and* $T_c$ *for* $c \in \mathbb{T}$ *is defined by* $T_c(t) = \int_c^t \frac{\Diamond_\alpha s}{r(s)}$.

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