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Double integral calculus of variations on time scales

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Abstract

We consider a version of the double integral calculus of variations on time scales, which includes as special cases the classical two-variable calculus of variations and the discrete two-variable calculus of variations. Necessary and sufficient conditions for a local extremum are established, among them an analogue of the Euler–Lagrange equation.

Keywords: Time scales; Partial delta derivatives; Double delta integrals; Euler–Lagrange equation

1. Introduction

Variational problems are similar to the important problem in the usual differential calculus in which we determine maximum or minimum values of a function $y = f(x)$ for values x in a certain interval of the reals $\mathbb{R}$ or in a region of $\mathbb{R}^n$. The main difference is that in variational problems we deal with so-called *functionals* instead of usual functions. Recall that any mapping $J : X \to \mathbb{R}$ of an arbitrary set X (in particular, X may be a set of functions) into the real numbers $\mathbb{R}$ is called a functional. Various entities in geometry, physics, mechanics, technology, and nature have a tendency to minimize (or maximize) some quantities. Those quantities can mathematically be described as functionals. Variational calculus gives methods for finding the minimal or maximal values of functionals, and problems that consist in finding minima or maxima of a functional are called variational problems. Several important variational problems such as the brachistochrone problem, the problem of geodesics, and the isoperimetric problem were first posed at the end of the 17th century (beginning in 1696). General methods of solving variational problems were created by L. Euler and J. Lagrange in the 18th century. Later on, variational calculus became an independent mathematical discipline with its own research methods. Since the concept of functional (that is a special case of the concept of operator) is one of the main subjects investigated in functional analysis, calculus of variations is considered at present as a branch of functional analysis.

Continuous single and multivariable calculus of variations possesses an extensive literature from which we indicate here only [1,2]. Discrete variable calculus of variations has started to be considered systematically only during the last two decades. An account on the single discrete variable case can be found in [3–6] whereas the two discrete variable

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case is concerned in [7]. In order to unify continuous and discrete analysis and to extend those areas to "in between" cases, Aulbach and Hilger [8,9] generalized the definition of a derivative and of an integral to functions whose domains of definition are time scales. A time scale is a nonempty closed subset of the reals. Time scales calculus allows to unify and extend many problems from the theories of differential and of difference equations (see [10,11]). Single time scale variable calculus of variations (that contains both continuous and discrete calculus of variations as special cases) was initiated in [12] and further developed in [13,14]. At present this topic is in progress. Recently, in [15] a two-variable calculus of variations on time scales was initiated by Ahlbrandt and Morian, where an Euler–Lagrange equation for double integral variational problems on time scales was obtained in case of rectangular regions of integration. In the present paper, we reformulate this problem for the case of so-called ω-type regions of integration, using the multivariable differential and integral calculus developed by the authors in [16–18].

This paper is organized as follows. In Section 2, following [16–18], we give a brief introduction into the two-variable time scales calculus and present a version of Green's formula for ω-type regions in a time scale plane. Section 3 formulates the statement of the double integral variational problem. In Section 4, the first and second variations of a functional are introduced and necessary and sufficient conditions for local minima of the functional are provided in terms of the first and second variations. Finally, in Section 5, we present a version of the Euler–Lagrange equation for two-dimensional variational calculus on time scales.

2. The two-variable time scales calculus

A time scale is an arbitrary nonempty closed subset of the real numbers. For a general introduction to the calculus of one time scale variable we refer the reader to the textbooks [10,11]. In this section, following [16–18], we give a brief introduction into the two-variable time scales calculus.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two given time scales and put $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x, y) : x \in \mathbb{T}_1, y \in \mathbb{T}_2\}$, which is a complete metric space with the metric (distance) d defined by

$$d((x, y), (x', y')) = \sqrt{(x - x')^2 + (y - y')^2} \quad \text{for } (x, y),\ (x', y') \in \mathbb{T}_1 \times \mathbb{T}_2.$$

For a given $\delta > 0$, the δ-neighborhood $U_\delta(x_0, y_0)$ of a given point $(x_0, y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ is the set of all points $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $d((x_0, y_0), (x, y)) < \delta$. Let σ_1 and σ_2 be the forward jump operators in $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. The first-order partial delta derivatives of $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ at a point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ are defined to be

$$\frac{\partial f(x_0, y_0)}{\Delta_1 x} = \lim_{x \to x_0, x \neq \sigma_1(x_0)} \frac{f(\sigma_1(x_0), y_0) - f(x, y_0)}{\sigma_1(x_0) - x}$$

and

$$\frac{\partial f(x_0, y_0)}{\Delta_2 y} = \lim_{y \to y_0, y \neq \sigma_2(y_0)} \frac{f(x_0, \sigma_2(y_0)) - f(x_0, y)}{\sigma_2(y_0) - y}.$$

These derivatives will be denoted also by $f^{\Delta_1}(x_0, y_0)$ and $f^{\Delta_2}(x_0, y_0)$, respectively. If f has partial derivatives $\frac{\partial f(x,y)}{\Delta_1 x}$ and $\frac{\partial f(x,y)}{\Delta_2 y}$, then we can also consider their partial derivatives. These are called second-order partial derivatives. We write

$$\frac{\partial^2 f(x, y)}{\Delta_1 x^2} \quad \text{and} \quad \frac{\partial^2 f(x, y)}{\Delta_2 y \Delta_1 x}, \quad \text{or} \quad f^{\Delta_1 \Delta_1}(x, y) \quad \text{and} \quad f^{\Delta_1 \Delta_2}(x, y)$$

for the partial delta derivatives of $\frac{\partial f(x,y)}{\Delta_1 x}$ with respect to x and with respect to y, respectively. Thus

$$\frac{\partial^2 f(x, y)}{\Delta_1 x^2} = \frac{\partial}{\Delta_1 x}\left(\frac{\partial f(x, y)}{\Delta_1 x}\right) \quad \text{and} \quad \frac{\partial^2 f(x, y)}{\Delta_2 y \Delta_1 x} = \frac{\partial}{\Delta_2 y}\left(\frac{\partial f(x, y)}{\Delta_1 x}\right).$$

Higher-order partial delta derivatives are defined similarly. By [16, Theorem 6.1] we have the following result that gives us a sufficient condition for the independence of mixed partial delta derivatives of the order of differentiation.

Theorem 2.1. *Let a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ have the mixed partial delta derivatives $\frac{\partial^2 f(x,y)}{\Delta_1 x \Delta_2 y}$ and $\frac{\partial^2 f(x,y)}{\Delta_2 y \Delta_1 x}$ in some neighborhood of the point $(x_0, y_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$. If these derivatives are continuous at (x_0, y_0), then*

$$\frac{\partial^2 f(x_0, y_0)}{\Delta_1 x \Delta_2 y} = \frac{\partial^2 f(x_0, y_0)}{\Delta_2 y \Delta_1 x}.$$

We now introduce double Riemann delta integrals over regions in $\mathbb{T}_1 \times \mathbb{T}_2$. First we define double Riemann integrals over rectangles (for details see [17]). Suppose $a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $[a, b)$ is the half-closed bounded interval in $\mathbb{T}_1$, and $[c, d)$ is the half-closed bounded interval in $\mathbb{T}_2$. Let us introduce a "rectangle" (or "delta rectangle") in $\mathbb{T}_1 \times \mathbb{T}_2$ by

$$R = [a, b) \times [c, d) = \{(x, y) : x \in [a, b), y \in [c, d)\}. \tag{2.1}$$

Let

$$\{x_0, x_1, \dots, x_n\} \subset [a, b], \quad \text{where } a = x_0 < x_1 < \cdots < x_n = b$$

and

$$\{y_0, y_1, \dots, y_k\} \subset [c, d], \quad \text{where } c = y_0 < y_1 < \cdots < y_k = d.$$

The numbers n and k may be arbitrary positive integers. We call the collection of intervals $P_1 = \{[x_{i-1}, x_i) : 1 \le i \le n\}$ a Δ-partition (or delta partition) of $[a, b)$ and denote the set of all Δ-partitions of $[a, b)$ by $\mathcal{P}([a, b))$. Similarly, the collection of intervals $P_2 = \left\{[y_{j-1}, y_j) : 1 \le j \le k\right\}$ is called a Δ-partition of $[c, d)$ and the set of all Δ-partitions of $[c, d)$ is denoted by $\mathcal{P}([c, d))$. Let us set

$$R_{ij} = [x_{i-1}, x_i) \times [y_{j-1}, y_j), \quad \text{where } 1 \le i \le n, 1 \le j \le k. \tag{2.2}$$

We call the collection

$$P = \left\{R_{ij} : 1 \le i \le n, 1 \le j \le k\right\} \tag{2.3}$$

a Δ-partition of R, generated by the Δ-partitions P_1 and P_2 of $[a, b)$ and $[c, d)$, respectively, and write $P = P_1 \times P_2$. The rectangles R_{ij}, $1 \le i \le n$, $1 \le j \le k$, are called the subrectangles of the partition P. The set of all Δ-partitions of R is denoted by $\mathcal{P}(R)$.

We need the following auxiliary result. See [11, Lemma 5.7] for the proof.

Lemma 2.2. *For any $\delta > 0$ there exists at least one $P_1 \in \mathcal{P}([a, b))$ generated by a set*

$$\{x_0, x_1, \dots, x_n\} \subset [a, b], \quad \textit{where } a = x_0 < x_1 < \cdots < x_n = b$$

so that for each $i \in \{1, 2, \dots, n\}$ either $x_i - x_{i-1} \le \delta$ or $x_i - x_{i-1} > \delta$ and $\sigma_1(x_{i-1}) = x_i$.

We denote by $\mathcal{P}_\delta([a, b))$ the set of all $P_1 \in \mathcal{P}([a, b))$ that possess the property indicated in Lemma 2.2. Similarly we define $\mathcal{P}_\delta([c, d))$. Further, by $\mathcal{P}_\delta(R)$ we denote the set of all $P \in \mathcal{P}(R)$ such that

$$P = P_1 \times P_2, \quad \text{where } P_1 \in \mathcal{P}_\delta([a, b)) \text{ and } P_2 \in \mathcal{P}_\delta([c, d)).$$

Definition 2.3. Let f be a bounded function on R and $P \in \mathcal{P}(R)$ be given by (2.2) and (2.3). In each "rectangle" R_{ij} with $1 \le i \le n$, $1 \le j \le k$, choose an arbitrary point (ξ_{ij}, η_{ij}) and form the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{k} f(\xi_{ij}, \eta_{ij})(x_i - x_{i-1})(y_j - y_{j-1}). \tag{2.4}$$

We call S a *Riemann Δ-sum* of f corresponding to $P \in \mathcal{P}(R)$. We say that f is Riemann Δ-integrable over R if there exists a number I with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann Δ-sum S of f corresponding to any $P \in \mathcal{P}_\delta(R)$ independent of the way in which we choose $(\xi_{ij}, \eta_{ij}) \in R_{ij}$ for $1 \le i \le n$, $1 \le j \le k$. The number I is the double Riemann Δ-integral of f over R, denoted by $\iint_R f(x, y) \Delta_1 x \Delta_2 y$. We write $I = \lim_{\delta \to 0} S$.

It is easy to see that the number I from Definition 2.3 is unique if it exists. Hence the double Riemann Δ-integral is well defined. Note also that in Definition 2.3 we need not assume the boundedness of f in advance. However, it easily follows that the Riemann Δ-integrability of a function f over R implies its boundedness on R.

In our definition of $\iint_R f(x, y)\Delta_1 x\Delta_2 y$ with $R = [a, b) \times [c, d)$ we assumed that $a < b$ and $c < d$. We extend the definition to the case $a \le b$ and $c \le d$ by setting

$$\iint_R f(x, y)\Delta_1 x\Delta_2 y = 0 \quad \text{if } a = b \text{ or } c = d. \tag{2.5}$$

Theorem 2.4. *Assume $a, b \in \mathbb{T}_1$ with $a \le b$ and $c, d \in \mathbb{T}_2$ with $c \le d$. Every*

constant function $f(t, s) \equiv A$ for $(x, y) \in R = [a, b) \times [c, d)$

is Δ-integrable over R and

$$\iint_R f(x, y)\Delta_1 x\Delta_2 y = A(b - a)(d - c). \tag{2.6}$$

Proof. Let $a < b$ and $c < d$. Consider a partition P of $R = [a, b) \times [c, d)$ of the type (2.2) and (2.3). Thus we have from (2.4)

$$S = \sum_{i=1}^{n}\sum_{j=1}^{k} A(x_i - x_{i-1})(y_j - y_{j-1}) = A(b - a)(d - c),$$

so f is Δ-integrable and (2.6) holds. For $a = b$ or $c = d$, (2.6) follows by (2.5). $\square$

Theorem 2.5. *Let $x_0 \in \mathbb{T}_1$ and $y_0 \in \mathbb{T}_2$. Every function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is Δ-integrable over $R(x_0, y_0) = [x_0, \sigma_1(x_0)) \times [y_0, \sigma_2(y_0))$, and*

$$\iint_{R(x_0,y_0)} f(x, y)\Delta_1 x\Delta_2 y = \mu_1(x_0)\mu_2(y_0) f(x_0, y_0), \tag{2.7}$$

where $\mu_1(x_0) = \sigma_1(x_0) - x_0$ and $\mu_2(y_0) = \sigma_2(y_0) - y_0$.

Proof. If $\mu_1(x_0) = 0$ or $\mu_2(y_0) = 0$, then (2.7) is obvious as both sides of (2.7) are equal to zero in this case. If $\mu_1(x_0) > 0$ and $\mu_2(y_0) > 0$, then a single partition of $R(x_0, y_0)$ is $P = \{[x_0, \sigma_1(x_0)) \times [y_0, \sigma_2(y_0))\} = \{(x_0, y_0)\}$, so that from (2.4)

$$S = f(x_0, y_0)(\sigma_1(x_0) - x_0)(\sigma_2(y_0) - y_0) = f(x_0, y_0)\mu_1(x_0)\mu_2(y_0).$$

Therefore f is Δ-integrable over $R(x_0, y_0)$ and (2.7) holds. $\square$

Theorem 2.6. *Let $a, b \in \mathbb{T}_1$ with $a \le b$ and $c, d \in \mathbb{T}_2$ with $c \le d$. Then we have:*

(i) *If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then a bounded function f on $R = [a, b) \times [c, d)$ is Δ-integrable if and only if f is Riemann integrable on R in the classical sense, and then*

$$\iint_R f(x, y)\Delta_1 x\Delta_2 y = \iint_R f(x, y)\mathrm{d}x\mathrm{d}y,$$

where the integral on the right is the ordinary Riemann integral.

(ii) *If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, then every function f defined on $R = [a, b) \times [c, d)$ is Δ-integrable over R, and*

$$\iint_R f(x, y)\Delta_1 x\Delta_2 y = \begin{cases} \displaystyle\sum_{k=a}^{b-1}\sum_{l=c}^{d-1} f(k, l) & \text{if } a < b \text{ and } c < d \\ 0 & \text{if } a = b \text{ or } c = d. \end{cases} \tag{2.8}$$

Proof. Clearly, the above given Definition 2.3 coincides in case $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ with the usual Riemann definition of the integral. Notice that the classical definition of Riemann's integral does not depend on whether the rectangle R and the subrectangles of its partition are taken closed, half-closed, or open. Moreover, if $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then $\mathcal{P}_\delta(R)$ consists of all partitions of R with norm (mesh) less than or equal to $\delta\sqrt{2}$. So part (i) is valid. To prove part (ii), let $a < b$ and $c < d$. Then $b = a + p$ and $d = c + q$ for some $p, q \in \mathbb{N}$. Obviously, for all $\delta \in (0, 1)$, the set $\mathcal{P}_\delta(R)$ will contain the single partition P^* of R given by (2.2) and (2.3) with $n = p$, $k = q$, and

$$x_0 = a, \qquad x_1 = a + 1, \ldots, x_p = a + p \quad \text{and} \quad y_0 = c, \qquad y_1 = c + 1, \ldots, y_q = c + q.$$

Then R_{ij} contains the single point (x_{i-1}, y_{j-1}):

$$R_{ij} = [x_{i-1}, x_i) \times [y_{j-1}, y_j) = \left\{(x_{i-1}, y_{j-1})\right\} \quad \text{for all } 1 \le i \le p,\ 1 \le j \le q.$$

Therefore from (2.4)

$$S = \sum_{i=1}^{p}\sum_{j=1}^{q} f(x_{i-1}, y_{j-1})(x_i - x_{i-1})(y_j - y_{j-1}) = \sum_{i=1}^{p}\sum_{j=1}^{q} f(x_{i-1}, y_{j-1}) = \sum_{k=a}^{b-1}\sum_{l=c}^{d-1} f(k, l)$$

for all partitions in $\mathcal{P}_\delta(R)$ with arbitrary $\delta \in (0, 1)$. Hence f is Δ-integrable over $R = [a, b) \times [c, d)$ and (2.8) holds for $a < b$ and $c < d$. If $a = b$ or $c = d$, then relation (2.5) shows the validity of (2.8). □

Note that in the two-variable case four types of integrals can be defined:

(i) $\Delta\Delta$-integral over $[a, b) \times [c, d)$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha, \beta) \times [\gamma, \delta)$;
(ii) $\nabla\nabla$-integral over $(a, b] \times (c, d]$, which is defined by using partitions consisting of subrectangles of the form $(\alpha, \beta] \times (\gamma, \delta]$;
(iii) $\Delta\nabla$-integral over $[a, b) \times (c, d]$, which is defined by using partitions consisting of subrectangles of the form $[\alpha, \beta) \times (\gamma, \delta]$;
(iv) $\nabla\Delta$-integral over $(a, b] \times [c, d)$, which is defined by using partitions consisting of subrectangles of the form $(\alpha, \beta] \times [\gamma, \delta)$.

For brevity the first integral is called simply as Δ-integral, and in this paper we are dealing solely with such double Δ-integrals.

Now we present some properties of double Δ-integrals over rectangles. A function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is said to be continuous at $(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x, y) - f(x', y')| < \varepsilon$ for all $(x', y') \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfying $d((x, y), (x', y')) < \delta$. If (x, y) is an isolated point of $\mathbb{T}_1 \times \mathbb{T}_2$, then the definition implies that every function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is continuous at (x, y). In particular, every function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is continuous at each point of $\mathbb{Z} \times \mathbb{Z}$.

Theorem 2.7. *Every continuous function on $K = [a, b] \times [c, d]$ is Δ-integrable over $R = [a, b) \times [c, d)$.*

Theorem 2.8. *Let f be a function that is Δ-integrable over $R = [a, b) \times [c, d)$. Further, let $a', b' \in [a, b]$ with $a' < b'$ and $c', d' \in [c, d]$ with $c' < d'$. Then f is Δ-integrable over $R' = [a', b') \times [c', d')$.*

Theorem 2.9 (*Linearity*). *Let f and g be Δ-integrable functions on $R = [a, b) \times [c, d)$, and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also Δ-integrable on R and*

$$\iint_R [\alpha f(x, y) + \beta g(x, y)]\, \Delta_1 x \Delta_2 y = \alpha \iint_R f(x, y)\Delta_1 x \Delta_2 y + \beta \iint_R g(x, y)\Delta_1 x \Delta_2 y.$$

Theorem 2.10. *If f and g are Δ-integrable on R, then so is their product fg.*

Theorem 2.11 (*Additivity*). *Let the rectangle $R = [a, b) \times [c, d)$ be a union of two disjoint rectangles of the forms $R_1 = [a_1, b_1) \times [c_1, d_1)$ and $R_2 = [a_2, b_2) \times [c_2, d_2)$. Then f is Δ-integrable over R if and only if f is Δ-integrable over each of R_1 and R_2. In this case*

$$\iint_R f(x, y)\Delta_1 x \Delta_2 y = \iint_{R_1} f(x, y)\Delta_1 x \Delta_2 y + \iint_{R_2} f(x, y)\Delta_1 x \Delta_2 y.$$

Theorem 2.12. *If f and g are Δ-integrable functions on R satisfying the inequality $f(x,y) \le g(x,y)$ for all $(x,y) \in R$, then*

$$\iint_R f(x,y)\Delta_1 x \Delta_2 y \le \iint_R g(x,y)\Delta_1 x \Delta_2 y.$$

Theorem 2.13. *If f is a Δ-integrable function on R, then so is $|f|$ and*

$$\left| \iint_R f(x,y)\Delta_1 x \Delta_2 y \right| \le \iint_R |f(x,y)| \, \Delta_1 x \Delta_2 y.$$

Theorem 2.14 (*Mean Value Theorem*). *Let f and g be Δ-integrable functions on R, and let g be nonnegative (or nonpositive) on R. Let us set*

$$m = \inf\{f(x,y) : (x,y) \in R\} \quad \text{and} \quad M = \sup\{f(x,y) : (x,y) \in R\}.$$

Then there exists a real number $\Lambda \in [m, M]$ such that

$$\iint_R f(x,y) g(x,y)\Delta_1 x \Delta_2 y = \Lambda \iint_R g(x,y)\Delta_1 x \Delta_2 y.$$

An effective way for evaluating multiple integrals is to reduce them to iterated (successive) integrations with respect to each of the variables.

Theorem 2.15. *Let f be Δ-integrable over $R = [a,b) \times [c,d)$ and suppose that the single integral*

$$I(x) = \int_c^d f(x,y)\Delta_2 y \tag{2.9}$$

exists for each $x \in [a,b)$. Then the iterated integral $\int_a^b I(x)\Delta_1 x$ exists, and

$$\iint_R f(x,y)\Delta_1 x \Delta_2 y = \int_a^b \Delta_1 x \int_c^d f(x,y)\Delta_2 y. \tag{2.10}$$

Remark 2.16. We can interchange in Theorem 2.15 the roles of x and y, i.e., we may assume the existence of the double integral and the existence of the single integral

$$K(y) = \int_a^b f(x,y)\Delta_1 x \tag{2.11}$$

for each $y \in [c,d)$. Then the iterated integral $\int_c^d K(y)\Delta_2 y$ exists, and

$$\iint_R f(x,y)\Delta_1 x \Delta_2 y = \int_c^d \Delta_2 y \int_a^b f(x,y)\Delta_1 x. \tag{2.12}$$

Remark 2.17. If together with $\iint_R f(x,y)\Delta_1 x \Delta_2 y$ there exist both single integrals (2.9) and (2.11), then the formulas (2.10) and (2.12) hold simultaneously, i.e.,

$$\int_a^b \Delta_1 x \int_c^d f(x,y)\Delta_2 y = \int_c^d \Delta_2 y \int_a^b f(x,y)\Delta_1 x = \iint_R f(x,y)\Delta_1 x \Delta_2 y.$$

Remark 2.18. If the function f is continuous on $[a,b] \times [c,d]$, then the existence of all the above mentioned integrals is guaranteed. In this case any of the formulas (2.10) and (2.12) may be used to calculate the double integral.

Now we define double Δ-integrals over so-called ω-type subsets of $\mathbb{T}_1 \times \mathbb{T}_2$ as follows (see [17] for double Δ-integrals over more general subsets like Jordan Δ-measurable subsets of $\mathbb{T}_1 \times \mathbb{T}_2$).

Definition 2.19. We say that $E \subset \mathbb{T}_1 \times \mathbb{T}_2$ is a *set of the type* ω (or ω-type set) if it can be represented in at least one way as a union

$$E = \bigcup_{k=1}^{m} R_k \tag{2.13}$$

of a finite number of rectangles $R_1, R_2, \ldots, R_m$ of the form (2.1) that are pairwise disjoint and adjoining to each other. Next, we say that a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is Δ-integrable over the ω-type set E if f is Δ-integrable over each of the rectangles R_k for $1 \le k \le m$. Then the number

$$\iint_E f(x, y)\Delta_1 x \Delta_2 y = \sum_{k=1}^{m} \iint_{R_k} f(x, y)\Delta_1 x \Delta_2 y \tag{2.14}$$

is called the double Δ-integral of f over E.

It is easily seen, by using Theorem 2.11, that the sum (2.14) does not depend on how E is represented as a union of a finite number of rectangles of the form (2.1) which are disjoint and adjoining to each other.

Finally, we present the concept of line integrals on time scales and, using it, a version of Green's formula for time scales (for details see [18]).

Definition 2.20. Together with the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, let $\mathbb{T}$ be a third time scale with the delta differentiation operator Δ. Further, let $\alpha \le \beta$ be points in $\mathbb{T}$ and $[\alpha, \beta]$ be the closed interval in $\mathbb{T}$, and let $\varphi : [\alpha, \beta] \to \mathbb{T}_1$ and $\psi : [\alpha, \beta] \to \mathbb{T}_2$ be continuous (in the time scale topology) on $[\alpha, \beta]$. Then the pair of functions

$$x = \varphi(t), \qquad y = \psi(t), \quad t \in [\alpha, \beta] \subset \mathbb{T} \tag{2.15}$$

is said to define a (*time scale continuous*) *curve* Γ in $\mathbb{T}_1 \times \mathbb{T}_2$. If $(\varphi(\alpha), \psi(\alpha)) = (\varphi(\beta), \psi(\beta))$, then the curve is said to be *closed*.

We can think of Γ as an *oriented curve*, in the sense that a point $(x', y') = (\varphi(t'), \psi(t')) \in \Gamma$ is regarded as *distinct* from a point $(x'', y'') = (\varphi(t''), \psi(t'')) \in \Gamma$ if $t' \ne t''$ and as *preceding* (x'', y'') if $t' < t''$. The oriented curve Γ is then said to be "traversed in the direction of increasing t". The curve differing from Γ only by the direction in which it is traversed will be denoted by $-\Gamma$.

Definition 2.21. We say that the curve Γ given by (2.15) is Δ-*smooth* if φ and ψ are continuous on $[\alpha, \beta]$ and Δ-differentiable on $[\alpha, \beta)$ and their Δ-derivatives φ^Δ and ψ^Δ are Δ-integrable over $[\alpha, \beta)$.

Let two functions $M(x, y)$ and $N(x, y)$ be defined and continuous on the curve Γ (for example, for the function $M(x, y)$, this means that for each $A_0 \in \Gamma$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $|M(A) - M(A_0)| < \varepsilon$ whenever $A \in \Gamma$ and $d(A, A_0) < \delta$, where $d(A, A_0)$ denotes the Euclidean distance between the points A and A_0). Next, let Γ be Δ-smooth. Then we define the line delta integral by

$$\int_\Gamma M(x, y)\Delta_1 x + N(x, y)\Delta_2 y = \int_\alpha^\beta [M(\varphi(t), \psi(t))\varphi^\Delta(t) + N(\varphi(t), \psi(t))\psi^\Delta(t)]\Delta t.$$

Remark 2.22. We call the curve Γ given by (2.15) *piecewise* Δ-*smooth* if φ and ψ are continuous on $[\alpha, \beta]$ and there is a partition $\alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_m = \beta$ of $[\alpha, \beta]$ such that φ and ψ have Δ-integrable Δ-derivatives on each of the intervals $[\gamma_{i-1}, \gamma_i)$, $i \in \{1, 2, \ldots, m\}$. In case of a piecewise Δ-smooth curve Γ, it is natural to define line Δ-integrals along this curve as sums of line Δ-integrals along all Δ-smooth parts constituting the curve Γ.

Similarly to line delta integrals we can also define *line nabla integrals*. Suppose that the curve Γ is given by the parametric equation (2.15), where φ and ψ are continuous on $[\alpha, \beta]$ and ∇-differentiable on $(\alpha, \beta]$. If φ^∇ and ψ^∇ are ∇-integrable over $(\alpha, \beta]$ and if the functions M and N are continuous on Γ, then we define

$$\int_\Gamma M(x, y)\nabla_1 x + N(x, y)\nabla_2 y = \int_\alpha^\beta [M(\varphi(t), \psi(t))\varphi^\nabla(t) + N(\varphi(t), \psi(t))\psi^\nabla(t)]\nabla t.$$

Definition 2.23. Let R be a "rectangle" in $\mathbb{T}_1 \times \mathbb{T}_2$ as given by (2.1). Let us set

$$L_1 = \{(x, c) : x \in [a, b]\}, \qquad L_2 = \{(b, y) : y \in [c, d]\},$$
$$L_3 = \{(x, d) : x \in [a, b]\}, \qquad L_4 = \{(a, y) : y \in [c, d]\}.$$

Each of L_j for $j \in \{1, 2, 3, 4\}$ is an oriented "line segment"; e.g., the positive orientation of L_1 arises according to the increase of x from a to b and the positive orientation of L_2 arises according to the increase of y from c to d. The set (closed curve)

$$\Gamma := L_1 \cup L_2 \cup (-L_3) \cup (-L_4)$$

is called the *positively oriented fence* of R. Positivity of orientation of Γ means that the rectangle R remains on the "left" side as we describe the fence curve Γ.

Definition 2.24. Let $E \subset \mathbb{T}_1 \times \mathbb{T}_2$ be an ω-type set of the form (2.13). Let Γ_k be the positively oriented fence of the rectangle R_k. Let us set $X = \bigcup_{k=1}^{m} \Gamma_k$. Further, let X_0 consist of a finite number of line segments each of which serves as a common part of fences of two adjoining rectangles belonging to $\{R_1, R_2, \ldots, R_m\}$. Then the set $\Gamma = X \setminus X_0$ forms a positively oriented closed "polygonal curve", which we call the *positively oriented fence of the set E* (the set E remains on the "left" as we describe the fence curve Γ).

We are now able to formulate the following theorem (for its proof see [18]).

Theorem 2.25 (*Green's Formula*). *Let $E \subset \mathbb{T}_1 \times \mathbb{T}_2$ be an ω-type set and let Γ be its positively oriented fence. If the functions M and N are continuous and have continuous partial delta derivatives $\partial M/\Delta_2 y$ and $\partial N/\Delta_1 x$ on $E \cup \Gamma$, then*

$$\iint_E \left(\frac{\partial N}{\Delta_1 x} - \frac{\partial M}{\Delta_2 y} \right) \Delta_1 x \Delta_2 y = \int_\Gamma M \mathrm{d}^* x + N \mathrm{d}^* y, \tag{2.16}$$

where the "star line integrals" on the right side in (2.16) denote the sum of line delta integrals taken over the line segment constituents of Γ directed to the right or upwards and line nabla integrals of f taken over the line segment constituents of Γ directed to the left or downwards.

3. The double integral variational problem

Recall that a single variable function on a time scale is called *rd-continuous* provided it is continuous at right-dense points and its left-sided limit exists (finite) at left-dense points. Let C_{rd} denote the set of functions $f(x, y)$ on $\mathbb{T}_1 \times \mathbb{T}_2$ with the following properties:

(i) f is rd-continuous in x for fixed y;
(ii) f is rd-continuous in y for fixed x;
(iii) if $(x_0, y_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ with x_0 right-dense or maximal and y_0 right-dense or maximal, then f is continuous at (x_0, y_0);
(iv) if x_0 and y_0 are both left-dense, then the limit of $f(x, y)$ exists (finite) as (x, y) approaches (x_0, y_0) along any path in $\{(x, y) \in \mathbb{T}_1 \times \mathbb{T}_2 : x < x_0, y < y_0\}$.

By $C_{\mathrm{rd}}^{(1)}$ we denote the set of all continuous functions for which both the Δ_1-partial derivative and the Δ_2-partial derivative exist and are of class C_{rd}.

Let $E \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a set of type ω and let Γ be its positively oriented fence. Further, let a function

$$L(x, y, u, p, q), \quad \text{where } (x, y) \in E \cup \Gamma \text{ and } (u, p, q) \in \mathbb{R}^3$$

be given. We require that, in the indicated domain of variation of the independent variables, the function should be continuous, together with its partial delta derivatives of the first and second order with respect to x, y and partial usual derivatives of the first and second order with respect to u, p, q. Consider the functional

$$J(u) = \iint_E L(x, y, u(\sigma_1(x), \sigma_2(y)), u^{\Delta_1}(x, \sigma_2(y)), u^{\Delta_2}(\sigma_1(x), y)) \Delta_1 x \Delta_2 y, \tag{3.1}$$

whose domain of definition $D(J)$ consists of functions $u \in C_{\mathrm{rd}}^{(1)}(E \cup \Gamma)$ satisfying the

$$\text{“boundary condition” } u = g(x, y) \quad \text{on } \Gamma, \tag{3.2}$$

where g is a fixed function defined and continuous on the fence Γ of E. We call functions $u \in D(J)$ *admissible*. The problem of the variational calculus now consists of the following: Given a functional J of the form (3.1) with its domain of definition $D(J)$, it is required to find an element $\hat{u} \in D(J)$ which satisfies

$$\text{either } J(\hat{u}) = \inf_{u \in D(J)} J(u) \quad \text{or} \quad J(\hat{u}) = \sup_{u \in D(J)} J(u). \tag{3.3}$$

The problem of maximizing the functional J is identical with the problem of minimizing the functional $-J$. Therefore, in what follows, we will treat only the minimum problem. We will assume that there exists at least one admissible function u_0. Note that this assumption is essential: In contrast to the case of one variable, it is possible here (even if $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$) to choose a function $g(x, y)$, continuous on Γ, such that no function u_0 is admissible. In this case the domain $D(J)$ is empty, and the problem of minimizing the functional J loses its meaning. If the function u_0 exists, then the domain $D(J)$ contains a set of functions of the form $u(x, y) = u_0(x, y) + \eta(x, y)$, where $\eta \in C_{\mathrm{rd}}^{(1)}(E \cup \Gamma)$ and $\eta = 0$ on Γ. Any such η is called an *admissible variation*.

The above problems (3.3) are problems of finding *absolute extrema*, but we can easily define a weak or strong neighborhood of a given function and state the problem of finding *local* (or *relative*) *extrema*. For $f \in C_{\mathrm{rd}}^{(1)}(E \cup \Gamma)$ we define the norm

$$\|f\| = \sup_{(x,y) \in E \cup \Gamma} |f(x, y)| + \sup_{(x,y) \in E} |f^{\Delta_1}(x, \sigma_2(y))| + \sup_{(x,y) \in E} |f^{\Delta_2}(\sigma_1(x), y)|.$$

A function $\hat{u} \in D(J)$ is called a (*weak*) *local minimum* of J provided there exists $\delta > 0$ such that $J(\hat{u}) \le J(u)$ for all $u \in D(J)$ with $\|u - \hat{u}\| < \delta$. If $J(\hat{u}) < J(u)$ for all such $u \neq \hat{u}$, then $\hat{u}$ is said to be *proper*.

4. First and second variations

For a fixed element $u \in D(J)$ and a fixed admissible variation η we define a function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\varepsilon) = \Phi(\varepsilon; u, \eta) = J(u + \varepsilon\eta) \quad \text{for } \varepsilon \in \mathbb{R}.$$

From (3.1), by virtue of the conditions imposed on L, it follows that $\Phi(\varepsilon)$ is twice continuously differentiable, and the first and second derivatives of Φ can be obtained by differentiating under the integral sign. The *first and second variations* of the functional J at the point u are defined by

$$J_1(u, \eta) = \Phi'(0; u, \eta) \quad \text{and} \quad J_2(u, \eta) = \Phi''(0; u, \eta),$$

respectively. For fixed u, the variations $J_1(u, \eta)$ and $J_2(u, \eta)$ are functionals of η. Note that $J_1(u, \eta)$ and $J_2(u, \eta)$ are denoted also by $\delta J(u, \eta)$ and $\delta^2 J(u, \eta)$, respectively.

The following two theorems are standard and offer necessary and sufficient conditions for local minima of J in terms of the first and second variations of J.

Theorem 4.1 (*Necessary Conditions*). *If $\hat{u} \in D(J)$ is a local minimum of J, then*

$$J_1(\hat{u}, \eta) = 0 \quad \textit{and} \quad J_2(\hat{u}, \eta) \ge 0 \quad \textit{for all admissible variations } \eta.$$

Proof. Assume that the functional J has a local minimum at $\hat{u} \in D(J)$. We take an arbitrary fixed admissible variation η and define the function

$$\varphi(\varepsilon) = J(\hat{u} + \varepsilon\eta), \quad \text{where } \varepsilon \in \mathbb{R}. \tag{4.1}$$

Therefore we have $\varphi'(0) = J_1(\hat{u}, \eta)$ and $\varphi''(0) = J_2(\hat{u}, \eta)$. By Taylor's theorem,

$$\varphi(\varepsilon) = \varphi(0) + \frac{\varphi'(0)}{1!}\varepsilon + \frac{\varphi''(\alpha)}{2!}\varepsilon^2, \quad \text{where } |\alpha| \in (0, |\varepsilon|). \tag{4.2}$$

If $|\varepsilon|$ is sufficiently small, then we have that the norm of the difference

$$\|(\hat{u}+\varepsilon\eta)-\hat{u}\| = |\varepsilon|\|\eta\|$$

will be as small as we please, and then, from the definition of a local minimum,

$$J(\hat{u}+\varepsilon\eta) \geq J(\hat{u}), \quad \text{i.e., } \varphi(\varepsilon) \geq \varphi(0).$$

This inequality implies that the function φ of the real variable ε has a local minimum for $\varepsilon = 0$. But then, necessarily, $\varphi'(0) = 0$ (this easily follows also from (4.2)) or, equivalently, $J_1(\hat{u}, \eta) = 0$. Now from (4.2) by the equality $\varphi'(0) = 0$, we have

$$\varphi(\varepsilon) - \varphi(0) = \frac{1}{2}\varphi''(\alpha)\varepsilon^2$$

and therefore $\varphi''(\alpha) \geq 0$ for all ε whose absolute value is sufficiently small. Letting here $\varepsilon \to 0$ and noting that $\alpha \to 0$ as $\varepsilon \to 0$ and that φ'' is continuous, we get $\varphi''(0) \geq 0$ or, equivalently, $J_2(\hat{u}, \eta) \geq 0$. □

Theorem 4.2 (*Sufficient Condition*). *Let $\hat{u} \in D(J)$ be such that $J_1(\hat{u}, \eta) = 0$ for all admissible variations η. If $J_2(u, \eta) \geq 0$ for all $u \in D(J)$ and all admissible variations η, then J has an absolute minimum at the point $\hat{u}$. If $J_2(u, \eta) \geq 0$ for all u in some neighborhood of the point $\hat{u}$ and all admissible variations η, then the functional J has a local minimum at $\hat{u}$.*

Proof. Define the function φ as in (4.1). From (4.2) we have for $\varepsilon = 1$

$$\varphi(1) = \varphi(0) + \frac{\varphi'(0)}{1!} + \frac{\varphi''(\alpha)}{2!}, \quad \text{where } \alpha \in (0, 1). \tag{4.3}$$

Next, we have

$$\varphi(1) = J(\hat{u}+\eta), \qquad \varphi(0) = J(\hat{u}), \qquad \varphi'(0) = J_1(\hat{u}, \eta) = 0,$$
$$\varphi''(\alpha) = \left[\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} J(\hat{u}+\varepsilon\eta)\right]_{\varepsilon=\alpha} = \left[\frac{\mathrm{d}^2}{\mathrm{d}\beta^2} J(\hat{u}+\alpha\eta+\beta\eta)\right]_{\beta=0} = J_2(\hat{u}+\alpha\eta, \eta),$$

so that (4.3) gives

$$J(\hat{u}+\eta) = J(\hat{u}) + \frac{1}{2}J_2(\hat{u}+\alpha\eta, \eta) \quad \text{for all admissible variations } \eta, \tag{4.4}$$

where $\alpha \in (0, 1)$ depends on $\hat{u}$ and η. Now the proof of the theorem can be completed as follows. In the first case we have

$$J_2(\hat{u}+\alpha\eta, \eta) \geq 0 \quad \text{for all admissible variations } \eta.$$

If $u \in D(J)$, then putting $\eta = u - \hat{u}$ provides from (4.4) that $J(u) \geq J(\hat{u})$. Consider now the second case. There exists $r > 0$ such that for $u \in D(J)$ and $\|u - \hat{u}\| < r$ we have $J_2(u, \eta) \geq 0$ for all admissible variations η. We take such an element u and again put $\eta = u - \hat{u}$. Then

$$J(u) = J(\hat{u}) + \frac{1}{2}J_2(\hat{u}+\alpha\eta, \eta).$$

We have

$$\|(\hat{u}+\alpha\eta)-\hat{u}\| = \|\alpha\eta\| = |\alpha|\|\eta\| \leq \|\eta\| = \|u-\hat{u}\| < r.$$

Hence it follows that $J_2(\hat{u}+\alpha\eta, \eta) \geq 0$, and, consequently, $J(u) \geq J(\hat{u})$. □

In view of the above two results it will be important to find another representation of the first and second variations. This is done in the following lemma.

Lemma 4.3. *Let $u \in D(J)$. The first and second variations of J at u are given by*

$$J_1(u,\eta) = \iint_E \{L_u(\cdot)\eta(\sigma_1(x),\sigma_2(y)) + L_p(\cdot)\eta^{\Delta_1}(x,\sigma_2(y)) + L_q(\cdot)\eta^{\Delta_2}(\sigma_1(x),y)\}\Delta_1 x\Delta_2 y, \tag{4.5}$$

$$\begin{aligned} J_2(u,\eta) = &\iint_E \{L_{uu}(\cdot)\,[\eta(\sigma_1(x),\sigma_2(y))]^2 + L_{pp}(\cdot)[\eta^{\Delta_1}(x,\sigma_2(y))]^2 \\ &+ L_{qq}(\cdot)[\eta^{\Delta_2}(\sigma_1(x),y)]^2 + 2L_{up}(\cdot)\eta(\sigma_1(x),\sigma_2(y))\eta^{\Delta_1}(x,\sigma_2(y)) \\ &+ 2L_{uq}(\cdot)\eta(\sigma_1(x),\sigma_2(y))\eta^{\Delta_2}(\sigma_1(x),y) + 2L_{pq}(\cdot)\eta^{\Delta_1}(x,\sigma_2(y))\eta^{\Delta_2}(\sigma_1(x),y)\}\Delta_1 x\Delta_2 y \end{aligned} \tag{4.6}$$

for all admissible variations η, where

$$(\cdot) = (x, y, u(\sigma_1(x),\sigma_2(y)), u^{\Delta_1}(x,\sigma_2(y)), u^{\Delta_2}(\sigma_1(x),y)). \tag{4.7}$$

Proof. By definition we have

$$J_1(u,\eta) = \Phi'(0) \quad \text{and} \quad J_2(u,\eta) = \Phi''(0),$$

where

$$\Phi(\varepsilon) = J(u+\varepsilon\eta) = \iint_E L(\cdot\cdot)\Delta_1 x\Delta_2 y,$$

with

$$\begin{aligned}(\cdot\cdot) = &(x, y, u(\sigma_1(x),\sigma_2(y)) + \varepsilon\eta(\sigma_1(x),\sigma_2(y)), u^{\Delta_1}(x,\sigma_2(y)) \\ &+ \varepsilon\eta^{\Delta_1}(x,\sigma_2(y)), u^{\Delta_2}(\sigma_1(x),y) + \varepsilon\eta^{\Delta_2}(\sigma_1(x),y)).\end{aligned}$$

We can differentiate under the integral sign and thus obtain formulas (4.5)–(4.7). □

5. Euler's condition

Let E be an ω-type subset of $\mathbb{T}_1 \times \mathbb{T}_2$ and Γ be the positively oriented fence of E. Let us set

$$E^o = \{(x,y) \in E : (\sigma_1(x),\sigma_2(y)) \in E\}.$$

The following lemma is an extension of the fundamental lemma of double integral variational analysis to time scales.

Lemma 5.1 (*Dubois–Reymond*). *If $M(x,y)$ is continuous on $E \cup \Gamma$ with*

$$\iint_E M(x,y)\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y = 0$$

for every admissible variation η, then

$$M(x,y) = 0 \quad \text{for all } (x,y) \in E^o.$$

Proof. We assume that the function M is not zero at some point $(x_0, y_0) \in E^o$; suppose $M(x_0, y_0) > 0$. Continuity ensures that $M(x,y)$ is positive in a rectangle

$$\Omega = [x_0, x_1) \times [y_0, y_1) \subset E$$

for some points $x_1 \in \mathbb{T}_1$, $y_1 \in \mathbb{T}_2$ such that $\sigma_1(x_0) \le x_1$ and $\sigma_2(y_0) \le y_1$. We set

$$\eta(x,y) = \begin{cases} (x-x_0)^2[x-\sigma_1(x_1)]^2(y-y_0)^2[y-\sigma_2(y_1)]^2 & \text{for } (x,y) \in \Omega, \\ 0 \quad \text{for } (x,y) \in E \setminus \Omega. \end{cases}$$

This function is zero on Γ and belongs to $C^{(1)}_{\mathrm{rd}}(E \cup \Gamma)$. We have

$$\iint_E M(x,y)\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y = \iint_\Omega M(x,y)\eta(\sigma_1(x),\sigma_2(y))\Delta_1 x\Delta_2 y > 0.$$

This contradiction proves the assertion of the lemma. □

Now, using Lemma 5.1, we can derive Euler's necessary condition.

Theorem 5.2 (*Euler's Necessary Condition*). *Suppose that an admissible function $\hat{u}$ provides a local minimum for J and that the function $\hat{u}$ has continuous partial delta derivatives of the second order. Then $\hat{u}$ satisfies the Euler–Lagrange equation*

$$L_u(\cdot) - \frac{\partial}{\Delta_1 x} L_p(\cdot) - \frac{\partial}{\Delta_2 y} L_q(\cdot) = 0 \quad \text{for } (x, y) \in E^o, \tag{5.1}$$

where

$$(\cdot) = (x, y, \hat{u}(\sigma_1(x), \sigma_2(y)), \hat{u}^{\Delta_1}(x, \sigma_2(y)), \hat{u}^{\Delta_2}(\sigma_1(x), y)). \tag{5.2}$$

Proof. By Theorem 4.1 we have $J_1(\hat{u}, \eta) = 0$ for all admissible variations η. Hence Lemma 4.3 gives

$$\iint_E \{L_u(\cdot)\eta(\sigma_1(x), \sigma_2(y)) + L_p(\cdot)\eta^{\Delta_1}(x, \sigma_2(y)) + L_q(\cdot)\eta^{\Delta_2}(\sigma_1(x), y)\}\Delta_1 x \Delta_2 y = 0, \tag{5.3}$$

where $(\cdot)$ is given in (5.2). Now we notice that

$$\begin{aligned}
&\iint_E \{L_p(\cdot)\eta^{\Delta_1}(x, \sigma_2(y)) + L_q(\cdot)\eta^{\Delta_2}(\sigma_1(x), y)\}\Delta_1 x \Delta_2 y \\
&\quad = \iint_E \left\{ \frac{\partial}{\Delta_1 x}\left[L_p(\cdot)\eta(x, \sigma_2(y))\right] + \frac{\partial}{\Delta_2 y}\left[L_q(\cdot)\eta(\sigma_1(x), y)\right] \right\} \Delta_1 x \Delta_2 y \\
&\qquad - \iint_E \left\{ \frac{\partial}{\Delta_1 x} L_p(\cdot) + \frac{\partial}{\Delta_2 y} L_q(\cdot) \right\} \eta(\sigma_1(x), \sigma_2(y))\Delta_1 x \Delta_2 y.
\end{aligned}$$

On the other hand, by Theorem 2.25 (Green's formula),

$$\begin{aligned}
&\iint_E \left\{ \frac{\partial}{\Delta_1 x}\left[L_p(\cdot)\eta(x, \sigma_2(y))\right] + \frac{\partial}{\Delta_2 y}\left[L_q(\cdot)\eta(\sigma_1(x), y)\right] \right\} \Delta_1 x \Delta_2 y \\
&\quad = \int_\Gamma \eta(x, \sigma_2(y))L_p(\cdot)\mathrm{d}^* y - \eta(\sigma_1(x), y)L_q(\cdot)\mathrm{d}^* x = 0
\end{aligned}$$

since $\eta = 0$ on Γ. Consequently, we get from (5.3)

$$\iint_E \left\{ L_u(\cdot) - \frac{\partial}{\Delta_1 x} L_p(\cdot) - \frac{\partial}{\Delta_2 y} L_q(\cdot) \right\} \eta(\sigma_1(x), \sigma_2(y))\Delta_1 x \Delta_2 y = 0$$

for all admissible variations η. Therefore, by Lemma 5.1, we have (5.1). □

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