Pergamon

Computers Math. Applic. Vol. 36, No. 10–12, pp. 179–192, 1998

Printed in Great Britain
0898-1221/98 $19.00 + 0.00

PII: S0898-1221(98)00210-7

Discrete Linear Hamiltonian Eigenvalue Problems

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Abstract—This paper introduces general discrete linear Hamiltonian eigenvalue problems and characterizes the eigenvalues. Assumptions are given, among them the new notion of strict controllability of a discrete system, that imply isolatedness and lower boundedness of the eigenvalues. Due to the quite general assumptions, discrete Sturm-Liouville eigenvalue problems of higher order are included in the presented theory. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Linear Hamiltonian difference system, Discrete Sturm-Liouville eigenvalue problem, Strict controllability, Reid roundabout theorem, Comparison theorem.

1. INTRODUCTION

In this paper, we introduce discrete linear Hamiltonian eigenvalue problems, i.e., eigenvalue problems which consist of a linear Hamiltonian difference system depending on an eigenvalue parameter $\lambda \in \mathbb{R}$ subject to self-adjoint boundary conditions. The **main result** on these problems states that, under certain assumptions, the eigenvalues may be arranged as follows:

$$-\infty < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots,$$

i.e., that the set of eigenvalues is bounded below and that the eigenvalues are isolated in the sense that for any $\lambda \in \mathbb{R}$ one may pick an $\varepsilon = \varepsilon(\lambda) > 0$ such that the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$ contains at most one eigenvalue. The **central notion** connected to this isolatedness is the new concept of strict controllability of discrete systems which is also introduced in this paper. The **main tools** on handling these eigenvalue problems and on proving the above result is a theorem that gives a useful characterization of the eigenvalues (in terms of some matrix being singular), an index theorem (which calculates the local change of the number of some matrix-valued function's negative eigenvalues), a Reid roundabout theorem (that characterizes so-called positive definiteness of discrete quadratic functionals), and a comparison theorem (which states that positive definiteness of one functional together with certain assumptions imply positive definiteness of some other functional). Finally, it should be emphasized that our general assumptions allow us to include **discrete Sturm-Liouville eigenvalue problems** of higher order so that these important problems may be treated with the same techniques.

Let us shortly give an overview on the existing **literature** of the subject. Discrete Sturm-Liouville difference equations of order two as well as linear Hamiltonian difference systems have been an object of recent interest. Linear Hamiltonian difference systems were introduced by Erbe and Yan in [1] and examined in three proceeding papers [2–4] by the same authors, however, under assumptions that only include the case of Sturm-Liouville difference equations of order two but

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not of higher order. Further important results in this matter have been obtained by Ahlbrandt, Došlý, Heifetz, Hooker, Patula, Peil, Peterson, and Ridenhour in [5–11]. In a recent series of publications by the author [12–17] (one of them is a joint work with Došlý), linear Hamiltonian difference systems were considered under assumptions that include the important case of Sturm-Liouville difference equations of higher order and that give so-called Reid roundabout theorems for those problems. This work may be considered as one origin of the results proved in the present paper. The other origin is the treatment of continuous linear Hamiltonian eigenvalue problems as is done in the paper [18] by Baur and Kratz and in the monograph [19] on the subject by Kratz. This work also contains the above cited index theorem which may be successfully applied in our discrete case also. Finally, while the study of eigenvalue problems in the existing literature basically reduces to discrete Sturm-Liouville eigenvalue problems of order two (see the books by Agarwal [20, Chapter 11] and by Kelley and Peterson [21, Chapter 7]), a special Sturm-Liouville difference equation of higher order depending on an eigenvalue parameter has been considered in the recent paper [22] by Kratz; however, there is no theory for eigenvalue problems subject to general linear Hamiltonian difference systems.

A brief discussion of this paper's setup is in order. The following section introduces discrete linear Hamiltonian eigenvalue problems and gives some preliminaries on linear Hamiltonian difference systems. In Section 3, we present the main result of this paper and give the assumptions that are needed; among them we introduce the concept of strict controllability of discrete systems and the notion of the so-called strict controllability index, which has no obvious analogue in the "continuous theory". Section 4 contains a characterization of the eigenvalues, and this characterization is also improved in some sense if the boundary conditions under consideration are separated. While Section 5 recalls two important auxiliary results (the index theorem from [19, Theorem 3.4.1] and the Reid roundabout theorem from [14, Theorem 3]), Section 6 contains a series of lemmas that are needed for proving the isolatedness of the eigenvalues. Finally, a comparison theorem is proved in Section 7, and as an application of it we show that the eigenvalues are bounded below.

2. PRELIMINARIES ON DISCRETE EIGENVALUE PROBLEMS

First of all, let us agree upon some terminology. While $\operatorname{Ker} M$, $\operatorname{Im} M$, $\operatorname{def} M$, $\operatorname{ind} M$, $M^{\top}$, and $M^{\dagger}$ denote the kernel, the image, the dimension of the kernel, the index (i.e., the number of negative eigenvalues), the transpose, and the Moore-Penrose Inverse (see, e.g., [23, Theorem 1.5]) of the matrix M, respectively, $M > 0$ and $M \geq 0$ mean that the (symmetric) matrix M is positive definite and positive semidefinite, respectively. Let $n \in \mathbb{N}$, $N \in \mathbb{N} \cup \{0\}$, $J := [0, N] \cap \mathbb{Z}$, $J^* := [0, N+1] \cap \mathbb{Z}$. We abbreviate a sequence $(z_k)_{k \in J^*}$ by z and use the forward difference operator Δ defined by $\Delta z_k := z_{k+1} - z_k$, $k \in J$.

Let there be given $n \times n$-matrices A_k, B_k, C_k for all $k \in J$ so that

$$I - A_k \text{ is invertible and } H_k = \begin{pmatrix} -C_k & A_k^{\top} \\ A_k & B_k \end{pmatrix} \text{ is symmetric for all } k \in J.$$

The system

$$\Delta \begin{pmatrix} -u_k \\ x_k \end{pmatrix} = H_k \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix}, \qquad 0 \leq k \leq N, \tag{H}$$

(where $x_k, u_k \in \mathbb{R}^n$ for all $k \in J^*$) is then called a *linear Hamiltonian difference system*. If the $n \times n$-matrix-valued functions $A_k(\lambda)$, $B_k(\lambda)$, $C_k(\lambda)$ depend for all $k \in J$

continuously and differentiable

on a parameter $\lambda \in \mathbb{R}$ (so that the above assumptions are satisfied for $I - A_k(\lambda)$ and for

$$H_k(\lambda) = \begin{pmatrix} -C_k(\lambda) & A_k^{\top}(\lambda) \\ A_k(\lambda) & B_k(\lambda) \end{pmatrix}$$

for each $\lambda \in \mathbb{R}$), then we consider the systems

$$\Delta \begin{pmatrix} -u_k \\ x_k \end{pmatrix} = H_k(\lambda) \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix}, \qquad 0 \le k \le N. \tag{H$_\lambda$}$$

Moreover, let there be given $2n \times 2n$-matrices R and R^* with

$$\operatorname{rank}(R \quad R^*) = 2n \qquad \text{and} \qquad RR^{*\top} = R^* R^\top.$$

We are interested in so-called *self-conjoined boundary conditions* (see [19, Definition 2.1.1 and Proposition 2.1.1])

$$R^* \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + R \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = 0. \tag{R}$$

Now, this paper deals with *discrete linear Hamiltonian eigenvalue problems* of the form

$$(\mathrm{H}_\lambda), \quad \lambda \in \mathbb{R} \quad \text{and} \quad (\mathrm{R}), \tag{E}$$

i.e.,

$$\begin{gathered} \left.\begin{aligned} \Delta x_k &= A_k(\lambda) x_{k+1} + B_k(\lambda) u_k \\ \Delta u_k &= C_k(\lambda) x_{k+1} - A_k^\top(\lambda) u_k \end{aligned}\right\}, \quad 0 \le k \le N, \qquad (\lambda \in \mathbb{R}), \\ R^* \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + R \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = 0. \end{gathered} \tag{E}$$

As usual, a number $\lambda \in \mathbb{R}$ is called an eigenvalue of (E) if (H$_\lambda$) has a nontrivial solution (x, u) satisfying (R), and this solution is then called an eigenfunction corresponding to the eigenvalue λ. Moreover, the set of all eigenfunctions is called the eigenspace, and its dimension is referred to as being the multiplicity of the eigenvalue.

We shortly summarize some basic definitions and results from [14] on linear Hamiltonian difference systems that will be needed later on.

DEFINITION 1. *(Conjoined Basis; see [14, Definition 1].) If the $n \times n$-matrices X_k, U_k (instead of the vectors x_k, u_k) solve (H) with*

$$\operatorname{rank}(X_k^\top \quad U_k^\top) = n \qquad \text{and} \qquad X_k^\top U_k = U_k^\top X_k, \qquad \text{for all } k \in J^*,$$

then (X, U) is called a conjoined basis of (H). Two conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ are called normalized whenever

$$X_k^\top \tilde{U}_k - U_k^\top \tilde{X}_k = I \ (\text{the } n \times n\text{-identity-matrix}), \qquad \text{for all } k \in J^*$$

holds. The conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ of (H) with

$$X_0 = \tilde{U}_0 = 0 \quad \text{and} \quad U_0 = -\tilde{X}_0 = I$$

are known as the special normalized conjoined bases of (H) at 0. ∎

LEMMA 1. *(See [19, Corollary 3.3.9] and [14, Lemma 3].) For any $m \in J^*$ and any conjoined basis (X, U) of (H), there exists another conjoined basis $(\tilde{X}, \tilde{U})$ of (H) such that (X, U) and $(\tilde{X}, \tilde{U})$ are normalized and such that $\tilde{X}_m$ is invertible.*

Furthermore, two matrix-valued solutions (X, U) and $(\tilde{X}, \tilde{U})$ are normalized conjoined bases of (H) iff (X^*, U^*) with

$$X^* = \begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix} \qquad \text{and} \qquad U^* = \begin{pmatrix} I & 0 \\ U & \tilde{U} \end{pmatrix},$$

is a conjoined basis of the system

$$\Delta \begin{pmatrix} -u_k \\ x_k \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -C_k & 0 & A_k^\top \\ 0 & 0 & 0 & 0 \\ 0 & A_k & 0 & B_k \end{pmatrix} \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix}, \qquad 0 \le k \le N,$$

where the occurring matrix is of size $4n \times 4n$. ∎

Definition 2. *(Disconjugacy; see [14, Definition 2].) The discrete quadratic functional*

$$\mathcal{F}(x,u) := \sum_{k=0}^{N} \{x_{k+1}^\top C_k x_{k+1} + u_k^\top B_k u_k\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^\top R^\dagger R^* R^\dagger R \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}$$

is called positive definite (we write $\mathcal{F} > 0$) if $\mathcal{F}(x,u) > 0$ holds for all admissible pairs (x,u) (i.e., that satisfy $\Delta x_k = A_k x_{k+1} + B_k u_k$ for all $k \in J$) with $x \neq 0$ and $\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \in \operatorname{Im} R^\top$. If in this definition $R = 0$ and $\mathcal{F} > 0$, then (H) is called disconjugate on J^*. ∎

Definition 3. *(Controllability; see [12, Definition 3] and [14, Definition 5].) The system (H) is called controllable on J^* if there exists $k \in J^*$ such that for all solutions (x,u) of (H) and for all $m \in J$ with $m + k \in J^*$, we have that*

$$x_m = x_{m+1} = \cdots = x_{m+k} = 0$$

implies $x = u = 0$ on J^*. The minimal integer $\kappa \in J^*$ with this property is then called the controllability index of (H). ∎

3. STRICT CONTROLLABILITY AND MAIN RESULTS

We open this section with the following key definition.

Definition 4. *(Strict Controllability). The set of systems $\{(H_\lambda) : \lambda \in \mathbb{R}\} =: (H_\mathbb{R})$ is called strictly controllable on J^* if*

(i) *(H_λ) is controllable on J^* for all $\lambda \in \mathbb{R}$ (see Definition 3), and if*
(ii) *there exists $k \in J$ such that for all $\lambda \in \mathbb{R}$, for all solutions (x,u) of (H_λ), and for all $m \in J$ with $m + k \in J$*

$$\dot{H}_m(\lambda) \begin{pmatrix} x_{m+1} \\ u_m \end{pmatrix} = \dot{H}_{m+1}(\lambda) \begin{pmatrix} x_{m+2} \\ u_{m+1} \end{pmatrix} = \cdots = \dot{H}_{m+k}(\lambda) \begin{pmatrix} x_{m+k+1} \\ u_{m+k} \end{pmatrix} = 0,$$

implies $x = u = 0$ on J^*. The minimal integer $\kappa_s \in J$ with this property is then called the strict controllability index of $(H_\mathbb{R})$. ∎

For stating our main results, we wish to label the following *assumptions*.

(V_1) $(H_\mathbb{R})$ is strictly controllable on J^*.
(V_2) $\lambda_1 \le \lambda_2$ always implies $H_k(\lambda_1) \le H_k(\lambda_2)$ for all $k \in J$.
(V_3) There exists $\underset{\sim}{\lambda} \in \mathbb{R}$ such that $\mathcal{F}(\cdot; \underset{\sim}{\lambda}) > 0$ and such that $\lambda \le \underset{\sim}{\lambda}$ always implies for all $k \in J$

$$\operatorname{Ker} B_k(\underset{\sim}{\lambda}) \subset \operatorname{Ker} B_k(\lambda) \qquad \text{and} \qquad B_k(\lambda) \left\{ B_k^\dagger(\lambda) - B_k^\dagger(\underset{\sim}{\lambda}) \right\} B_k(\lambda) \ge 0.$$

Now our main result reads as follows.

THEOREM 1. *Assume (V_1), (V_2), and (V_3). Then, if there exist eigenvalues of (E), they may be arranged by*

$$-\infty < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots .$$

More precisely,

(i) *(V_1) and (V_2) imply that the eigenvalues are isolated, and*

(ii) *(V_2) and (V_3) imply that the eigenvalues are bounded below by $\underset{\sim}{\lambda}$ (which is not an eigenvalue) provided (H_λ) is controllable on J^* for all $\lambda \in \mathbb{R}$.* ∎

The remaining sections are devoted to the proof of the above theorem. However, here we wish to make some remarks concerning the concept of strict controllability.

REMARK 1. Suppose $A_k(\lambda) \equiv: A_k$ and $B_k(\lambda) \equiv: B_k$ are constant for all $k \in J$. Then condition (ii) of Definition 4 (with strict controllability index $\kappa_s \in J$) already implies condition (i), i.e., controllability of (H_λ) on J^* for all $\lambda \in \mathbb{R}$, and the controllability indices $\kappa(\lambda)$ of (H_λ) satisfy $\max_{\lambda\in\mathbb{R}} \kappa(\lambda) \le \kappa_s + 1 \in J^*$. To prove this, assume (ii), let there be given $\lambda \in \mathbb{R}$, a solution (x, u) of (H_λ), and $m \in J$ with $m + \kappa_s + 1 \in J^*$ such that

$$x_m = x_{m+1} = x_{m+2} = \cdots = x_{m+\kappa_s+1} = 0$$

holds. Therefore,

$$\dot{C}_m(\lambda)x_{m+1} = \dot{C}_{m+1}(\lambda)x_{m+2} = \cdots = \dot{C}_{m+\kappa_s}(\lambda)x_{m+\kappa_s+1} = 0,$$

and hence (note $\dot{H}_k(\lambda) = \begin{pmatrix} -\dot{C}_k(\lambda) & 0 \\ 0 & 0 \end{pmatrix}$ for $k \in J$),

$$\dot{H}_m(\lambda)\begin{pmatrix} x_{m+1} \\ u_m \end{pmatrix} = \dot{H}_{m+1}(\lambda)\begin{pmatrix} x_{m+2} \\ u_{m+1} \end{pmatrix} = \cdots = \dot{H}_{m+\kappa_s}(\lambda)\begin{pmatrix} x_{m+\kappa_s+1} \\ u_{m+\kappa_s} \end{pmatrix} = 0.$$

Condition (ii) thus implies $x = u = 0$ on J^* so that controllability of (H_λ) on J^* with controllability index $\kappa(\lambda) \le \kappa_s + 1$ follows. ∎

REMARK 2. Suppose as in the previous remark that $A_k(\lambda)$ and $B_k(\lambda)$ are independent of $\lambda \in \mathbb{R}$ for all $k \in J$. Furthermore, assume that $\dot{C}_k(\lambda)$ is nonsingular for all $k \in J$ and all $\lambda \in \mathbb{R}$. Then controllability of (H_λ) on J^* for all $\lambda \in \mathbb{R}$ with controllability indices $\kappa(\lambda) \in J$ implies strict controllability of $(\mathrm{H}_\mathbb{R})$ on J^* with strict controllability index $\kappa_s \le \kappa := \max_{\lambda\in\mathbb{R}} \kappa(\lambda)$. To show this, let $\lambda \in \mathbb{R}$, let (x, u) be a solution of (H_λ), let $m \in J$ with $m + \kappa \in J$ (this yields $m + 1 \in J$ since $\kappa(\lambda) \ge 1$ trivially), and assume

$$\dot{H}_m(\lambda)\begin{pmatrix} x_{m+1} \\ u_m \end{pmatrix} = \dot{H}_{m+1}(\lambda)\begin{pmatrix} x_{m+2} \\ u_{m+1} \end{pmatrix} = \cdots = \dot{H}_{m+\kappa}(\lambda)\begin{pmatrix} x_{m+\kappa+1} \\ u_{m+\kappa} \end{pmatrix} = 0.$$

Then, we have

$$\dot{C}_m(\lambda)x_{m+1} = \dot{C}_{m+1}(\lambda)x_{m+2} = \cdots = \dot{C}_{m+\kappa}(\lambda)x_{m+\kappa+1} = 0,$$

and hence, invertibility of $\dot{C}_m(\lambda), \dot{C}_{m+1}(\lambda), \ldots, \dot{C}_{m+\kappa}(\lambda)$ yields

$$x_{m+1} = x_{m+2} = \cdots = x_{m+\kappa+1} = 0.$$

Now controllability of (H_λ) on J^* with controllability index $\kappa(\lambda) \le \kappa$ together with $m + 1 \in J$ and $m + 1 + \kappa \in J^*$ imply $x = u = 0$ on J^*. Altogether, strict controllability of $(\mathrm{H}_\mathbb{R})$ on J^* with strict controllability index $\kappa_s \le \kappa$ follows. ∎

REMARK 3. The purpose of this central remark is to show that Sturm-Liouville difference equations of order $2n$ depending linearly on an eigenvalue parameter $\lambda \in \mathbb{R}$ satisfy (V_1) provided $N \geq 2n-1$ holds. To start with, let

$$r_k^{(\nu)} \in \mathbb{R}, \quad 0 \leq \nu \leq n, \qquad \text{and} \qquad r_k^{(n)} \neq 0, \quad \text{for all } k \in \mathbb{Z}.$$

We consider the $n \times n$-matrices

$$A_k(\lambda) \equiv A_k = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad B_k(\lambda) \equiv B_k = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \frac{1}{r_k^{(n)}} \end{pmatrix},$$

$$C_k(\lambda) = \begin{pmatrix} r_k^{(0)} & & & \\ & r_k^{(1)} & & \\ & & \ddots & \\ & & & r_k^{(n-1)} \end{pmatrix} - \lambda \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad k \in J, \qquad \lambda \in \mathbb{R}$$

and the corresponding systems (H_λ), $\lambda \in \mathbb{R}$. Note that the $2n \times 2n$-matrices $\dot{H}_k(\lambda)$ are given by

$$\dot{H}_k(\lambda) = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad k \in J,$$

(so that (V_2) is satisfied). Let $N \geq 2n-1$. Clearly (H_λ) is controllable on J^* for each $\lambda \in \mathbb{R}$ (see, e.g., [12, Remark 2(i)]) with controllability index $n \leq N+1$, i.e., $n \in J^*$. Let $\lambda \in \mathbb{R}$ and pick a solution (x, u) of (H_λ). It is very well known (see, e.g., [4, Section 3]) that in this case

$$x_k = \begin{pmatrix} y_k \\ \Delta y_{k-1} \\ \Delta^2 y_{k-2} \\ \vdots \\ \Delta^{n-1} y_{k+1-n} \end{pmatrix}, \qquad \text{for all } k \in J^*,$$

holds with a solution y_k ($1-n \leq k \leq N+1$) of the linear self-adjoint difference equation of order $2n$

$$\sum_{\mu=0}^{n} (-\Delta)^\mu \left\{ r_k^{(\mu)} \Delta^\mu y_{k+1-\mu} \right\} = \lambda y_{k+1}, \qquad 0 \leq k \leq N-n, \tag{SL_λ}$$

a so-called Sturm-Liouville difference equation. (In fact, (SL_λ) and (H_λ) are even equivalent in the sense that a solution of (SL_λ) yields a solution of (H_λ) and the other way around; see, e.g., [4, Section 3].) Now we assume that for some $m \in J$ with $m + 2n - 1 \in J$

$$\dot{H}_m(\lambda) \begin{pmatrix} x_{m+1} \\ u_m \end{pmatrix} = \dot{H}_{m+1}(\lambda) \begin{pmatrix} x_{m+2} \\ u_{m+1} \end{pmatrix} = \cdots = \dot{H}_{m+2n-1}(\lambda) \begin{pmatrix} x_{m+2n} \\ u_{m+2n-1} \end{pmatrix} = 0$$

holds. This implies

$$y_{m+1} = y_{m+2} = \cdots = y_{m+2n} = 0,$$

and since y is a solution of a linear difference equation of order $2n$ being zero at $2n$ consecutive values, it has to be zero always, i.e.,

$$y_k = 0, \qquad \text{for all } 1-n \leq k \leq N+1,$$

and $x = 0$ on J^* follows. Controllability of (H_λ) on J^* now implies $x = u = 0$ on J^*. Thus ($H_\mathbb{R}$) is strictly controllable on J^* with strict controllability index (no smaller index works) $\kappa_s = 2n - 1 \in J$. ∎

4. CHARACTERIZATION OF EIGENVALUES

THEOREM 2. *(Characterization of Eigenvalues.) Let $\lambda \in \mathbb{R}$, and let (X, U) and $(\tilde{X}, \tilde{U})$ be normalized conjoined bases of (H_λ). Then, λ is an eigenvalue of (E) if and only if the $2n \times 2n$-matrix*

$$\Lambda := R^* \begin{pmatrix} -X_0 & -\tilde{X}_0 \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix} + R \begin{pmatrix} U_0 & \tilde{U}_0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix}$$

is singular, and then $\operatorname{def} \Lambda$ is the multiplicity of the eigenvalue λ.

PROOF. Let (x, u) be a nontrivial solution of (H_λ). We put

$$d := \begin{pmatrix} X_0 & \tilde{X}_0 \\ U_0 & \tilde{U}_0 \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} \tilde{U}_0^\top & -\tilde{X}_0^\top \\ -U_0^\top & X_0^\top \end{pmatrix} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \neq 0$$

(observe that Definition 1 yields the invertibility of the occurring matrix), and thus

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix} d, \qquad \text{for } k \in J^*,$$

since the initial value problem under consideration has a unique solution (observe that $I - A_k(\lambda)$ are assumed to be invertible matrices for all $k \in J$). Now, we have

$$\begin{aligned} R^* \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + R \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} &= R^* \begin{pmatrix} -X_0 & -\tilde{X}_0 \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix} d + R \begin{pmatrix} U_0 & \tilde{U}_0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} d \\ &= \left\{ R^* \begin{pmatrix} -X_0 & -\tilde{X}_0 \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix} + R \begin{pmatrix} U_0 & \tilde{U}_0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} \right\} d = \Lambda d. \end{aligned}$$

Thus, (x, u) solves (R), i.e., λ is an eigenvalue of (E), if and only if $\Lambda d = 0$ holds with $d \neq 0$, and this proves our assertion. ∎

Next we wish to simplify this criterion in the case of so-called separated boundary conditions. By this we mean that the boundary conditions (R) may be equivalently written with $2n \times 2n$-matrices

$$R^* = \begin{pmatrix} -R_0^* & 0 \\ 0 & R_{N+1}^* \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} R_0 & 0 \\ 0 & R_{N+1} \end{pmatrix},$$

such that the $n \times n$-matrices R_0, R_0^*, R_{N+1}, R_{N+1}^* satisfy (as usual)

$$\begin{gathered} \operatorname{rank}(R_0 \quad R_0^*) = \operatorname{rank}(R_{N+1} \quad R_{N+1}^*) = n, \\ R_0 {R_0^*}^\top = R_0^* R_0^\top, \qquad R_{N+1} {R_{N+1}^*}^\top = R_{N+1}^* R_{N+1}^\top. \end{gathered}$$

In this special case, we have the following result.

COROLLARY 1. *(Separated Boundary Conditions.) Assume that separated (and self-conjoined) boundary conditions are given. Let (X, U) be the conjoined basis of (H_λ), $\lambda \in \mathbb{R}$, with*

$$X_0 = -R_0^\top \qquad \text{and} \qquad U_0 = {R_0^*}^\top.$$

Then, $\lambda \in \mathbb{R}$ is an eigenvalue of (E) if and only if the $n \times n$-matrix

$$R_{N+1}^* X_{N+1} + R_{N+1} U_{N+1}$$

is singular.

PROOF. Let $\lambda \in \mathbb{R}$. For the above conjoined basis of (H_λ), there exists another conjoined basis $(\tilde{X}, \tilde{U})$ of (H_λ) so that (X, U) and $(\tilde{X}, \tilde{U})$ are normalized (see Lemma 1). According to Theorem 2, λ is an eigenvalue of (E) iff

$$\begin{aligned}\Lambda &= R^* \begin{pmatrix} -X_0 & -\tilde{X}_0 \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix} + R \begin{pmatrix} U_0 & \tilde{U}_0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} \\ &= \begin{pmatrix} -R_0^* & 0 \\ 0 & R_{N+1}^* \end{pmatrix} \begin{pmatrix} R_0^\top & -\tilde{X}_0 \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix} + \begin{pmatrix} R_0 & 0 \\ 0 & R_{N+1} \end{pmatrix} \begin{pmatrix} R_0^{*\top} & \tilde{U}_0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} \\ &= \begin{pmatrix} -R_0^* R_0^\top + R_0 R_0^{*\top} & U_0^\top \tilde{X}_0 - X_0^\top \tilde{U}_0 \\ R_{N+1}^* X_{N+1} + R_{N+1} U_{N+1} & R_{N+1}^* \tilde{X}_{N+1} + R_{N+1} \tilde{U}_{N+1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -I \\ R_{N+1}^* X_{N+1} + R_{N+1} U_{N+1} & R_{N+1}^* \tilde{X}_{N+1} + R_{N+1} \tilde{U}_{N+1} \end{pmatrix}\end{aligned}$$

(observe Definition 1) is singular, and this happens iff $R_{N+1}^* X_{N+1} + R_{N+1} U_{N+1}$ is singular. ∎

We wish to conclude this section with the following example.

EXAMPLE 1. Let $n = 2$ and consider the eigenvalue problem (E) given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} -\lambda & 0 \\ 0 & 0 \end{pmatrix},$$
$$R_0^* = R_{N+1}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_0 = \begin{pmatrix} -7 & 3 \\ 3 & 0 \end{pmatrix}, \quad R_{N+1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Remark 3, (V_2) is satisfied and (V_1) holds provided $N \geq 2n - 1 = 3$. However, now we let $N = 2$. Then, due to Corollary 1, $\lambda \in \mathbb{R}$ is an eigenvalue of (E) iff

$$X_3(\lambda) = \begin{pmatrix} 4\lambda - 6 & 3 - 2\lambda \\ 4\lambda - 6 & 3 - 2\lambda \end{pmatrix}$$

is singular. Therefore $\mathbb{R}$ is the set of eigenvalues of (E). ∎

5. TWO AUXILIARY RESULTS

In this short section, we cite two results that will be needed in the proof of Theorem 1.

LEMMA 2. *(Index Theorem; see [19, Theorem 3.4.1].) Let $m \in \mathbb{N}$, let there be given $m \times m$-matrices R, R^*, X, U with*

$$\operatorname{rank}(R \quad R^*) = \operatorname{rank}(X^\top \quad U^\top) = m \qquad \text{and} \qquad RR^{*\top} = R^* R^\top, \quad X^\top U = U^\top X,$$

and let $X(\lambda)$, $U(\lambda)$ be $m \times m$-matrix-valued functions on $\mathbb{R}$ with

$$\begin{gathered} X^\top(\lambda) U(\lambda) = U^\top(\lambda) X(\lambda), \qquad \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon], \qquad \text{for some } \varepsilon > 0, \\ X(\lambda) \to X, \qquad U(\lambda) \to U, \qquad \text{as } \lambda \to \lambda_0, \\ X(\lambda) \text{ invertible for all } \lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}. \end{gathered}$$

Suppose that

$$U(\lambda) X^{-1}(\lambda) \text{ decreases strictly on } [\lambda_0 - \varepsilon, \lambda_0) \text{ and on } (\lambda_0, \lambda_0 + \varepsilon],$$

and denote for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$

$$\begin{gathered} M(\lambda) = R^* R^\top + R U(\lambda) X^{-1}(\lambda) R^\top, \\ \Lambda(\lambda) = R X(\lambda) + R^* U(\lambda), \qquad \Lambda = RX + R^* U. \end{gathered}$$

Then, $\operatorname{ind} M(\lambda_0^-) = \lim_{\lambda\to\lambda_0^-} \{\operatorname{ind} M(\lambda)\}$ *and* $\operatorname{ind} M(\lambda_0^+) = \lim_{\lambda\to\lambda_0^+} \{\operatorname{ind} M(\lambda)\}$ *both exist,*

$$\Lambda(\lambda) \text{ is invertible for all } \lambda \in [\lambda_0 - \delta, \lambda + \delta] \setminus \{\lambda_0\} \text{ for some } \delta \in (0, \varepsilon),$$

and the formula

$$\operatorname{def}\Lambda = \operatorname{ind} M(\lambda_0^+) - \operatorname{ind} M(\lambda_0^-) + \operatorname{def} X$$

holds.

PROOF. We refer to [19, Theorem 3.4.1] (observe also [19, Corollary 3.4.4]). ∎

LEMMA 3. *(Reid Roundabout Theorem; see [14, Theorem 3].) Suppose the system (H) is controllable on* J^* *(see Definition 3). Let* (X, U) *and* $(\tilde{X}, \tilde{U})$ *be the special normalized conjoined bases of (H) at 0 (see Definition 1). Then,* $\mathcal{F} > 0$ *(see Definition 2) if and only if*

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \qquad X_k X_{k+1}^{\dagger} (I - A_k)^{-1} B_k \geq 0, \qquad \text{for all } k \in J,$$
$$X_{N+1} \text{ invertible},$$
$$M := R^* R^{\mathsf{T}} + R \begin{pmatrix} I & 0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} \begin{pmatrix} 0 & I \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix}^{-1} R^{\mathsf{T}} > 0, \qquad \text{on } \operatorname{Im} R$$

holds.

PROOF. We refer to [14, Theorem 3] and remark that $R\{R^{\dagger} R^* R^{\dagger} R\} R^{\mathsf{T}} = R^* R^{\mathsf{T}}$ holds. ∎

6. ISOLATEDNESS OF EIGENVALUES

In this section, we wish to establish Theorem 1(i). Consider the following condition.

For all $\lambda_0 \in \mathbb{R}$ there exists $\varepsilon > 0$ such that $X_{N+1}(\lambda)$ is invertible and $\begin{pmatrix} I & 0 \\ U_{N+1}(\lambda) & \tilde{U}_{N+1}(\lambda) \end{pmatrix} \begin{pmatrix} 0 & I \\ X_{N+1}(\lambda) & \tilde{X}_{N+1}(\lambda) \end{pmatrix}^{-1}$ is strictly decreasing for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$, where $(X(\lambda), U(\lambda))$ and $(\tilde{X}(\lambda), \tilde{U}(\lambda))$ are the special normalized conjoined bases of (H) at 0 for every $\lambda \in \mathbb{R}$. (I)

Of course, condition (I) implies by the index theorem, Lemma 2 (observe also Lemma 1 and the continuity of the $H_k(\lambda)$, $k \in J$), that the singular points of

$$\begin{aligned} \Lambda(\lambda) &= R^* \begin{pmatrix} 0 & I \\ X_{N+1}(\lambda) & \tilde{X}_{N+1}(\lambda) \end{pmatrix} + R \begin{pmatrix} I & 0 \\ U_{N+1}(\lambda) & \tilde{U}_{N+1}(\lambda) \end{pmatrix} \\ &= R^* \begin{pmatrix} -X_0(\lambda) & -\tilde{X}_0(\lambda) \\ X_{N+1}(\lambda) & \tilde{X}_{N+1}(\lambda) \end{pmatrix} + R \begin{pmatrix} U_0(\lambda) & \tilde{U}_0(\lambda) \\ U_{N+1}(\lambda) & \tilde{U}_{N+1}(\lambda) \end{pmatrix}, \end{aligned}$$

i.e., (according to Theorem 2) the eigenvalues of (E), are isolated. Therefore our goal is to show that (V_1) and (V_2) imply (I). This we will achieve by showing some lemmas.

LEMMA 4. *Suppose* $(X(\lambda), U(\lambda))$ *is a conjoined basis of* (H_λ) *for each* $\lambda \in \mathbb{R}$ *with* $\dot{X}_0(\lambda) = \dot{U}_0(\lambda) = 0$. *Then,*

$$X_k^{\mathsf{T}}(\lambda) \dot{U}_k(\lambda) - U_k^{\mathsf{T}}(\lambda) \dot{X}_k(\lambda) = - \sum_{m=0}^{k-1} \begin{pmatrix} X_{m+1}(\lambda) \\ U_m(\lambda) \end{pmatrix}^{\mathsf{T}} \dot{H}_m(\lambda) \begin{pmatrix} X_{m+1}(\lambda) \\ U_m(\lambda) \end{pmatrix}$$

holds for all $k \in J^* \setminus \{0\}$ *and for all* $\lambda \in \mathbb{R}$.

PROOF. Let $\lambda, \mu \in \mathbb{R}$ and $m \in J$. Then,

$$\begin{aligned}
&\Delta\left[X_m^\top(\mu)\{U_m(\lambda)-U_m(\mu)\} - U_m^\top(\mu)\{X_m(\lambda)-X_m(\mu)\}\right] \\
&\quad = \Delta\left[X_m^\top(\mu)U_m(\lambda) - U_m^\top(\mu)X_m(\lambda)\right] = \Delta\left[\begin{pmatrix} X_m(\mu) \\ U_m(\mu)\end{pmatrix}^\top \begin{pmatrix} U_m(\lambda) \\ -X_m(\lambda)\end{pmatrix}\right] \\
&\quad = \left\{\Delta\begin{pmatrix} X_m(\mu) \\ U_m(\mu)\end{pmatrix}^\top\right\}\begin{pmatrix} U_{m+1}(\lambda) \\ -X_{m+1}(\lambda)\end{pmatrix} + \begin{pmatrix} X_m(\mu) \\ U_m(\mu)\end{pmatrix}^\top\left\{\Delta\begin{pmatrix} U_m(\lambda) \\ -X_m(\lambda)\end{pmatrix}\right\} \\
&\quad = \left\{\Delta\begin{pmatrix} X_m(\mu) \\ U_m(\mu)\end{pmatrix}^\top\right\}\left\{\begin{pmatrix} U_m(\lambda) \\ -X_{m+1}(\lambda)\end{pmatrix} + \begin{pmatrix} \Delta U_m(\lambda) \\ 0\end{pmatrix}\right\} \\
&\qquad + \left\{\begin{pmatrix} X_{m+1}(\mu) \\ U_m(\mu)\end{pmatrix} - \begin{pmatrix} \Delta X_m(\mu) \\ 0\end{pmatrix}\right\}^\top\left\{\Delta\begin{pmatrix} U_m(\lambda) \\ -X_m(\lambda)\end{pmatrix}\right\} \\
&\quad = \left\{\Delta\begin{pmatrix} X_m(\mu) \\ U_m(\mu)\end{pmatrix}^\top\right\}\begin{pmatrix} U_m(\lambda) \\ -X_{m+1}(\lambda)\end{pmatrix} + \begin{pmatrix} X_{m+1}(\mu) \\ U_m(\mu)\end{pmatrix}^\top\left\{\Delta\begin{pmatrix} U_m(\lambda) \\ -X_m(\lambda)\end{pmatrix}\right\} \\
&\quad = \left\{\Delta\begin{pmatrix} -U_m(\mu) \\ X_m(\mu)\end{pmatrix}^\top\right\}\begin{pmatrix} X_{m+1}(\lambda) \\ U_m(\lambda)\end{pmatrix} - \begin{pmatrix} X_{m+1}(\mu) \\ U_m(\mu)\end{pmatrix}^\top\left\{\Delta\begin{pmatrix} -U_m(\lambda) \\ X_m(\lambda)\end{pmatrix}\right\} \\
&\quad = \left\{H_m(\mu)\begin{pmatrix} X_{m+1}(\mu) \\ U_m(\mu)\end{pmatrix}\right\}^\top\begin{pmatrix} X_{m+1}(\lambda) \\ U_m(\lambda)\end{pmatrix} - \begin{pmatrix} X_{m+1}(\mu) \\ U_m(\mu)\end{pmatrix}^\top\left\{H_m(\lambda)\begin{pmatrix} X_{m+1}(\lambda) \\ U_m(\lambda)\end{pmatrix}\right\} \\
&\quad = -\begin{pmatrix} X_{m+1}(\mu) \\ U_m(\mu)\end{pmatrix}^\top\{H_m(\lambda)-H_m(\mu)\}\begin{pmatrix} X_{m+1}(\lambda) \\ U_m(\lambda)\end{pmatrix}.
\end{aligned}$$

Now, division by $\lambda - \mu$ and letting μ tend to λ yields

$$\Delta\left[X_m^\top(\lambda)\dot U_m(\lambda) - U_m^\top(\lambda)\dot X_m(\lambda)\right] = -\begin{pmatrix} X_{m+1}(\lambda) \\ U_m(\lambda)\end{pmatrix}^\top \dot H_m(\lambda)\begin{pmatrix} X_{m+1}(\lambda) \\ U_m(\lambda)\end{pmatrix}$$

so that $\dot X_0(\lambda) = \dot U_0(\lambda) = 0$ prove the validity of our assertion. ∎

LEMMA 5. *Suppose $(X(\lambda), U(\lambda))$ and $(\tilde X(\lambda), \tilde U(\lambda))$ are normalized conjoined bases of (H_λ) for each $\lambda \in \mathbb{R}$ with $\dot X_0(\lambda) = \dot U_0(\lambda) = \dot{\tilde X}_0(\lambda) = \dot{\tilde U}_0(\lambda) = 0$. Let $k \in J^*$. Assume that $X_k(\lambda)$ is invertible on some nontrivial open interval $\mathcal{I}$. Put*

$$Q_k(\lambda) := \begin{pmatrix} I & 0 \\ U_k(\lambda) & \tilde U_k(\lambda)\end{pmatrix}\begin{pmatrix} 0 & I \\ X_k(\lambda) & \tilde X_k(\lambda)\end{pmatrix}^{-1}, \qquad \lambda \in \mathcal{I}.$$

Then (V_2) implies that $Q_k(\lambda)$ decreases on $\mathcal{I}$. Moreover, (V_1) and (V_2) imply that $Q_k(\lambda)$ decreases strictly on $\mathcal{I}$ provided $k > \kappa_s$ holds, where $\kappa_s \in J$ is the strict controllability index of $(H_{\mathbb{R}})$.

PROOF. Let $k \in J^* \setminus \{0\}$ and $\lambda \in \mathcal{I}$. We may apply Lemma 4 with the conjoined basis $(X^*(\lambda), U^*(\lambda))$

$$X^*(\lambda) = \begin{pmatrix} 0 & I \\ X(\lambda) & \tilde X(\lambda)\end{pmatrix} \qquad \text{and} \qquad U^*(\lambda) = \begin{pmatrix} I & 0 \\ U(\lambda) & \tilde U(\lambda)\end{pmatrix}$$

of the "big" system from Lemma 4 so that for $d \in \mathbb{R}^{2n}$

$$\begin{aligned}
d^{\mathsf T}\dot Q_k(\lambda)d &= d^{\mathsf T}\left\{\dot U_k^*(\lambda)X_k^{*^{-1}}(\lambda)-U_k^*(\lambda)X_k^{*^{-1}}(\lambda)\dot X_k^*(\lambda)X_k^{*^{-1}}(\lambda)\right\}d\\
&=\left\{X_k^{*^{-1}}(\lambda)d\right\}^{\mathsf T}\left\{X_k^{*^{\mathsf T}}(\lambda)\dot U_k^*(\lambda)-U_k^{*^{\mathsf T}}(\lambda)\dot X_k^*(\lambda)\right\}X_k^{*^{-1}}(\lambda)d\\
&=-\left\{X_k^{*^{-1}}(\lambda)d\right\}^{\mathsf T}\left\{\sum_{m=0}^{k-1}\begin{pmatrix}X_{m+1}^*(\lambda)\\ U_m^*(\lambda)\end{pmatrix}^{\mathsf T}\begin{pmatrix}0&0&0&0\\0&-\dot C_m(\lambda)&0&\dot A_m^{\mathsf T}(\lambda)\\0&0&0&0\\0&\dot A_m(\lambda)&0&\dot B_m(\lambda)\end{pmatrix}\begin{pmatrix}X_{m+1}^*(\lambda)\\ U_m^*(\lambda)\end{pmatrix}\right\}\\
&\quad\times X_k^{*^{-1}}(\lambda)d\\
&=-\left\{X_k^{*^{-1}}(\lambda)d\right\}^{\mathsf T}\left\{\sum_{m=0}^{k-1}\begin{pmatrix}X_{m+1}(\lambda)&\tilde X_{m+1}(\lambda)\\ U_m(\lambda)&\tilde U_m(\lambda)\end{pmatrix}^{\mathsf T}\dot H_m(\lambda)\begin{pmatrix}X_{m+1}(\lambda)&\tilde X_{m+1}(\lambda)\\ U_m(\lambda)&\tilde U_m(\lambda)\end{pmatrix}\right\}\\
&\quad\times X_k^{*^{-1}}(\lambda)d\\
&=-\sum_{m=0}^{k-1}\begin{pmatrix}x_{m+1}\\ u_m\end{pmatrix}^{\mathsf T}\dot H_m(\lambda)\begin{pmatrix}x_{m+1}\\ u_m\end{pmatrix}\le 0
\end{aligned}$$

holds provided we assume (V_2) and use the solution (x,u) of (H_λ) defined by

$$\begin{pmatrix}x_m\\ u_m\end{pmatrix}:=\begin{pmatrix}X_m(\lambda)&\tilde X_m(\lambda)\\ U_m(\lambda)&\tilde U_m(\lambda)\end{pmatrix}\begin{pmatrix}0&I\\ X_k(\lambda)&\tilde X_k(\lambda)\end{pmatrix}^{-1}d.$$

Now we assume (V_1) and (V_2), let $k>\kappa_s$, and suppose $d^{\mathsf T}\dot Q_k(\lambda)d=0$. This yields

$$\dot H_m(\lambda)\begin{pmatrix}x_{m+1}\\ u_m\end{pmatrix}=0,\qquad \text{for all } 0\le m\le k-1.$$

It follows that

$$\dot H_0(\lambda)\begin{pmatrix}x_1\\ u_0\end{pmatrix}=\dot H_1(\lambda)\begin{pmatrix}x_2\\ u_1\end{pmatrix}=\cdots=\dot H_{\kappa_s}(\lambda)\begin{pmatrix}x_{\kappa_s+1}\\ u_{\kappa_s}\end{pmatrix}=0$$

holds. Strict controllability of ($H_{\mathbb R}$) on J^* with strict controllability index $\kappa_s\in J$ now forces $x=u=0$ on J^* so that $d=0$ and hence $\dot Q_k(\lambda)<0$ follows. ∎

LEMMA 6. *(V_1) and (V_2) imply (I).*

PROOF. For every $\lambda\in\mathbb R$, we denote the special normalized conjoined bases of (H_λ) at 0 by $(X(\lambda),U(\lambda))$ and $(\tilde X(\lambda),\tilde U(\lambda))$. Let $\lambda_0\in\mathbb R$. We pick a conjoined basis $(\hat X,\hat U)$ of (H_{λ_0}) such that $(X(\lambda_0),U(\lambda_0))$ and $(\hat X,\hat U)$ are normalized and such that $\hat X_{N+1}$ is invertible (observe Lemma 1). Let $(\hat X(\lambda),\hat U(\lambda))$ be the conjoined basis of (H_λ) with $\hat X_0(\lambda)\equiv\hat X_0$ and $\hat U_0(\lambda)\equiv\hat U_0$, $\lambda\in\mathbb R$. Due to continuity, $\hat X_{N+1}(\lambda)$ is invertible on some nontrivial open interval that contains λ_0, and on this interval we have strict monotonicity of

$$\begin{aligned}
&\begin{pmatrix}I&0\\ -\hat U_{N+1}(\lambda)&U_{N+1}(\lambda)\end{pmatrix}\begin{pmatrix}0&I\\ -\hat X_{N+1}(\lambda)&X_{N+1}(\lambda)\end{pmatrix}^{-1}\\
&\qquad=\begin{pmatrix}\hat X_{N+1}^{-1}(\lambda)X_{N+1}(\lambda)&-\hat X_{N+1}^{-1}(\lambda)\\ -\left\{\hat X_{N+1}^{-1}(\lambda)\right\}^{\mathsf T}&\hat U_{N+1}(\lambda)\hat X_{N+1}^{-1}(\lambda)\end{pmatrix}
\end{aligned}$$

by Lemma 5 so that $\hat X_{N+1}^{-1}(\lambda)X_{N+1}(\lambda)$ is strictly decreasing on this interval also. Thus, there exists $\varepsilon>0$ such that $X_{N+1}(\lambda)$ is invertible on $[\lambda_0-\varepsilon,\lambda_0+\varepsilon]\setminus\{\lambda_0\}$. We now may apply Lemma 5 once again to obtain that

$$\begin{pmatrix}I&0\\ U_{N+1}(\lambda)&\tilde U_{N+1}(\lambda)\end{pmatrix}\begin{pmatrix}0&I\\ X_{N+1}(\lambda)&\tilde X_{N+1}(\lambda)\end{pmatrix}^{-1}$$

decreases strictly on $[\lambda_0-\varepsilon,\lambda_0+\varepsilon]\setminus\{\lambda_0\}$. This shows that (I) holds and hence the proof of Theorem 1(i) is done. ∎

7. LOWER BOUNDEDNESS OF EIGENVALUES

The purpose of this section is to provide a proof of Theorem 1(ii). We need the following auxiliary result.

LEMMA 7. *Let there be given $m \times m$-matrices A, $\underset{\sim}{A}$, B, $\underset{\sim}{B}$, C, $\underset{\sim}{C}$ such that*

$$H = \begin{pmatrix} -C & A^{\mathsf T} \\ A & B \end{pmatrix} \qquad \text{and} \qquad \underset{\sim}{H} = \begin{pmatrix} -\underset{\sim}{C} & \underset{\sim}{A}^{\mathsf T} \\ \underset{\sim}{A} & \underset{\sim}{B} \end{pmatrix}$$

are symmetric. Suppose that

$$\underset{\sim}{H} \geq H, \qquad \operatorname{Ker} \underset{\sim}{B} \subset \operatorname{Ker} B \qquad \text{and} \qquad B\left(B^{\dagger} - \underset{\sim}{B}^{\dagger}\right) B \geq 0$$

hold. Then, we have

$$x^{\mathsf T} C x + u^{\mathsf T} B u \geq x^{\mathsf T} \underset{\sim}{C} x + \underset{\sim}{u}^{\mathsf T} \underset{\sim}{B}\underset{\sim}{u}$$

for all $x, u, \underset{\sim}{u} \in \mathbb{R}^m$ with $Bu - \underset{\sim}{B}\underset{\sim}{u} = (\underset{\sim}{A} - A)x$.

PROOF. By [19, Lemma 3.1.10], $\underset{\sim}{H} \geq H$ implies $\underset{\sim}{B} \geq B$ and the existence of a matrix D with

$$\underset{\sim}{A} - A = (\underset{\sim}{B} - B)D \qquad \text{and} \qquad D^{\mathsf T}(\underset{\sim}{B} - B)D \leq C - \underset{\sim}{C}.$$

According to [14, Remark 2(iii)], $\operatorname{Ker} \underset{\sim}{B} \subset \operatorname{Ker} B$ is equivalent to

$$B = B\underset{\sim}{B}^{\dagger}\underset{\sim}{B} = \underset{\sim}{B}\underset{\sim}{B}^{\dagger}B.$$

Let $x, u, \underset{\sim}{u} \in \mathbb{R}^m$ with $Bu - \underset{\sim}{B}\underset{\sim}{u} = (\underset{\sim}{A} - A)x = (\underset{\sim}{B} - B)Dx$. Then,

$$\begin{aligned}
x^{\mathsf T} C x + u^{\mathsf T} B u - x^{\mathsf T} \underset{\sim}{C} x - \underset{\sim}{u}^{\mathsf T} \underset{\sim}{B}\underset{\sim}{u} &= x^{\mathsf T}(C - \underset{\sim}{C})x + u^{\mathsf T} B u - \underset{\sim}{u}^{\mathsf T} \underset{\sim}{B}\underset{\sim}{B}^{\dagger}\underset{\sim}{B}\underset{\sim}{u} \\
&\geq x^{\mathsf T} D^{\mathsf T}(\underset{\sim}{B} - B)Dx + u^{\mathsf T} B u - \left\{Bu + (B - \underset{\sim}{B})Dx\right\}^{\mathsf T} \\
&\qquad \times \underset{\sim}{B}^{\dagger}\left\{Bu + (B - \underset{\sim}{B})Dx\right\} \\
&= x^{\mathsf T} D^{\mathsf T}\left\{\underset{\sim}{B} - B - (B - \underset{\sim}{B})\underset{\sim}{B}^{\dagger}(B - \underset{\sim}{B})\right\} Dx \\
&\qquad + u^{\mathsf T}\left\{B - B\underset{\sim}{B}^{\dagger}B\right\} u - 2x^{\mathsf T} D^{\mathsf T}(B - \underset{\sim}{B})\underset{\sim}{B}^{\dagger} B u \\
&= x^{\mathsf T} D^{\mathsf T}\left\{\underset{\sim}{B} - B - B\underset{\sim}{B}^{\dagger}B - \underset{\sim}{B}\underset{\sim}{B}^{\dagger}\underset{\sim}{B} + 2B\underset{\sim}{B}^{\dagger}\underset{\sim}{B}\right\} Dx \\
&\qquad + u^{\mathsf T}\left\{B - B\underset{\sim}{B}^{\dagger}B\right\} u - 2x^{\mathsf T} D^{\mathsf T}\left\{B\underset{\sim}{B}^{\dagger}B - \underset{\sim}{B}\underset{\sim}{B}^{\dagger}B\right\} u \\
&= x^{\mathsf T} D^{\mathsf T}(B - B\underset{\sim}{B}^{\dagger}B)Dx + u^{\mathsf T}(B - B\underset{\sim}{B}^{\dagger}B)u \\
&\qquad + 2x^{\mathsf T} D^{\mathsf T}(B - B\underset{\sim}{B}^{\dagger}B)u \\
&= (Dx + u)^{\mathsf T}(B - B\underset{\sim}{B}^{\dagger}B)(Dx + u) \\
&= (Dx + u)^{\mathsf T} B(B^{\dagger} - \underset{\sim}{B}^{\dagger})B(Dx + u) \geq 0. \qquad \blacksquare
\end{aligned}$$

THEOREM 3. *(Comparison Theorem.) Suppose that conditions (V_2) and (V_3) hold. Then, $\mathcal{F}(\cdot;\lambda) > 0$ for all $\lambda \leq \underset{\sim}{\lambda}$.*

PROOF. Suppose $\mathcal{F}(\cdot;\underset{\sim}{\lambda}) > 0$ and let $\lambda \le \underset{\sim}{\lambda}$. By ($V_2$) and ($V_3$), we have for all $k \in J$

$$H_k(\underset{\sim}{\lambda}) \ge H_k(\lambda), \qquad \operatorname{Ker} B_k(\underset{\sim}{\lambda}) \subset \operatorname{Ker} B_k(\lambda),$$
$$B_k(\lambda)\left\{B_k^\dagger(\lambda) - B_k^\dagger(\underset{\sim}{\lambda})\right\} B_k(\lambda) \ge 0.$$

Let (x,u) be such that $\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \in \operatorname{Im} R^\top$, $x \ne 0$, and $\Delta x_k = A_k(\lambda)x_{k+1} + B_k(\lambda)u_k$, $k \in J$. Define

$$\underset{\sim}{u}_k := B_k^\dagger(\underset{\sim}{\lambda})B_k(\lambda)u_k - \left\{I - B_k^\dagger(\underset{\sim}{\lambda})B_k(\lambda)\right\} D_k x_{k+1}, \qquad k \in J,$$

where $A_k(\underset{\sim}{\lambda}) - A_k(\lambda) = \left\{B_k(\underset{\sim}{\lambda}) - B_k(\lambda)\right\} D_k$ according to the proof of Lemma 7. Then,

$$B_k(\lambda)u_k - B_k(\underset{\sim}{\lambda})\underset{\sim}{u}_k = \left\{B_k(\underset{\sim}{\lambda}) - B_k(\lambda)\right\} D_k x_{k+1} = \left\{A_k(\underset{\sim}{\lambda}) - A_k(\lambda)\right\} x_{k+1}$$

and thus $\Delta x_k = A_k(\underset{\sim}{\lambda})x_{k+1} + B_k(\underset{\sim}{\lambda})\underset{\sim}{u}_k$ for all $k \in J$, so that an application of Lemma 7 yields

$$\begin{aligned}
0 &< \mathcal{F}(x,u;\underset{\sim}{\lambda}) \\
&= \sum_{k=0}^{N} \left\{x_{k+1}^\top C_k(\underset{\sim}{\lambda})x_{k+1} + \underset{\sim}{u}_k^\top B_k(\underset{\sim}{\lambda})\underset{\sim}{u}_k\right\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^\top R^\dagger R^* R^\dagger R \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \\
&\le \sum_{k=0}^{N} \left\{x_{k+1}^\top C_k(\lambda)x_{k+1} + u_k^\top B_k(\lambda)u_k\right\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^\top R^\dagger R^* R^\dagger R \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \\
&= \mathcal{F}(x,u;\lambda).
\end{aligned}$$

Hence $\mathcal{F}(\cdot;\lambda) > 0$ also. ∎

Now we are able to finish the proof of Theorem 1(ii)—and hence, of Theorem 1—as follows. Assume (V_2), (V_3), and controllability of (H_λ) on J^* for all $\lambda \in \mathbb{R}$. For $\lambda \in \mathbb{R}$, let $(X(\lambda), U(\lambda))$ and $(\tilde{X}(\lambda), \tilde{U}(\lambda))$ be the special normalized conjoined bases of (H_λ) at 0 and define

$$M(\lambda) := R^*R^\top + R\begin{pmatrix} I & 0 \\ U_{N+1}(\lambda) & \tilde{U}_{N+1}(\lambda) \end{pmatrix}\begin{pmatrix} 0 & I \\ X_{N+1}(\lambda) & \tilde{X}_{N+1}(\lambda) \end{pmatrix}^{-1} R^\top$$

whenever the inverse exists. Now we pick $\lambda_0 \le \underset{\sim}{\lambda}$. Thus, $\mathcal{F}(\cdot;\lambda_0) > 0$ according to the above comparison result, Theorem 3. Our Reid roundabout theorem, Lemma 3, now yields that $X_{N+1}(\lambda_0)$ is invertible and that $M(\lambda_0) > 0$ holds on $\operatorname{Im} R$. Of course, $X_{N+1}(\lambda)$ is invertible in some nontrivial open interval containing λ_0, $\begin{pmatrix} I & 0 \\ U_{N+1}(\lambda) & \tilde{U}_{N+1}(\lambda) \end{pmatrix}\begin{pmatrix} 0 & I \\ X_{N+1}(\lambda) & \tilde{X}_{N+1}(\lambda) \end{pmatrix}^{-1}$ is strictly decreasing there due to Lemma 5, and $\operatorname{ind} M(\lambda_0^+) = \operatorname{ind} M(\lambda_0^-) = 0$ so that we may apply the index theorem, Lemma 2, to obtain

$$\begin{aligned}
\operatorname{def} \Lambda(\lambda_0) &= \operatorname{ind} M(\lambda_0^+) - \operatorname{ind} M(\lambda_0^-) + \operatorname{def} \begin{pmatrix} 0 & I \\ X_{N+1}(\lambda_0) & \tilde{X}_{N+1}(\lambda_0) \end{pmatrix} \\
&= \operatorname{def} X_{N+1}(\lambda_0) = 0.
\end{aligned}$$

Thus, the crucial matrix $\Lambda(\lambda_0)$ from our result on characterization of eigenvalues, Theorem 2, is nonsingular, and hence λ_0 is not an eigenvalue. Therefore there exists a smallest eigenvalue λ_1—if there exists an eigenvalue at all—and it satisfies the inequality $\lambda_1 > \underset{\sim}{\lambda}$.

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