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Linear Fractional Programming Problems on Time Scales

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Abstract. *In this paper, we develop a time scales approach to formulate and solve linear fractional programming problems. This time scales approach unifies the discrete and continuous linear fractional programming models and extends them to other cases "in between". Our approach enables us to derive a pair of primal and dual linear fractional programming models on arbitrary time scales. We also establish and prove the weak duality theorem and the optimality condition for arbitrary time scales, while the strong duality theorem is established for isolated time scales. Examples are provided to illustrate the presented theory.*

Key words: Time Scales, Linear Fractional Programming Problem, Primal Problem, Dual Problem, Weak Duality Theorem, Optimality Condition, Strong Duality Theorem.

AMS Subject Classifications: 90C05, 90C11, 90C32

1. Introduction

It is well known that discrete-time linear programming problems have numerous applications in areas such as portfolio optimization, crew scheduling, manufacturing, transportation, telecommunication, agriculture, and so on. Continuous-time linear programming problems were first studied by Bellman [5] as a bottleneck process. He established the weak duality theorem and optimality conditions. A computational approach has been presented by Bellman and Dreyfus [6]. The strong duality theorem was studied by Tyndall [32, 33] and Levinson [28]. Grinold [26] has established strong duality without discretizing the continuous problem. A numerical solution to continuous-time linear programming was considered by Buie and Abrham [24].Wen, Lur, and Lai [38] have presented

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an approximation approach to solve continuous-time problems.

If the objective function appears as a ratio of two continuous-time linear objective functions, then the problem is known as a continuous-time linear fractional program. As it is an ideal discipline for applications of optimization problems, the theory and algorithms as well as various applications of this problem have gotten specific consideration in recent decades. Zalmai [42–46] has studied continuous-time fractional programming problems. A stochastic class of these types of problems was treated by Stancu and Tigan [30]. An approximation method to solve these kinds of problems was considered by Wen et al. [34, 37, 39, 40]. Wen et al. established in [34, 37, 39, 40] weak and strong duality theorems for these kinds of problems. Continuous-time generalized fractional models were studied by Ching and Wen [35, 36], and they have investigated the basic theory and an interval-type approach to solve fractional models. Wu [41] has proposed the parametric formulation of a class of these types of problems and has established duality theorems.

The theory of time scales, on the other hand, was first introduced by Stefan Hilger in 1988 in his PhD dissertation, see [27]. The purpose of this theory is to unify discrete and continuous analysis and to offer an extension to cases “in between.” Many applications in mathematical modelling exist for this theory, e.g., to optimal control [4, 20–23, 29], population biology [8–11], calculus of variations [7, 12, 15], and economics [3, 13, 16, 17, 31].

In this paper, we continue our study on time scales programming problems from [1, 2] and derive a new formulation for a class of linear fractional programming problems on arbitrary time scales. The by-product of our work is to extend continuous-time linear fractional programming problems and the results in [34, 37, 39–41] to a general form of linear fractional programming problems on arbitrary time scales. The paper is organized as follows. In Section 2, some examples related to time scales calculus are given. In Section 3, we recall some recent results by the authors [1] about linear primal and dual programs on time scales. In Section 4, the basic structure of the primal model is formulated, and the Charnes–Cooper transformation [25] is used to rewrite the problem as a linear programming problem on time scales. In Section 5, the obtained linear problem is rewritten as an ordinary linear programming problem as studied in [1], and its dual is found. Section 6 uses the recently established results from [1] to prove the weak duality theorem and the optimality condition on arbitrary time scales, while the strong duality theorem is stated and proved for isolated time scales. Examples are presented in Section 7 in order to demonstrate our theoretical results. In Section 8, some conclusions are given.

2. Time Scales Calculus

In this section, instead of introducing the basic definitions, derivative, and integral on time scales, we refer the reader to the monographs [14, 18, 19], in which comprehensive details and complete proofs are given. For readers not familiar with the time scales calculus, we give the following few examples. Throughout, $\mathbb{T}$ is the time scale, σ is the forward jump operator, μ is the graininess, $f:\mathbb{T}\to\mathbb{R}$ is a function, $f^{\sigma}=f\circ\sigma$ is the advance of f, f^{Δ} is the delta derivative of f, and $\int_a^b f(t)\Delta t$ is the time scales integral of f between $a,b\in\mathbb{T}$.

Example 2.1. If $\mathbb{T}=\mathbb{R}$, then

$$\sigma(t) = t, \quad \mu(t) \equiv 0, \quad f^{\Delta}(t) = f'(t) \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt, \quad \text{where} \quad a, b \in \mathbb{T} \text{ with } a < b,$$

is the usual Riemann integral of classical calculus.

Example 2.2. If $\mathbb{T} = \{t_k \in \mathbb{R} : k \in \mathbb{N}_0\}$ with $t_k < t_{k+1}$ for all $k \in \mathbb{N}_0$ consists only of isolated points, then

$$\sigma(t_k) = t_{k+1}, \quad \mu(t_k) = t_{k+1} - t_k, \quad f^{\Delta}(t_k) = \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k} \quad \text{for} \quad k \in \mathbb{N}_0,$$

and

$$\int_{t_m}^{t_n} f(t)\Delta t = \sum_{k=m}^{n-1} \mu(t_k)\, f(t_k), \quad \text{where} \quad m, n \in \mathbb{N}_0 \text{ with } m < n. \tag{1}$$

The examples in Section 7 are specific cases of Example 2.2 as follows.

Example 2.3. Let $h > 0$. If $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, then

$$\sigma(t) = t + h, \quad \mu(t) \equiv h, \quad f^{\Delta}(t) = \frac{f(t+h) - f(t)}{h} \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_a^b f(t)\,\Delta t = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh), \quad \text{where} \quad a, b \in \mathbb{T} \text{ with } a < b.$$

Example 2.4. If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1, \quad \mu(t) \equiv 1, \quad f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t) \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_a^b f(t)\,\Delta t = \sum_{k=a}^{b-1} f(k), \quad \text{where} \quad a, b \in \mathbb{T} \text{ with } a < b.$$

Example 2.5. Let $q > 1$. If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, then

$$\sigma(t) = qt, \quad \mu(t) = (q-1)t, \quad f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_{q^m}^{q^n} f(t)\Delta t = (q-1) \sum_{k=m}^{n-1} q^k f(q^k), \quad \text{where} \quad m, n \in \mathbb{N}_0 \text{ with } m < n.$$

3. Linear Programming Problems

Throughout this paper, $\mathbb{T}$ stands for a time scale, we assume $0 \in \mathbb{T}$, we let $T \in \mathbb{T}$, and we use $\mathfrak{I}$ to denote the time scales interval

$$\mathfrak{I} = [0, T] \cap \mathbb{T}.$$

By E_k, we denote the space of all rd-continuous functions from $\mathfrak{I}$ into $\mathbb{R}^k$. In [1], the authors have introduced the primal time scales programming problem as

$$\left.\begin{aligned} &\text{Maximize} \quad U(x) = \int_0^{\sigma(T)} f^\top(t)\, x(t)\Delta t, \\ &\text{subject to} \quad B(t)\, x(t) \le g(t) + \int_0^t K(t,s)\, x(s)\Delta s, \quad t \in \mathfrak{I}, \\ &\text{and} \quad x \in E_n, \quad x(t) \ge 0, \quad t \in \mathfrak{I}, \end{aligned}\right\} \tag{2}$$

where $f \in E_n$, $g \in E_m$, and B and K are rd-continuous $m \times n$ matrix-valued functions. Moreover, in [1], the dual time scales programming problem is introduced as

$$\left.\begin{aligned} &\text{Minimize} \quad V(z) = \int_0^{\sigma(T)} f^\top(t)\, z(t)\Delta t, \\ &\text{subject to} \quad B^\top(t)\, z(t) \ge f(t) + \int_{\sigma(t)}^{\sigma(T)} K^\top(s,t)\, z(s)\Delta s, \quad t \in \mathfrak{I}, \\ &\text{and} \quad z \in E_m, \quad z(t) \ge 0, \quad t \in \mathfrak{I}. \end{aligned}\right\} \tag{3}$$

A feasible solution of (2) (or (3)) is any one that satisfies the given constraints. An optimal solution to (2) (or (3)) is a feasible solution with the largest (or smallest) objective function value. In [1], the following results are established.

Theorem 3.1. (Weak duality theorem) *If x and z are arbitrary feasible solutions of (2) and (3), respectively, then $U(x) \le V(z)$.*

Theorem 3.2. (Optimality condition) *If there exist feasible solutions x^* and z^* of (2) and (3), respectively, such that $U(x^*) = V(z^*)$, then x^* and z^* are optimal solutions of their respective problems.*

Theorem 3.3. (Strong duality theorem) *Assume $\mathbb{T}$ is an isolated time scale. If (2) has an optimal solution x^*, then (3) has an optimal solution z^* such that $U(x^*) = V(z^*)$.*

4. The Primal Fractional Problem

In this section, we consider linear fractional programming problems on time scales as an extension of continuous-time linear fractional programming problems and extend the results in [34, 37, 39–41 to the general time scales model for arbitrary time scales. The primal time scales linear fractional programming model is formulated as

$$\left.\begin{aligned} &\text{Maximize} \quad U(x) = \frac{\alpha + \int_0^{\sigma(T)} f^\top(t)\, x(t)\Delta t}{\gamma + \int_0^{\sigma(T)} h^\top(t)\, x(t)\Delta t}, \\ &\text{subject to} \quad B(t)\, x(t) \le g(t) + \int_0^t K(t,s)\, x(s)\Delta s, \quad t \in \Im, \\ &\text{and} \quad x \in E_n, \quad x(t) \ge 0, \quad t \in \Im, \end{aligned}\right\} \tag{4}$$

where $\alpha, \gamma \in \mathbb{R}$, $\gamma > 0$, $f, h \in E_n$, $g \in E_m$, $g, h \ge 0$, and B and K are matrix-valued functions of size $m \times n$. Together with (4), we consider the problem

$$\left.\begin{aligned} &\text{Maximize} \quad W(y,\lambda) = \lambda\alpha + \int_0^{\sigma(T)} f^\top(t)\, y(t)\Delta t, \\ &\text{subject to} \quad B\, y(t) \le \lambda g(t) + \int_0^t K(t,s)\, y(s)\Delta s, \quad t \in \Im, \\ &\qquad \lambda\gamma + \int_0^{\sigma(T)} h^\top(t)\, y(t)\Delta t = 1, \\ &\text{and} \quad y \in E_n, \quad y(t) \ge 0, \quad t \in \Im, \quad \lambda \in \mathbb{R}, \quad \lambda \ge 0. \end{aligned}\right\} \tag{5}$$

In the remainder of this section, we establish some relationships between (4) and (5).

Theorem 4.1. *Let x be feasible for equation (4) and define*

$$\lambda := \frac{1}{\gamma + \int_0^{\sigma(T)} h^\top(t)\, x(t)\Delta t} \quad \text{and} \quad y := \lambda x. \tag{6}$$

Then (y,λ) is feasible for (5) and

$$W(y,\lambda) = U(x).$$

Proof. Assume that x is feasible for (4) and define (y,λ) by (6). Since $\gamma > 0$ and $h(t) \ge 0$ for $t \in \Im$, we have $\lambda > 0$. Now

$$B(t)\, x(t) \le g(t) + \int_0^t K(t,s)\, x(s)\Delta s,$$

for $t \in \Im$, implies that

$$\begin{aligned} B(t)y(t) = \lambda B(t)\, x(t) &\le \lambda\left[g(t) + \int_0^t K(t,s)x(s)\Delta s \right] \\ &= \lambda g(t) + \int_0^t K(t,s)\, y(s)\Delta s, \end{aligned}$$

for $t \in \Im$, and $x \in E_n$, $x(t) \ge 0$ for $t \in \Im$ imply $y \in E_n$, $y(t) \ge 0$ for $t \in \Im$. Moreover, we have

$$1 = \lambda\left[\gamma + \int_0^{\sigma(T)} h^\top(t)\, x(t)\Delta t \right] = \lambda\gamma + \int_0^{\sigma(T)} h^\top(t)\, y(t)\Delta t.$$

Thus, (y,λ) is feasible for (5). Finally,

$$W(y,\lambda) = \lambda\alpha + \int_0^{\sigma(T)} f^\top(t)y(t)\Delta t = \lambda\left[\alpha + \int_0^{\sigma(T)} f^\top(t)\,x(t)\Delta t\right]$$
$$= \frac{\alpha + \int_0^{\sigma(T)} f^\top(t)\,x(t)\Delta t}{\gamma + \int_0^{\sigma(T)} h^\top(t)\,x(t)\Delta t} = U(x),$$

completing the proof. ■

Theorem 4.2. *Let (y,λ) be feasible for equation (5).*
i. *If $\lambda > 0$, then $x := \frac{y}{\lambda}$ is feasible for (4) and*
$$U(x) = W(y,\lambda).$$
ii. *If $\lambda = 0$, then $x_c := \frac{c\gamma}{1-c}\,y$, $0 \le c < 1$, is feasible for (4) and*
$$U(x_c) = \frac{\alpha(1-c)}{\gamma} + cW(y,\lambda).$$

Proof. Assume that (y,λ) is feasible for (5). First, suppose $\lambda > 0$ and define $x := \frac{y}{\lambda}$. Now
$$B(t)y(t) \le \lambda g(t) + \int_0^t K(t,s)y(s)\Delta s$$
for $t \in \Im$ implies
$$B(t)x(t) = \frac{1}{\lambda}B(t)y(t) \le \frac{1}{\lambda}\left[\lambda g(t) + \int_0^t K(t,s)\,y(s)\Delta s\right]$$
$$= g(t) + \int_0^t K(t,s)\,x(s)\Delta s,$$
for $t \in \Im$, and $y \in E_n$, $y(t) \ge 0$ for $t \in \Im$ imply $x \in E_n$, $x(t) \ge 0$ for $t \in \Im$. Thus, x is feasible for (4). Finally, since
$$1 = \lambda\gamma + \int_0^{\sigma(T)} h^\top(t)\,y(t)\Delta t = \lambda\left[\gamma + \int_0^{\sigma(T)} h^\top(t)\,x(t)\Delta t\right],$$
we get
$$U(x) = \frac{\alpha + \int_0^{\sigma(T)} f(t)x(t)\Delta t}{\gamma + \int_0^{\sigma(T)} h(t)x(t)\Delta t} = \lambda\left[\alpha + \int_0^{\sigma(T)} f(t)x(t)\Delta t\right]$$
$$= \lambda\alpha + \int_0^{\sigma(T)} f(t)y(t)\Delta t = W(y,\lambda),$$
completing the proof of i. Next, suppose $\lambda = 0$ and define $x_c := \frac{c\gamma}{1-c}y$, where $0 \le c < 1$. Now $\gamma > 0$, $g(t) \ge 0$ for $t \in \Im$, and
$$B(t)\,y(t) \le \int_0^t K(t,s)\,y(s)\Delta s$$
for $t \in \Im$ imply

$$B(t)x_c(t) = \frac{c\gamma}{1-c}B(t)y(t) \leq \frac{c\gamma}{1-c}\int_0^t K(t,s)y(s)\Delta s$$
$$= \int_0^t K(t,s)x_c(s)\Delta s \leq g(t) + \int_0^t K(t,s)x_c(s)\Delta s,$$

for $t \in \Im$, and $y \in E_n$, $y(t) \geq 0$ for $t \in \Im$ imply $x \in E_n$, $x_c(t) \geq 0$ for $t \in \Im$. Thus, x_c is feasible for (4). Finally, since

$$\int_0^{\sigma(T)} h^{\top}(t)\, y(t)\Delta t = 1,$$

we get

$$U(x_c) = \frac{\alpha + \int_0^{\sigma(T)} f(t)x_c(t)\Delta t}{\gamma + \int_0^{\sigma(T)} h(t)x_c(t)\Delta t} = \frac{\alpha + \frac{c\gamma}{1-c}\int_0^{\sigma(T)} f(t)y(t)\Delta t}{\gamma + \frac{c\gamma}{1-c}}$$
$$= \frac{\alpha(1-c) + c\gamma\int_0^{\sigma(T)} f(t)y(t)\Delta t}{\gamma} = \frac{\alpha(1-c)}{\gamma} + cW(y,\lambda),$$

completing the proof of ii. ■

Lemma 4.1. *Suppose x is feasible for (4) and $(y,0)$ is feasible for (5). Then $x_{(r)} := x + ry$, $r > 0$, is also feasible for (4) and*

$$U(x_{(r)}) = \frac{U(x) + r\lambda W(y,0)}{1 + r\lambda},$$

where λ is defined in (6).

Proof. Using the assumptions and notation of the statement, we have

$$B(t)x_{(r)}(t) = B(t)x(t) + rB(t)y(t)$$
$$\leq g(t) + \int_0^t K(t,s)x(s)\Delta s + r\int_0^t K(t,s)y(s)\Delta s$$
$$= g(t) + \int_0^t K(t,s)x_{(r)}(s)\Delta s$$

for $t \in \Im$, so that $x_{(r)}$ is seen to be feasible for (4). Finally,

$$U(x_{(r)}) = \frac{\alpha + \int_0^{\sigma(T)} f(t)x_{(r)}(t)\Delta t}{\gamma + \int_0^{\sigma(T)} h(t)x_{(r)}(t)\Delta t}$$
$$= \frac{\alpha + \int_0^{\sigma(T)} f(t)x(t)\Delta t + r\int_0^{\sigma(T)} f(t)y(t)\Delta t}{\gamma + \int_0^{\sigma(T)} h(t)x(t)\Delta t + r}$$
$$= \frac{U(x) + r\lambda W(y,0)}{1 + r\lambda},$$

completing the proof. ■

Theorem 4.3. *Let x^* be optimal for (4) and define*

$$\lambda^* := \frac{1}{\gamma + \int_0^{\sigma(T)} h^\top(t)\, x^*(t)\Delta t} \quad \text{and} \quad y^* := \lambda^* x^*. \tag{7}$$

Then (y^*, λ^*) is optimal for (5) and

$$W(y^*, \lambda) = U(x^*).$$

Proof. Assume that x^* is optimal for (4) and define (y^*, λ^*) by (7). By Theorem 4.1, (y^*, λ^*) is feasible for (5) and $W(y^*, \lambda^*) = U(x^*)$. Now let (y, λ) be an arbitrary feasible solution of (5). First, assume $\lambda > 0$. By Theorem 4.2 i, $x := \frac{y}{\lambda}$ is feasible for (4) and $U(x) = W(y, \lambda)$. Since $U(x) \le U(x^*)$, we get

$$W(y, \lambda) = U(x) \le U(x^*) = W(y^*, \lambda^*).$$

Next, assume $\lambda = 0$. By Theorem 4.2 ii, $x_c = \frac{c\gamma}{1-c} y$ is feasible for (4), for any $0 \le c < 1$, and $U(x_c) = \frac{\alpha(1-c)}{\gamma} + cW(y, 0)$. Since $U(x_c) \le U(x^*)$ for all $0 \le c < 1$, we get

$$W(y^*, \lambda^*) = U(x^*) \ge U(x_C) = \frac{\alpha(1-c)}{\gamma} + cW(y, \lambda), \forall\ 0 \le c < 1,$$

and thus, by letting $c \to 1^-$, we find

$$W(y^*, \lambda^*) \ge W(y, \lambda).$$

Hence, for all feasible (y, λ), we have $W(y, \lambda) \le W(y^*, \lambda^*)$, showing that (y^*, λ^*) is optimal for (5). ■

Theorem 4.4. *Let (y^*, λ^*) be optimal for (5).*

i. *If $\lambda^* > 0$, then $x^* := \frac{y^*}{\lambda^*}$ is optimal for (4) and*

$$U(x^*) = W(y^*, \lambda^*).$$

ii. *If $\lambda^* = 0$ but (y, λ) is not optimal for (5) for any $\lambda > 0$, then (4) has no optimal solution.*

Proof. Assume that (y^*, λ^*) is optimal for (5). First, suppose $\lambda^* > 0$ and define $x^* := \frac{y^*}{\lambda^*}$. By Theorem 4.2 i, x^* is feasible for (4) and $U(x^*) = W(y^*, \lambda^*)$. Now let x be an arbitrary feasible solution for (4) and define (y, λ) by (6). By Theorem 4.1, (y, λ) is feasible for (5) and $W(y, \lambda) = U(x)$. Since $W(y, \lambda) \le W(y^*, \lambda^*)$, we get

$$U(x) = W(y, \lambda) \le W(y^*, \lambda^*) = U(x^*),$$

and thus x^* is optimal for (4). Next, suppose $\lambda^* = 0$ and $W(y, \lambda) < W(y^*, 0)$ for all feasible (y, λ) with $\lambda > 0$. Let x be an arbitrary feasible solution for (4) and define (y, λ) by (6). By Theorem 4.1, we have $U(x) = W(y, \lambda)$. By Lemma 4.1, $x_{(1)} = x + y^*$ is also feasible for (4) and $U(x_{(1)}) = \frac{U(x)+\lambda W(y^*,0)}{1+\lambda}$. Since $W(y, \lambda) < W(y^*, 0)$ and $\lambda > 0$, we get

$$U(x_{(1)}) = \frac{U(x) + \lambda W(y^*, 0)}{1 + \lambda} > \frac{U(x) + \lambda W(y, \lambda)}{1 + \lambda} = \frac{U(x) + \lambda U(x)}{1 + \lambda} = U(x),$$

showing that no feasible x can be optimal for (4). ■

5. The Dual Fractional Problem

In this section, we rewrite (5) as a linear problem of the form (2) and find its dual (3), which we shall express later as (10). Define

$$\tilde{\mathbb{T}} = [0, \sigma(T)] \cap \mathbb{T}] \cup \{-1, \sigma(T) + 1\},$$

and

$$\tilde{\mathfrak{I}} = \mathfrak{I} \cup \{-1, \sigma(T)\}.$$

By $\tilde{E}_k$, we denote the space of all rd-continuous functions from $\tilde{\mathfrak{I}}$ into $\mathbb{R}^k$. Note that, on $\tilde{\mathbb{T}}$, we have

$$\sigma(-1) = 0 \quad \text{and} \quad \sigma(\sigma(T)) = \sigma(T) + 1.$$

Defining

$$f(-1) = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad h(-1) = \begin{pmatrix} \gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad B(-1) = 0,$$

$$K(t, -1) = \begin{pmatrix} g(t) & 0 \end{pmatrix}, \quad t \in \mathsf{I}, \quad y(-1) = \begin{pmatrix} \lambda \\ * \\ \vdots \\ * \end{pmatrix},$$

we find that

$$\lambda\alpha + \int_0^{\sigma(T)} f^\top(t)\, y(t)\Delta t = \int_{-1}^{\sigma(T)} f^\top(t) y(t)\Delta t,$$

$$\lambda\gamma + \int_0^{\sigma(T)} h^\top(t)\, y(t)\Delta t = \int_{-1}^{\sigma(T)} h^\top(t)\, y(t)\Delta t,$$

and

$$\lambda g(t) + \int_0^{\sigma(T)} K(t,s)\, y(t)\Delta t = \int_{-1}^{\sigma(T)} K(t,s)\, y(t)\Delta t, \quad t \in \mathfrak{I}.$$

Hence, by putting

$$f(\sigma(T)) = 0, \quad g(\sigma(T)) = 0, \quad B(\sigma(T)) = 0,$$

$$K(\sigma(T), t) = \begin{pmatrix} h^\top(t) \\ -h^\top(t) \\ 0 \end{pmatrix}, \quad t \in \mathfrak{I} \cup \{-1\},$$

$$\tilde{g}(\sigma(T)) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \tilde{g}(t) = 0 \quad \text{for} \quad t \neq \sigma(T)$$

(this is for the case $m > 1$; if $m = 1$, then we add a zero row to all other K and to B, hence getting a problem with $\tilde{m} = 2$), we may rewrite (5) as

$$\left.\begin{aligned} &\text{Maximize} \quad \int_{-1}^{\sigma(\sigma(T))} f^{\top}(t)\, y(t)\Delta t, \\ &\text{such that} \quad B(t)\, y(t) \leq \tilde{g}(t) + \int_{-1}^{t} K(t,s)\, y(s)\Delta s, \quad t \in \tilde{\mathfrak{I}}, \\ &\text{and} \quad y \in \tilde{E}_n, \quad y(t) \geq 0, \quad t \in \mathfrak{I}. \end{aligned}\right\} \tag{8}$$

From Section 3, the dual of (8) is found as

$$\left.\begin{aligned} &\text{Mnimize} \quad \int_{-1}^{\sigma(\sigma(T))} \tilde{g}^{\top}(t)\, z(t)\Delta t, \\ &\text{such that} \quad B^{\top}(t)\, z(t) \geq f(t) + \int_{\sigma(t)}^{\sigma(T)} K^{\top}(s,t)\, z(s)\Delta s, \quad t \in \tilde{\mathfrak{I}}, \\ &\text{and} \qquad z \in \tilde{E}_{\tilde{m}}, \quad z(t) \geq 0, \quad t \in \mathfrak{I}. \end{aligned}\right\} \tag{9}$$

Introducing now the notation

$$z(\sigma(T)) = \begin{pmatrix} p \\ q \\ * \\ \vdots \\ * \end{pmatrix} \quad \text{and} \quad \beta = q - p,$$

we find

$$\int_{-1}^{\sigma(\sigma(T))} \tilde{g}^{\top}(t) z(t)\Delta t = \tilde{g}^{\top}(\sigma(T)) z(\sigma(T)) = q - p = \beta,$$

$$\int_{\sigma(t)}^{\sigma(\sigma(T))} K^{\top}(s,t) z(s)\Delta s = \int_{\sigma(t)}^{\sigma(T)} K^{\top}(s,t) z(s)\Delta s - \beta h(t), \quad t \in \mathfrak{I} \cup \{-1\},$$

and

$$\int_{\sigma(-1)}^{\sigma(T)} K^{\top}(s,-1)\, z(s)\Delta s = \int_{0}^{\sigma(T)} \begin{pmatrix} g^{\top}(s) z(s) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Delta s,$$

so equation (9) can be rewritten as

$$\left.\begin{aligned} &\text{Minimize} \quad V(z,\beta) = \beta \\ &\qquad B^{\top}(t)\, z(t) + \beta h(t) \geq f(t) + \int_{\sigma(t)}^{\sigma(T)} K^{\top}(s,t) z(s)\Delta s, \quad t \in \Im, \\ &\text{subject to } -\beta\gamma + \int_{0}^{\sigma(T)} g^{\top}(t)\, z(t)\Delta t \leq -\alpha, \\ &z \in E_m, \quad z(t) \geq 0, \quad t \in \Im. \end{aligned}\right\} \tag{10}$$

6. Duality Theorems

In this section, we state and prove the weak duality theorem and the optimality condition theorem for linear fractional programming model on arbitrary time scales, while the strong duality theorem is established for isolated time scales.

Theorem 6.1. (Weak duality theorem) *If x and (z,β) are arbitrary feasible solutions of (4) and (10), respectively, then $U(x) \leq V(z,\beta)$.*

Proof. Assume that x is a feasible solution of (4) and that (z,β) is a feasible solution of (10). Define (y,λ) by (6). By Theorem 4.1, (y,λ) is feasible for (5) and $W(y,\lambda) = U(x)$. Hence, by Theorem 3.1, $W(y,\lambda) \leq V(z,\beta)$, so

$$U(x) = W(y,\lambda) \leq V(z,\beta),$$

completing the proof. ■

Theorem 6.2. (Optimality condition) *If there exist feasible solutions x^* and (z^*,β^*) of (4) and (10), respectively, such that $U(x^*) = V(z^*,\beta^*)$, then x^* and (z^*,β^*) are optimal solutions of their respective problems.*

Proof. Assume that x^* is a feasible solution of (4), that (z^*,β^*) is a feasible solution of (10), and that $U(x^*) = V(z^*,\beta^*)$. Define (y^*,λ^*) by (7). By Theorem 4.1, (y^*,λ^*) is feasible for (5) and $W(y^*,\lambda^*) = U(x^*)$. Hence,

$$W(y^*,\lambda^*) = U(x^*) = V(z^*,\beta^*).$$

By Theorem 3.2, (y^*,λ^*) is optimal for (5) and (z^*,β^*) is optimal for (10). Since $\lambda^* > 0$, by Theorem 4.4, $x^* = \frac{y^*}{\lambda^*}$ is optimal for (4). ■

Theorem 6.3. (Strong duality theorem) *Assume Υ is an isolated time scale. If (4) has an optimal solution x^*, then (10) has an optimal solution (z^*,β^*) such that $U(x^*) = V(z^*,\beta^*)$.*

Proof. Assume that (4) has an optimal solution x^* and define (y^*,λ^*) by (7). By Theorem 4.3, (y^*,λ^*) is optimal for (5) and $W(y^*,\lambda^*) = U(x^*)$. By Theorem 3.3, (10) has an optimal solution (z^*,β^*) such that $V(z^*,\beta^*) = W(y^*,\lambda^*)$. Hence,

$$U(x^*) = W(y^*,\lambda^*) = V(z^*,\beta^*),$$

and here the proof completes. ■

7. Illustrative Examples

In this section, three examples are given in order to illustrate our duality theorems on isolated time scales.

Example 7.1. Let $\mathbb{T} = \mathbb{Z}$ and $\Im = \{0,1,2,3,4\}$. Then, we consider the isolated time scales linear fractional programming primal model

$$\begin{cases} \text{Maximize} \quad U(x) = \dfrac{\frac{1}{3} + \int_0^{\sigma(4)} t\,x(t)\Delta t}{\frac{1}{2} + \frac{1}{2}\int_0^{\sigma(4)} t\,x(t)\Delta t} = \dfrac{\frac{1}{3} + \sum_{t=0}^{4} t\,x(t)}{\frac{1}{2} + \frac{1}{2}\sum_{t=0}^{4} t\,x(t)} \\ \text{subject to} \quad 6x(t) \le t + \int_0^t x(s)\Delta s = t + \sum_{s=0}^{t-1} x(s), \quad t \in \Im, \\ \text{and} \quad x(t) \ge 0, \quad t \in \Im, \end{cases}$$

where we have used σ and the integral given in Example 2.4. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} &x^*(0) = 0.000000, \quad x^*(1) = 0.166665, \quad x^*(2) = 0.361110, \\ &x^*(3) = 0.587964, \quad x^*(4) = 0.852625, \quad U(x^*) = 1.801792. \end{aligned}$$

On the other hand, the isolated time scales linear fractional programming dual model is

$$\begin{cases} \text{Minimize} \quad V(z) = \beta \\ \text{subject to} \quad \begin{aligned} &6z(t) + \beta\left(\tfrac{t}{2}\right) \ge t + \int_{\sigma(t)}^{\sigma(T)} z(s)\Delta s = t + \sum_{s=t+1}^{4} z(s), \quad t \in \Im, \\ &\tfrac{-\beta}{2} + \int_0^{\sigma(4)} tz(t)\Delta t = \tfrac{-\beta}{2} + \sum_{t=0}^{4} tz(t) \le -\tfrac{1}{3} \end{aligned} \\ \text{and} \quad z(t) \ge 0, \quad t \in \Im, \end{cases}$$

where we have used again Example 2.4. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} &z^*(0) = 0.037903, \quad z^*(1) = 0.046646, \quad z^*(2) = 0.054140, \\ &z^*(3) = 0.060564, \quad z^*(4) = 0.066069, \quad \beta^* = 1.801792, \end{aligned}$$

and the optimal value is $V(z^*, \beta^*) = 1.801792$, confirming $U(x^*) = V(z^*, \beta^*)$.

Example 7.2. Let $\mathbb{T} = 5\mathbb{Z}$ and $\Im = \{0,5,10,15,20\}$. Then, we consider the isolated time scales linear fractional programming primal model

$$\begin{cases} \text{Maximize} \quad U(x) = \dfrac{3 + \int_0^{\sigma(20)} t\,x(t)\Delta t}{2 + \int_0^{\sigma(20)} t^2\,x(t)\Delta t} = \dfrac{3 + \sum_{k=0}^{4} 25kx(5k)}{2 + 125\,\sum_{k=0}^{4} k^2x(5k)} \\ \text{subject to} \quad 7x(t) \leq t + \int_0^t x(s)\Delta s = t + 5\displaystyle\sum_{k=0}^{\frac{t}{5}-1} x(5k), \quad t \in \Im, \\ \text{and} \quad x(t) \geq 0, \quad t \in \Im, \end{cases}$$

where we have used σ and the integral given in Example 2.3. Using MATLAB command linprog or LINDO solver, we have

$$x^*(0) = 0.0, \quad x^*(5) = 0.0, \quad x^*(10) = 0.00,$$
$$x^*(15) = 0.0, \quad x^*(20) = 0.0, \quad U(x^*) = 1.5.$$

On the other hand, the isolated time scales linear fractional programming dual model is

$$\begin{cases} \text{Minimize} \quad V(z,\beta) = \beta \\ \text{subject to} \quad \begin{aligned} & 7z(t) + \beta t^2 \geq t + \int_{\sigma(t)}^{\sigma(T)} z(s)\Delta s = t + 5\sum_{k=\frac{t}{5}+1}^{4} z(5k), \quad t \in \Im, \\ & -2\beta + \int_0^{\sigma(4)} tz(t)\Delta t = -2\beta + 25\sum_{k=0}^{4} kz(5k) \leq -3, \end{aligned} \\ \text{and} \quad z(t) \geq 0, \quad t \in \Im, \end{cases}$$

where we have used again Example 2.3. Using MATLAB command linprog or LINDO solver, we have

$$z^*(0) = 0.0, \quad z^*(5) = 0.0, \quad z^*(10) = 0.0,$$
$$z^*(15) = 0.0, \quad z^*(20) = 0.0, \quad \beta^* = 1.5,$$

and the optimal value is $V(z^*,\beta^*) = 1.500000$, confirming $U(x^*) = V(z^*,\beta^*)$.

Example 7.3. Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $\Im = \{1,2,4\}$. Then, we consider the isolated time scales linear fractional programming primal model

$$\begin{cases} \text{Maximize} \quad U(x) = \dfrac{5+\int_1^{\sigma(4)} t\,x(t)\Delta t}{3+\int_1^{\sigma(4)} x(t)\Delta t} = \dfrac{5+\sum_{k=0}^{2} 4^k x(2^k)}{3+\sum_{k=0}^{2} 2^k x(2^k)} \\ \text{subject to} \quad 7x(t) \le t^3 + \int_1^t x(s)\Delta s = t^3 + \displaystyle\sum_{k=0}^{\log_2 t-1} 2^k x(2^k), \quad t \in \Im, \\ \text{and} \quad x(t) \ge 0, \quad t \in \Im, \end{cases}$$

where we have used σ and the integral given in Example 2.5. Using MATLAB command linprog or LINDO solver, we have

$$x^*(1) = 0.000000, \quad x^*(2) = 0.000000,$$
$$x^*(4) = 9.142772, \quad U(x^*) = 3.823105.$$

On the other hand, the isolated time scales linear fractional programming dual model is

$$\begin{cases} \text{Minimize} \quad V(z) = 6\int_1^{\sigma(2^2)} tz(t)\Delta t = 6\displaystyle\sum_{k=0}^{2} 4^k z(2^k) \\ \text{subject to} \quad \begin{aligned} &7z(t)+\beta \ge t + \int_{\sigma(t)}^{\sigma(4)} z(s)\Delta s = t + \sum_{k=1+\log_2 t}^{2} 2^k z(2^k), \quad t \in \Im, \\ &-3\beta + \int_1^{\sigma(4)} t^3 z(t)\Delta t = -3\beta + \sum_{t=0}^{2} 2^k (2^k)^3 z(2^k) \le -5, \end{aligned} \\ \text{and} \quad z(t) \ge 0, \quad t \in \Im, \end{cases}$$

where we have used again Example 2.5. Using MATLAB command linprog or LINDO solver, we have

$$z^*(1) = 0.000000, \quad z^*(2) = 0.000000,$$
$$z^*(4) = 0.025271, \quad \beta^* = 3.823105,$$

and the optimal value is $V(z^*, \beta^*) = 3.823105$, confirming $U(x^*) = V(z^*, \beta^*)$.

7. Conclusions

An efficient formulation and a computational approach have been successfully constructed in this paper in order to solve a general class of linear fractional programming problems on arbitrary time scales. A general form for the primal and the dual time scales models has been formulated. To guarantee that our time scales formulation is indeed a useful formulation, we have established the weak duality theorem and the optimality condition on arbitrary time scales, while the the strong duality theorem is given for isolated time scales. Obtaining the exact optimal solution is the key issue for solving these types of problems, which this paper achieves by using time scales formulation as particular cases for isolated time scales setting such as $\mathbb{T} = q^{\mathbb{N}_0}$ and $\mathbb{T} = h\mathbb{Z}$. Moreover, since the error bound of continuous-time linear fractional programming problems is dependent on both primal and dual models, another key issue for solving these types of problems is to solve both primal and dual models at the same time to abstain the error bound of the solution. Another contribution of this paper is obtaining the optimal solution by solving either the primal problem or the dual problem only, using isolated time scales, which reduces the large computational effort.

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References

[1] R. Al-Salih, and M. Bohner, Linear programming problems on time scales, *Applicable Analysis and Discrete Mathematics* **12**(1), (2018), 192–204.

[2] R. Al-Salih, and M. Bohner, Separated and state-constrained separated linear programming problems on time scales, *Boletim da Sociedade Paranaense de Matemática* **38**(4), (2019), 181–195. ·

[3] F. M. Atici, D. C. Biles, and A. Lebedinsky, An application of time scales to economics, *Mathematical and Computational Modelling* **43**(7-8), (2006), 718–726.

[4] Z. Bartosiewicz, Ü. Kotta, E. Pawłuszewicz, and M. Wyrwas, Control systems on regular time scales and their differential rings, *Mathematics of Control, Signals and Systems* **22**(3), (2011), 185–201.

[5] R. E. Bellman, *Dynamic Programming*, *Princeton Landmarks in Mathematics*, Princeton University Press, Princeton, NJ, 2010.

[6] R. E. Bellman, and S. E. Dreyfus, *Applied Dynamic Programming,* Princeton University Press, Princeton, NJ, 1962.

[7] M. Bohner, Calculus of variations on time scales, *Dynamical Systems and Applications* **13**(3-4), (2004), 339–349.

[8] M. Bohner, M. Fan, and J. Zhang, Periodicity of scalar dynamic equations and applications to population models, *Journal of Mathematical Analysis and Applications* **330**(1), (2007), 1–9.

[9] M. Bohner, and H. Warth, The Beverton-Holt dynamic equation, *Applicable Analysis* **86**(8), (2007), 1007–1015.

[10] M. Bohner, and S. Streipert, Optimal harvesting policy for the Beverton-Holt model, *Mathematical Biosciences and Engineering* **13**(4), (2016), 673–695.

[11] M. Bohner, F. Dannan, and S. Streipert, A nonautonomous Beverton-Holt equation of higher order, *Journal of Mathematical Analysis and Applications* **457**(1), (2018), 114–133.

[12] M. Bohner, R. A. C. Ferreira, and D. F. M. Torres, Integral inequalities and their applications to the calculus of variations on time scales, *Mathematical Inequalities and Applications* **13**(3), (2010), 511–522.

[13] M. Bohner, G. Gelles, and J. Heim, Multiplier-accelerator models on time scales, *International Journal of Statistical Economics* **4**(S10), (2010), 1–12.

[14] M. Bohner, and S. G. Georgiev, *Multivariable Calculus on Time Scales*, Springer, Cham, 2016.

[15] M. Bohner, and G. Sh. Guseinov, Double integral calculus of variations on time scales, *Computer Mathematics and Applications* **54**(1), (2007), 45–57.

[16] M. Bohner, J. Heim, and A. Liu, Solow models on time scales, *Cubo: A Matematical Journal* **15**(1), (2013), 13–31.

[17] M. Bohner, J. Heim, and A. Liu, Qualitative analysis of a Solow model on time scales, *Journal of Concrete and Applicable Mathematics* **13**(3-4), (2015), 183–197.

[18] M. Bohner, and A. Peterson, *Dynamic Equations on Time Scales-An Introduction with Applications*, Birkhäuser Inc., Boston, MA, 2001.

[19] M. Bohner, and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser Inc., Boston, MA, 2003.

[20] M. Bohner, and N. Wintz, The linear quadratic regulator on time scales, *International Journal of Difference Equations* **5**(2), (2010), 149–174.

[21] M. Bohner, and N. Wintz, The linear quadratic tracker on time scales, *International Journal of Dynamical Systems and Differential Equations* **3**(4), (2011), 423–447.

[22] M. Bohner, and N. Wintz, Controllability and observability of time-invariant linear dynamic systems, *Mathematica Bohemica* **137**(2), (2012), 149–163.

[23] M. Bohner, and N. Wintz, The Kalman filter for linear systems on time scales, *Journal of Mathematical Analysis and Applications* **406**(2), (2013), 419–436.

[24] R. N. Buie, and J. Abrham, Numerical solutions to continuous linear programming problems, *Z. Operations Research Ser. A-B* **17**(3), (1973), A107–A117.

[25] A. Charnes, and W. W. Cooper, Programming with linear fractional functionals, *Naval Research Logistics Quarterly* **9**, (1962), 181–186.

[26] R. C. Grinold, Continuous programming. I. Linear objectives, *Journal of Mathematical Analysis and Applications* **28**, (1969), 32–51.

[27] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, *Results in Mathematics* **18**(1-2), (1990), 18–56.

[28] N. Levinson, A class of continuous linear programming problems, *Journal of Mathematical Analysis and Applications* **16**, (1966), 73–83.

[29] E. Pawłuszewicz, Observability of nonlinear control systems on time scales, *International Journal of Systems Science* **43**(12), (2012), 2268–2274.

[30] I. M. Stancu-Minasian, and S. Tigan, Continuous time linear-fractional programming. The minimum-risk approach, *RAIRO Operations Research* **34**(4), (2000), 397–409.

[31] C. C. Tisdell, and A. Zaidi, Basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with an application to economic modelling, *Nonlinear Analysis* **68**(1), (2008), 3504–3524.

[32] W. F. Tyndall, A duality theorem for a class of continuous linear programming problems, *Journal of SIAM* **13**, (1965), 644–666.

[33] W. F. Tyndall, An extended duality theorem for continuous linear programming problems, *SIAM Journal of Applied Mathematics* **15**, (1967), 1294–1298.

[34] C-F. Wen, An interval-type algorithm for continuous-time linear fractional programming problems, *Taiwanese Journal of Mathematics* **16**(4), (2012), 1423–1452.

[35] C-F. Wen, Continuous-time generalized fractional programming problems. Part I : Basic theory, *Journal of Optimization Theory and Applications* **157**(2), (2013), 365–399.

[36] C-F. Wen, Continuous-time generalized fractional programming problems, Part II : An interval-type computational procedure, *Journal of Optimization Theory and Applications* **157**(3), (2013), 819–843.

[37] C-F. Wen, Y-Y. Lur, S-M. Guu, and E. S. Lee, On a recurrence algorithm for continuous-time linear fractional programming problems, *Computers and Mathematics With Applications* **59**(2), (2010), 829–852.

[38] C-F. Wen, Y-Y. Lur, and H-C. Lai, Approximate solutions and error bounds for a class of continuous-time linear programming problems, *Optimization* **61**(2), (2012), 163–185.

[39] C-F. Wen, and H-C. Wu, Using the Dinkelbach-type algorithm to solve the continuous-time linear fractional programming problems, *Journal of Global Optimization* **49**(2), (2011), 237–263.

[40] C-F. Wen, and H-C. Wu, Approximate solutions and duality theorems for continuous-time linear fractional programming problems, *Numerical Functional Analysis and Optimization* **33**(1), (2012), 80–129.

[41] H-C. Wu, Parametric continuous-time linear fractional programming problems, *Journal of Inequalities and Applications* **2015**, (2015), 251.

[42] G. J. Zalmai, Duality for a class of continuous-time homogeneous fractional programming problems, *Z. Operations Research Ser. A-B* **30**(1), (1986), A43–A48.

[43] G. J. Zalmai, Duality for a class of continuous-time fractional programming problems, *Utilitas Mathematica* **31**, (1987), 209–218.

[44] G. J. Zalmai, Optimality conditions and duality for a class of continuous-time generalized fractional programming problems, *Journal of Mathematical Analysis and Applications* **153**(2), (1990), 356–371.

[45] G. J. Zalmai, Continuous-time generalized fractional programming, *Optimization* **36**(3), (1996), 195–217.

[46] G. J. Zalmai, Optimality conditions and duality models for a class of nonsmooth constrained fractional optimal control problems, *Journal of Mathematical Analysis and Applications* 210(1), (1997), 114–149.

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