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Oscillation and spectral theory for linear Hamiltonian systems with nonlinear dependence on the spectral parameter

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In this paper, we consider linear Hamiltonian differential systems which depend in general nonlinearly on the spectral parameter and with Dirichlet boundary conditions. Our results generalize the known theory of linear Hamiltonian systems in two respects. Namely, we allow nonlinear dependence of the coefficients on the spectral parameter and at the same time we do not impose any controllability and strict normality assumptions. We introduce the notion of a finite eigenvalue and prove the oscillation theorem relating the number of finite eigenvalues which are less than or equal to a given value of the spectral parameter with the number of proper focal points of the principal solution of the system in the considered interval. We also define the corresponding geometric multiplicity of finite eigenvalues in terms of finite eigenfunctions and prove that the algebraic and geometric multiplicities coincide. The results are also new for Sturm–Liouville differential equations, being special linear Hamiltonian systems.

1 Introduction

Oscillation and spectral theory of self-adjoint linear differential equations or systems is a classical research topic. In this paper, we contribute to this theory by considering the linear Hamiltonian system

$$x' = A(t,\lambda)\,x + B(t,\lambda)\,u, \quad u' = C(t,\lambda)\,x - A^T(t,\lambda)\,u, \quad t \in [a,b], \tag{H$_\lambda$}$$

where $\lambda \in \mathbb{R}$ is the spectral parameter, with the Dirichlet boundary conditions

$$x(a,\lambda) = 0, \quad x(b,\lambda) = 0. \tag{1.1}$$

The coefficients are $n \times n$-matrix-valued functions such that the Hamiltonian

$$\mathcal{H}(t,\lambda) := \begin{pmatrix} -C(t,\lambda) & A^T(t,\lambda) \\ A(t,\lambda) & B(t,\lambda) \end{pmatrix} \tag{1.2}$$

defined on $[a,b] \times \mathbb{R}$ is symmetric, satisfies certain smoothness assumptions (see Section 2), and the monotonicity assumption

$$\mathcal{H}_\lambda(t,\lambda) := \frac{\partial}{\partial\lambda}\,\mathcal{H}(t,\lambda) \geq 0 \quad \text{for all} \quad (t,\lambda) \in [a,b] \times \mathbb{R},$$

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i.e., for each $t \in [a,b]$ the Hamiltonian $\mathcal{H}(t,\cdot)$ is nondecreasing in λ. Moreover, the matrix $B(t,\lambda)$ is assumed to satisfy the Legendre condition

$$B(t,\lambda) \geq 0 \quad \text{for all} \quad (t,\lambda) \in [a,b] \times \mathbb{R}. \tag{1.3}$$

The dependence of the Hamiltonian on the spectral parameter is allowed to be nonlinear.

In the classical theory, such as in [7], system (H$_\lambda$) is studied under controllability (or normality) and strict normality assumptions, see [7, Section 4.1]. Controllability means that the solutions $(x(\cdot,\lambda), u(\cdot,\lambda))$ of (H$_\lambda$) are not "degenerate" in the first component, that is, whenever $x(\cdot,\lambda) = 0$ on a subinterval of $[a,b]$, then also $u(\cdot,\lambda) = 0$ in this subinterval. On the other hand, strict normality means that the solutions of (H$_\lambda$) are not "degenerate" with respect to change in λ, that is, for any $\lambda \in \mathbb{R}$

$$\mathcal{H}_\lambda(\cdot,\lambda) \begin{pmatrix} x(\cdot,\lambda) \\ u(\cdot,\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.4}$$

on a subinterval of $[a,b]$ always implies that $(x(\cdot,\lambda), u(\cdot,\lambda)) = (0,0)$ in this subinterval. Then, in both cases, we have $(x(\cdot,\lambda), u(\cdot,\lambda)) = (0,0)$ on $[a,b]$ by the uniqueness argument.

In this paper, we are concerned with a so-called "oscillation theorem" for system (H$_\lambda$). By [7, Theorem 4.1.3], condition (1.3) and the above controllability assumption yield that for any $\lambda_0 \in \mathbb{R}$ the focal points of conjoined bases $(X(\cdot,\lambda_0), U(\cdot,\lambda_0))$ of system (H$_{\lambda_0}$), i.e., the points $t_0 \in [a,b]$ at which $X(t_0,\lambda_0)$ is singular, are isolated. The multiplicity of such a focal point is then the dimension of the kernel of $X(t_0,\lambda_0)$, i.e., the defect of $X(t_0,\lambda_0)$. The oscillation theorem then counts the number of these focal points in $(a,b]$ and compares this number with the number of eigenvalues of the eigenvalue problem

$$(\mathrm{H}_\lambda), \quad \lambda \in \mathbb{R}, \quad (1.1), \tag{E}$$

which are less than or equal to λ_0, see [7, Sections 7.1–7.2].

In [8] and followed by [18], [19], the concept of possibly "abnormal" linear Hamiltonian systems was introduced. Based on the result of [8, Theorem 3], saying that (1.3) implies a piecewise constant kernel of $X(\cdot,\lambda_0)$ on $[a,b]$, a new notion of proper focal points was given in [19]. More precisely, a point $t_0 \in (a,b]$ is a *proper focal point* of the conjoined basis $(X(\cdot,\lambda_0), U(\cdot,\lambda_0))$ of (H$_{\lambda_0}$), provided

$$m(t_0) := \operatorname{def} X(t_0,\lambda_0) - \operatorname{def} X(t_0^-,\lambda_0) \geq 1, \tag{1.5}$$

and then the number $m(t_0)$ is its multiplicity. Then, in the same paper, the oscillation theorem is proven in this "abnormal" setting for the system (H$_\lambda$) with the Hamiltonian given by

$$\mathcal{H}(t,\lambda) = \begin{pmatrix} -C(t) + \lambda\, W(t) & A^T(t) \\ A(t) & B(t) \end{pmatrix}, \quad \text{with} \quad \mathcal{H}_\lambda(t,\lambda) = \begin{pmatrix} W(t) & 0 \\ 0 & 0 \end{pmatrix} \geq 0, \tag{1.6}$$

that is, the corresponding system (H$_\lambda$) has the first equation independent of λ and the second equation linear in λ, with $W(t) \geq 0$ on $[a,b]$. Based on the work [10], the proof of the latter result was then refined in [9, Appendix B].

The main results of this paper (Theorems 3.4 and 3.5) generalize the two above mentioned oscillation theorems in two aspects. Namely, compared with [9], [19], we allow nonlinear dependence on λ in the Hamiltonian $\mathcal{H}(t,\lambda)$, and at the same time we remove the controllability and strict normality assumptions used in [7, Sections 7.1–7.2]. For this purpose, the notion of an eigenvalue of the eigenvalue problem (E) needs to be properly extended from the case of (1.6) used in [19] to the general case (1.2), including the geometric properties in terms of the corresponding eigenfunctions.

Remark 1.1 (i) The linear Hamiltonian system (H$_\lambda$) covers as a special case the second order Sturm–Liouville differential equation

$$\big(r(t,\lambda)\, x'\big)' + q(t,\lambda)\, x = 0, \quad t \in [a,b], \tag{SL}$$

see e.g., [7, Section 8.3]. In this case, $A(t,\lambda) \equiv 0$, $B(t,\lambda) = 1/r(t,\lambda)$, and $C(t,\lambda) = -q(t,\lambda)$, implying controllability of equation (SL). The main assumptions about the coefficients r and q stated in (2.1) below contain, except of continuity and/or differentiability conditions, the requirement that the function $q(t,\cdot)$ is nondecreasing in λ and that the function $r(t,\cdot)$ is positive and nonincreasing in λ on $\mathbb{R}$ for every $t \in [a,b]$. In particular, *strict monotonicity* of $q(t,\cdot)$ is not required in this paper. This corresponds to removing the strict normality assumption in our hypotheses compared to [7, Assumption (8.3.7), p. 245]. As a consequence, under the assumptions in (2.1), the oscillation theorem for equation (SL) proven in this paper via Theorem 3.5 generalizes [7, Theorem 8.3.7, p. 249] to the case of monotone $q(t,\cdot)$.

(ii) Note that the very same conditions on the coefficients r and q, namely positivity and monotonicity of $r(t,\cdot)$ and strict monotonicity of $q(t,\cdot)$ have been used in [11], [12], [16] in order to prove the accumulation of eigenvalues of (SL) at the boundary of the essential spectrum. The methods of this paper suggest that similar results also hold when $q(t,\cdot)$ is only monotone (nondecreasing) in λ when the traditional notion of an eigenvalue is replaced by the notion of a finite eigenvalue (see Definition 3.1).

(iii) The content of part (i) remains valid also for Sturm–Liouville differential equations of arbitrary even order

$$\sum_{j=0}^{n} (-1)^j \left(r_j(t,\lambda)\, x^{(j)}\right)^{(j)} = 0, \quad t \in [a,b], \tag{1.7}$$

as such equations are special cases of system (H_λ), in which $B(t,\lambda) = \operatorname{diag}\{0,\dots,1/r_n(t,\lambda)\}$, $C(t,\lambda) = \operatorname{diag}\{r_0(t,\lambda),\dots,r_{n-1}(t,\lambda)\}$, and $A(t,\lambda)$ is constant, see [7, Section 8.3] and [1, Chapter 7]. In this case, the assumptions in (2.1) imply that the coefficients $r_0(t,\cdot),\dots,r_n(t,\cdot)$ are nonincreasing in λ on $\mathbb{R}$ for every $t \in [a,b]$ and $r_n(t,\lambda)$ is positive. Therefore, strict monotonicity of the coefficient $r_0(t,\cdot)$ used in [7, Assumption (8.4.3), p. 254] is now removed. The resulting oscillation theorem for equation (1.7) then follows from Theorem 3.5.

(iv) The results in [11], [12], [16] and [7, Sections 8.3–8.4] concern general separated boundary conditions, which include the Dirichlet boundary conditions considered in this paper as a special case. However, the results in this paper can be extended to such separated (and even jointly varying) endpoints by a standard method, which is based on adding two isolated points to the interval $[a,b]$, see [3], [5].

(v) The main assumptions on the Hamiltonian $\mathcal{H}$ stated in (2.1) below and the absence of strict normality imply that the spectral parameter λ in the eigenvalue problem (E) can be restricted to a compact interval only. More precisely, if in the eigenvalue problem (E) we have $\lambda \in [\alpha,\beta]$, then we extend the Hamiltonian $\mathcal{H}$ to be constant in λ on the complement of $[\alpha,\beta]$, that is, for every $t \in [a,b]$ we set $\mathcal{H}(t,\lambda) \equiv \mathcal{H}(t,\alpha)$ for $\lambda < \alpha$ and $\mathcal{H}(t,\lambda) \equiv \mathcal{H}(t,\beta)$ for $\lambda > \beta$. Then the resulting eigenvalue problem with $\lambda \in \mathbb{R}$ has no finite eigenvalues in $(-\infty,\alpha) \cup (\beta,\infty)$, so that the main results of this paper count the finite eigenvalues in the given interval $[\alpha,\beta]$.

The paper is organized as follows. In the next section we recall some basic properties of solutions of linear Hamiltonian systems. In Section 3, we define finite eigenvalues of (E) and their algebraic multiplicities and state and prove the main results of this paper – the local and global oscillation theorems for system (H_λ). In Section 4, we present some applications of the main results, including a sufficient condition for the existence of finite eigenvalues (Theorem 4.6) and a characterization of the smallest finite eigenvalue (Theorem 4.8). These results are formulated in terms of positivity of the associated quadratic functional. Finally, in Section 5, we define finite eigenfunctions of (E) corresponding to finite eigenvalues and their geometric multiplicities, and we prove that the geometric multiplicity of each finite eigenvalue coincides with its algebraic multiplicity (Theorem 5.5).

2 Basic properties of linear Hamiltonian systems

We begin this section by stating the precise hypotheses about the coefficients of system (H_λ). In addition to (1.3), we assume that the given $2n \times 2n$ real symmetric Hamiltonian matrix $\mathcal{H}(\cdot,\cdot)$ satisfies the following. There exist a partition $a = \tau_0 < \cdots < \tau_m = b$ of $[a,b]$ and a partition $-\infty < \cdots < \lambda_k < \lambda_{k+1} < \cdots < \infty$ of $\mathbb{R}$ with no finite accumulation point such that

$$\left.\begin{array}{l} \bullet\ \mathcal{H} \text{ is continuous on } [\tau_i,\tau_{i+1}] \times \mathbb{R} \text{ for every } i = 0,\dots,m-1, \\ \bullet\ \mathcal{H}_\lambda \text{ is continuous on } [\tau_i,\tau_{i+1}] \times [\lambda_k,\lambda_{k+1}] \text{ for every } i = 0,\dots,m-1 \text{ and } k \in \mathbb{Z}, \\ \bullet\ \mathcal{H}_\lambda(t,\lambda) \geq 0 \text{ for every } (t,\lambda) \in [a,b] \times \mathbb{R}. \end{array}\right\} \tag{2.1}$$

Note that, as it is usual for piecewise continuity, the continuity of $\mathcal{H}_\lambda$ on the rectangles $[\tau_i,\tau_{i+1}] \times [\lambda_k,\lambda_{k+1}]$ means that $\mathcal{H}_\lambda$ is continuous on the open rectangles and that it may be defined on each boundary

$\partial[\tau_i, \tau_{i+1}] \times [\lambda_k, \lambda_{k+1}]$ separately in such a way that it is continuous on the closed rectangles. Thus, $\mathcal{H}_\lambda$ may have jumps at the boundaries of those rectangles.

Solutions of system (H_λ) depend on λ. Vector-valued solutions (as a $2n$-vector) and matrix-valued solutions (as a $2n \times n$-matrix) of system (H_λ) will be denoted by small and capital letters, respectively, typically by

$$z(\cdot,\lambda) = \begin{pmatrix} x(\cdot,\lambda) \\ u(\cdot,\lambda) \end{pmatrix}, \quad Z(\cdot,\lambda) = \begin{pmatrix} X(\cdot,\lambda) \\ U(\cdot,\lambda) \end{pmatrix},$$

which will also be abbreviated as $z(\cdot,\lambda) = (x(\cdot,\lambda), u(\cdot,\lambda))$ and $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$. The system ($\mathrm{H}_\lambda$) can then be written as

$$z' = \mathcal{J}\,\mathcal{H}(t,\lambda)\,z, \quad t \in [a,b], \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the $n \times n$-identity matrix. For a given $\lambda \in \mathbb{R}$, a solution $Z(\cdot,\lambda)$ of (H_λ) is called a *conjoined basis* if $Z^T(t,\lambda)\,\mathcal{J} Z(t,\lambda)$ is symmetric and $\operatorname{rank} Z(t,\lambda) = n$ at some (and hence at all) point $t \in [a,b]$. If $\tilde{Z}(\cdot,\lambda)$ and $Z(\cdot,\lambda)$ are two solutions of (H_λ), then their *Wronskian* is defined as the matrix $\tilde{Z}^T(\cdot,\lambda)\,\mathcal{J} Z(\cdot,\lambda)$, which is constant on $[a,b]$. If $\tilde{Z}^T(\cdot,\lambda)\,\mathcal{J} Z(\cdot,\lambda) \equiv I$, then $\tilde{Z}(\cdot,\lambda)$ and $Z(\cdot,\lambda)$ are called *normalized conjoined bases* of system (H_λ). It is well known that the oscillation theory works well only for conjoined bases of system (H_λ), see e.g., [1], [7], [13], [15], [17]. Also, the standard theory of continuous dependence of solutions of (in this case linear) differential equations on parameters allows to differentiate the solutions $Z(\cdot,\lambda)$ with respect to λ and have the formula $(Z')_\lambda = (Z_\lambda)'$, where the prime denotes the derivative with respect to t and the subscript λ the derivative with respect to λ. Finally, if $\tilde{Z}(\cdot,\lambda)$ and $Z(\cdot,\lambda)$ are any normalized conjoined bases of (H_λ), then

$$\Phi(\cdot,\lambda) := \big(\tilde{Z}(\cdot,\lambda)\ \ Z(\cdot,\lambda)\big) = \begin{pmatrix} \tilde{X}(\cdot,\lambda) & X(\cdot,\lambda) \\ \tilde{U}(\cdot,\lambda) & U(\cdot,\lambda) \end{pmatrix} \tag{2.2}$$

is a fundamental matrix of system (H_λ). Moreover, it is known that $\Phi(\cdot,\lambda)$ is symplectic, i.e.,

$$\Phi^T(\cdot,\lambda)\,\mathcal{J}\,\Phi(\cdot,\lambda) = \mathcal{J}, \qquad \text{so that} \qquad \Phi^{-1}(\cdot,\lambda) = -\mathcal{J}\,\Phi^T(\cdot,\lambda)\,\mathcal{J}. \tag{2.3}$$

These identities imply, among others, that the matrix $X(\cdot,\lambda)\,\tilde{X}^T(\cdot,\lambda)$ is symmetric on $[a,b]$.

Next we derive monotonicity formulas involving normalized conjoined bases $\tilde{Z}(\cdot,\lambda) = \big(\tilde{X}(\cdot,\lambda), \tilde{U}(\cdot,\lambda)\big)$ and $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ of system (H_λ), whose initial conditions do not depend on λ, i.e.,

$$\tilde{X}(a,\lambda) \equiv \tilde{X}(a), \quad \tilde{U}(a,\lambda) \equiv \tilde{U}(a), \tag{2.4}$$

$$X(a,\lambda) \equiv X(a), \quad U(a,\lambda) \equiv U(a). \tag{2.5}$$

Lemma 2.1 *Assume* (2.1). *Let* $\tilde{Z}(\cdot,\lambda) = \big(\tilde{X}(\cdot,\lambda), \tilde{U}(\cdot,\lambda)\big)$ *and* $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ *be normalized conjoined bases of system* (H_λ) *satisfying* (2.4) *and* (2.5). *If* $t \in [a,b]$ *and* $\lambda_0 \in \mathbb{R}$ *are such that* $\tilde{X}(t,\lambda_0)$ *is invertible, then*

$$\big(\tilde{X}^{-1}X\big)_\lambda(t,\lambda) = \int_a^t \zeta^T(t,\tau,\lambda)\,\mathcal{H}_\lambda(\tau,\lambda)\,\zeta(t,\tau,\lambda)\,\mathrm{d}\tau, \tag{2.6}$$

$$\big(\tilde{U}\tilde{X}^{-1}\big)_\lambda(t,\lambda) = -\int_a^t \xi^T(t,\tau,\lambda)\,\mathcal{H}_\lambda(\tau,\lambda)\,\xi(t,\tau,\lambda)\,\mathrm{d}\tau, \tag{2.7}$$

for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ *with some* $\varepsilon > 0$, *where*

$$\zeta(t,\tau,\lambda) := Z(\tau,\lambda) - \tilde{Z}(\tau,\lambda)\,\big(\tilde{X}^{-1}X\big)(t,\lambda), \tag{2.8}$$

$$\xi(t,\tau,\lambda) := \tilde{Z}(\tau,\lambda)\,\tilde{X}^{-1}(t,\lambda). \tag{2.9}$$

Consequently, the symmetric matrix function $\big(\tilde{X}^{-1}X\big)(t,\cdot)$ *is nondecreasing on* $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, *and the symmetric matrix function* $\big(\tilde{U}\tilde{X}^{-1}\big)(t,\cdot)$ *is nonincreasing on* $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.

P r o o f. By continuity, there exists $\varepsilon > 0$ such that $\tilde{X}(t,\lambda)$ is invertible for all $\lambda \in \Lambda := (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. On $[a,b] \times \Lambda$, we define the fundamental matrix $\Phi(\cdot,\cdot)$ of (H_λ) as in (2.2). Then, by (2.4) and (2.5), we have $\Phi_\lambda(a,\cdot) \equiv 0$ on Λ. Differentiating the equation $\Phi'(t,\lambda) = \mathcal{J}\,\mathcal{H}(t,\lambda)\,\Phi(t,\lambda)$ with respect to λ and using the variation of constants formula yields

$$\Phi_\lambda(t,\lambda) = \Phi(t,\lambda) \int_a^t \Phi^{-1}(\tau,\lambda)\,\mathcal{J}\,\mathcal{H}_\lambda(\tau,\lambda)\,\Phi(\tau,\lambda)\,\mathrm{d}\tau.$$

By using equation (2.3) for the inverse of a symplectic matrix and by $\mathcal{J}^2 = -I$, we arrive at

$$\Phi^T(t,\lambda)\,\mathcal{J}\,\Phi_\lambda(t,\lambda) = -\int_a^t \Phi^T(\tau,\lambda)\,\mathcal{H}_\lambda(\tau,\lambda)\,\Phi(\tau,\lambda)\,\mathrm{d}\tau. \tag{2.10}$$

Now the left-hand side of (2.10) can be expressed as (suppressing the arguments t and λ)

$$\Phi^T \mathcal{J}\,\Phi_\lambda = Y^T\,(VY^{-1})_\lambda\,Y, \quad \text{where} \quad Y := \begin{pmatrix} 0 & I \\ \tilde{X} & X \end{pmatrix}, \quad V := \begin{pmatrix} I & 0 \\ \tilde{U} & U \end{pmatrix},$$

and where

$$Y^{-1} = \begin{pmatrix} -\tilde{X}^{-1}X & \tilde{X}^{-1} \\ I & 0 \end{pmatrix}, \quad VY^{-1} = \begin{pmatrix} -\tilde{X}^{-1}X & \tilde{X}^{-1} \\ (\tilde{X}^T)^{-1} & \tilde{U}\tilde{X}^{-1} \end{pmatrix}. \tag{2.11}$$

This yields through equation (2.10) that

$$\big(VY^{-1}\big)_\lambda(t,\lambda) = -\big[Y^T(t,\lambda)\big]^{-1} \left(\int_a^t \Phi^T(\tau,\lambda)\,\mathcal{H}_\lambda(\tau,\lambda)\,\Phi(\tau,\lambda)\,\mathrm{d}\tau\right) Y^{-1}(t,\lambda). \tag{2.12}$$

Combining the second equation in (2.11) with formula (2.12) then yields the statement. □

Our next result shows that the monotonicity assumption (2.1) implies that the kernel of $X(t,\cdot)$ is piecewise constant on $\mathbb{R}$ for every $t \in [a,b]$ and every conjoined basis $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ of (H_λ) with initial conditions independent of λ. This is a generalization of [9, Lemma A.1] to system (H_λ).

Lemma 2.2 *Assume* (2.1). *Let* $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ *be a conjoined basis of* (H_λ) *satisfying* (2.5). *Then for each* $t \in [a,b]$, *the set* $\operatorname{Ker} X(t,\cdot)$ *is piecewise constant on* $\mathbb{R}$. *That is, for every* $\lambda_0 \in \mathbb{R}$ *there exists* $\delta > 0$ *such that*

$$\operatorname{Ker} X(t,\lambda) \equiv \operatorname{Ker} X\big(t,\lambda_0^-\big) \subseteq \operatorname{Ker} X(t,\lambda_0) \quad \textit{for all} \quad \lambda \in (\lambda_0 - \delta, \lambda_0), \tag{2.13}$$

$$\operatorname{Ker} X(t,\lambda) \equiv \operatorname{Ker} X\big(t,\lambda_0^+\big) \subseteq \operatorname{Ker} X(t,\lambda_0) \quad \textit{for all} \quad \lambda \in (\lambda_0, \lambda_0 + \delta). \tag{2.14}$$

P r o o f. Let $\lambda_0 \in \mathbb{R}$ be fixed. By [7, Proposition 4.1.1], there exists another conjoined basis $\tilde{Z}(\cdot,\lambda_0) = \big(\tilde{X}(\cdot,\lambda_0), \tilde{U}(\cdot,\lambda_0)\big)$ of system (H_{λ_0}) such that $\tilde{Z}(\cdot,\lambda_0)$ and $Z(\cdot,\lambda_0)$ are normalized conjoined bases of (H_{λ_0}), $\tilde{X}(t,\lambda_0)$ is invertible, and $\big(\tilde{X}^{-1}X\big)(t,\lambda_0) \geq 0$. For every $\lambda \in \mathbb{R}$ we choose $\tilde{Z}(\cdot,\lambda) = \big(\tilde{X}(\cdot,\lambda), \tilde{U}(\cdot,\lambda)\big)$ to be the conjoined basis of system (H_λ) given by the initial condition $\tilde{Z}(a,\lambda) \equiv \tilde{Z}(a,\lambda_0)$, so that this initial condition does also not depend on λ. Consequently, $\tilde{Z}(\cdot,\lambda)$ and $Z(\cdot,\lambda)$ are normalized conjoined bases of (H_λ) for any $\lambda \in \mathbb{R}$, and they satisfy (2.4) and (2.5). Since (2.1) is assumed, the matrix-valued function $\big(\tilde{X}^{-1}X\big)(t,\cdot)$ is nondecreasing on the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for some $\varepsilon > 0$. Let $c \in \operatorname{Ker} X(t,\lambda)$ for some $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$. Then the monotonicity of $\big(\tilde{X}^{-1}X\big)(t,\cdot)$ implies

$$0 \leq c^T\big(\tilde{X}^{-1}X\big)(t,\lambda_0)\,c \leq c^T\big(\tilde{X}^{-1}X\big)(t,\nu)\,c \leq c^T\big(\tilde{X}^{-1}X\big)(t,\lambda)\,c = 0 \quad \text{for all} \quad \nu \in (\lambda_0, \lambda].$$

Hence, $c^T\big(\tilde{X}^{-1}X\big)(t,\nu)\,c = 0$ and so $c \in \operatorname{Ker} X(t,\nu)$ for every $\nu \in (\lambda_0, \lambda]$. Therefore, we conclude that $\operatorname{Ker} X(t,\lambda) \subseteq \operatorname{Ker} X(t,\nu)$ for all $\lambda, \nu \in (\lambda_0, \lambda_0 + \varepsilon)$ with $\nu \leq \lambda$. This means that the set $\operatorname{Ker} X(t,\cdot)$ is nonincreasing on $(\lambda_0, \lambda_0 + \varepsilon)$, and by the continuity of $X(t,\cdot)$, it is nonincreasing on $[\lambda_0, \lambda_0 + \varepsilon)$, implying that formula (2.14) is satisfied for some sufficiently small $\delta \in (0,\varepsilon)$. For (2.13) we proceed in the same way except that we choose at the beginning $\tilde{Z}(\cdot,\lambda_0) = \big(\tilde{X}(\cdot,\lambda_0), \tilde{U}(\cdot,\lambda_0)\big)$ with $\big(\tilde{X}^{-1}X\big)(t,\lambda_0) \leq 0$. □

3 Finite eigenvalues and oscillation theorems

The result of Lemma 2.2 shows that for any conjoined basis $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ of (H$_\lambda$), the number $\operatorname{rank} X(t,\cdot)$ is constant on some left neighborhood of λ_0 and, possibly with a different value, on some right neighborhood of λ_0 for any fixed $\lambda_0 \in \mathbb{R}$. This allows to define correctly the notion of a finite eigenvalue as follows.

Let $\hat{Z}(\cdot,\lambda) = \big(\hat{X}(\cdot,\lambda), \hat{U}(\cdot,\lambda)\big)$ be the *principal solution* of system (H$_\lambda$), that is,

$$\hat{X}(a,\lambda) \equiv 0, \quad \hat{U}(a,\lambda) \equiv I.$$

Definition 3.1 (Finite eigenvalue) Under (2.1), a number $\lambda_0 \in \mathbb{R}$ is called a *finite eigenvalue* of (E), provided

$$\theta(\lambda_0) := \operatorname{rank} \hat{X}\big(b, \lambda_0^-\big) - \operatorname{rank} \hat{X}(b,\lambda_0) \geq 1. \tag{3.1}$$

In this case the number $\theta(\lambda_0)$ is called the *algebraic multiplicity* of λ_0.

Since $\operatorname{rank} M = n - \operatorname{def} M$ for any $n \times n$-matrix M, it follows that

$$\theta(\lambda_0) = \operatorname{def} \hat{X}(b,\lambda_0) - \operatorname{def} \hat{X}\big(b,\lambda_0^-\big).$$

Moreover, by formula (2.13) of Lemma 2.2, the number $\theta(\lambda_0)$ is always nonnegative.

Remark 3.2 (i) When $\hat{X}(b,\cdot)$ is invertible except at isolated values of λ, which is the case for "controllable" systems (H$_\lambda$) in [7, Sections 7.1–7.2], then $\operatorname{def} \hat{X}\big(b,\lambda_0^-\big) = 0$ for every $\lambda_0 \in \mathbb{R}$. Therefore, in this case a finite eigenvalue of (E) reduces to the classical eigenvalue, which is determined by the condition $\det \hat{X}(b,\lambda_0) = 0$, and its (algebraic) multiplicity is equal to $\theta(\lambda_0) = \operatorname{def} \hat{X}(b,\lambda_0)$.

(ii) Under (1.6), the above finite eigenvalue notion reduces to [9, Definition 2.6], see also [19, Proposition 3.3], because in this case the function $\hat{X}(b,\cdot)$ is entire in λ and so

$$\operatorname{rank} \hat{X}(b,\lambda_0^-) = \operatorname{rank} \hat{X}(b,\lambda_0^+) = \max_{\nu\in\mathbb{R}} \operatorname{rank} \hat{X}(b,\nu) \quad \text{for all} \quad \lambda_0 \in \mathbb{R}.$$

(iii) The term "finite" eigenvalue is motivated by the corresponding notion in the discrete time theory, see [2, Definition 2] and in particular [2, Remark 1(iv)–(v)], where this notion is connected to matrix pencils.

The following is a simple consequence of Lemma 2.2 and Definition 3.1.

Corollary 3.3 *Under* (2.1)*, the finite eigenvalues of* (E) *are isolated.*

Let be given a conjoined basis $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ of (H$_\lambda$) satisfying (2.5), that is, its initial conditions do not depend on λ. Under (1.3) and counting the multiplicities, we let

$$n_1(\lambda) := \text{ the number of proper focal points of } Z(\cdot,\lambda) \text{ in } (a,b], \text{ see (1.5).}$$

The following two theorems are the main results of this paper. The proof of Theorem 3.4 is postponed until the end of this section.

Theorem 3.4 (Local oscillation theorem) *Assume* (1.3) *and* (2.1)*. Suppose that* $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ *is a conjoined basis of* (H$_\lambda$) *satisfying* (2.5)*. Then for all* $\lambda \in \mathbb{R}$*, we have*

$$n_1(\lambda^+) = n_1(\lambda) < \infty, \quad n_1(\lambda^+) - n_1(\lambda^-) = \operatorname{rank} X(b,\lambda^-) - \operatorname{rank} X(b,\lambda) \geq 0. \tag{3.2}$$

Hence, the function n_1 *is nondecreasing on* $\mathbb{R}$*, the limit*

$$m := \lim_{\lambda\to-\infty} n_1(\lambda)$$

exists with $m \in \mathbb{N}_0$*, so that for a suitable* $\lambda_0 \in \mathbb{R}$*,* $\lambda_0 < 0$*, we have*

$$n_1(\lambda) \equiv m \quad \textit{and} \quad \operatorname{rank} X(b,\lambda^-) - \operatorname{rank} X(b,\lambda) \equiv 0 \quad \textit{for all} \quad \lambda \leq \lambda_0. \tag{3.3}$$

In contrast to Theorem 3.4, the following result applies to the principal solution $\hat{Z}(\cdot,\lambda) = \big(\hat{X}(\cdot,\lambda), \hat{U}(\cdot,\lambda)\big)$ of (H_λ). This theorem is a generalization of [19, Corollary 1.7] and [9, Theorem B.5] to systems (H_λ) depending nonlinearly on λ, and at the same time a generalization of [7, Theorem 4.2.3] to possibly abnormal systems (H_λ). Hence, for this moment, we let

$$n_1(\lambda) := \text{the number of proper focal points of } \hat{Z}(\cdot,\lambda) \text{ in } (a,b],$$
$$n_2(\lambda) := \text{the number of finite eigenvalues of (E) which are less than or equal to } \lambda.$$

Theorem 3.5 (Global oscillation theorem) *Assume* (1.3) *and* (2.1). *Then for all* $\lambda \in \mathbb{R}$,

$$n_2(\lambda^+) = n_2(\lambda) < \infty, \tag{3.4}$$
$$n_2(\lambda^+) - n_2(\lambda^-) = n_1(\lambda^+) - n_1(\lambda^-) \geq 0, \tag{3.5}$$

and there exists $m \in \mathbb{N}_0$ *such that*

$$n_1(\lambda) = n_2(\lambda) + m \quad \textit{for all} \quad \lambda \in \mathbb{R}. \tag{3.6}$$

Moreover, for a suitable $\lambda_0 \in \mathbb{R}$, $\lambda_0 < 0$, *we have*

$$n_2(\lambda) \equiv 0 \quad \textit{and} \quad n_1(\lambda) \equiv m \quad \textit{for all} \quad \lambda \leq \lambda_0, \tag{3.7}$$

so that the finite eigenvalues of (E) *are bounded from below.*

Proof. We apply the local oscillation theorem (Theorem 3.4) to the principal solution of (H_λ). By the definition of $n_2(\lambda)$ as the number of finite eigenvalues of (E) which are less than or equal to λ, it follows that the function n_2 is right-continuous on $\mathbb{R}$, i.e., condition (3.4) is established. Furthermore, by the definition of $n_2(\lambda)$ and $\theta(\lambda)$, we have

$$\begin{aligned} n_2(\lambda^+) - n_2(\lambda^-) &\overset{(3.4)}{=} n_2(\lambda) - n_2(\lambda^-) = \theta(\lambda) \overset{(3.1)}{=} \operatorname{rank} \hat{X}(b,\lambda^-) - \operatorname{rank} \hat{X}(b,\lambda) \\ &\overset{(3.2)}{=} n_1(\lambda^+) - n_1(\lambda^-) \geq 0, \end{aligned}$$

showing (3.5). Moreover, from (3.3), we have $m \in \mathbb{N}_0$ and $\lambda_0 \in \mathbb{R}$, $\lambda_0 < 0$, such that

$$n_1(\lambda) \equiv m \quad \text{and} \quad \theta(\lambda) \equiv 0 \quad \text{for all} \quad \lambda \leq \lambda_0, \tag{3.8}$$

which yields by (3.5) that $n_2(\lambda) \equiv 0$ for all $\lambda \leq \lambda_0$. Therefore, there are no finite eigenvalues which are less than or equal to λ_0 and hence, the finite eigenvalues of (E) are bounded from below. Finally, since (3.5) implies that the jumps in n_1 and n_2 are always the same and since these functions are right-continuous on $\mathbb{R}$ by (3.2) and (3.4), it follows that n_1 and n_2 differ on $\mathbb{R}$ by the constant m from (3.8). The proof is complete. □

The rest of this section is devoted to developing the tools for the proof of Theorem 3.4. First we recall an auxiliary statement about the number of proper focal points of a conjoined basis of one system (H_{λ_0}).

Lemma 3.6 *Let* $\lambda_0 \in \mathbb{R}$ *be fixed, and assume that* $\mathcal{H}(\cdot,\lambda_0)$ *is piecewise continuous and* $B(\cdot,\lambda_0) \geq 0$ *on* $[a,b]$. *Let* $a \leq \alpha < \beta \leq b$, *and suppose that* $\tilde{Z}(\cdot,\lambda_0) = \big(\tilde{X}(\cdot,\lambda_0), \tilde{U}(\cdot,\lambda_0)\big)$ *and* $Z(\cdot,\lambda_0) = (X(\cdot,\lambda_0), U(\cdot,\lambda_0))$ *are normalized conjoined bases of* (H_{λ_0}) *such that* $\tilde{X}(t,\lambda_0)$ *is invertible for all* $t \in [\alpha,\beta]$. *Let* $\tilde{m}$ *denote the number of proper focal points of* $Z(\cdot,\lambda_0)$ *in* $(\alpha,\beta]$. *Then* $0 \leq \tilde{m} \leq n$, *and*

$$\tilde{m} = \operatorname{ind}\big(\tilde{X}^{-1}X\big)(\alpha,\lambda_0) - \operatorname{ind}\big(\tilde{X}^{-1}X\big)(\beta,\lambda_0). \tag{3.9}$$

Proof. We refer to [9, Lemma B.2]. □

The above result can be regarded as the monotonicity of conjoined bases with respect to t, because the calculation $\big(\tilde{X}^{-1}X\big)'(\cdot,\lambda_0) = \big(\tilde{X}^{-1}B\tilde{X}^{T-1}\big)(\cdot,\lambda_0)$ and assumption (1.3) imply that the function $\big(\tilde{X}^{-1}X\big)(\cdot,\lambda_0)$ is nondecreasing on $[\alpha,\beta]$.

Lemma 3.7 *Assume* (1.3) *and* (2.1). *Let* $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ *be a conjoined basis of* (H_λ) *satisfying* (2.5). *For* $a \le \alpha < \beta \le b$, *we denote by* $m(\lambda)$ *the number of proper focal points of* $Z(\cdot,\lambda)$ *in* $(\alpha,\beta]$. *Then for all* $\lambda \in \mathbb{R}$, *we have*

$$m(\lambda^+) = m(\lambda) < \infty, \tag{3.10}$$

$$m(\lambda^+) - m(\lambda^-) = \operatorname{rank} X(\beta,\lambda^-) - \operatorname{rank} X(\beta,\lambda) - \operatorname{rank} X(\alpha,\lambda^-) + \operatorname{rank} X(\alpha,\lambda). \tag{3.11}$$

Proof. First we fix $\lambda_0 \in \mathbb{R}$. By [7, Proposition 4.1.1] and compactness of $[a,b]$, there are a partition $\alpha = \tau_0 < \tau_1 < \cdots < \tau_{k+1} = \beta$ of $[\alpha,\beta]$ and conjoined bases $\tilde Z_j(\cdot,\lambda_0) = \big(\tilde X_j(\cdot,\lambda_0), \tilde U_j(\cdot,\lambda_0)\big)$ of (H_{λ_0}) such that for every $j \in \{0,\dots,k\}$ the conjoined bases $\tilde Z_j(\cdot,\lambda_0)$ and $Z(\cdot,\lambda_0)$ are normalized, and $\tilde X_j(t,\lambda_0)$ is invertible on $[\tau_j,\tau_{j+1}]$. For any $\lambda \in \mathbb{R}$ and $j \in \{0,\dots,k\}$ we let $\tilde Z_j(\cdot,\lambda) = \big(\tilde X_j(\cdot,\lambda), \tilde U_j(\cdot,\lambda)\big)$ be the conjoined basis of (H_λ) given by the initial conditions

$$\tilde X_j(a,\lambda) \equiv \tilde X_j(a,\lambda_0), \quad \tilde U_j(a,\lambda) \equiv \tilde U_j(a,\lambda_0). \tag{3.12}$$

Then $\tilde Z_j(\cdot,\lambda)$ and $Z(\cdot,\lambda)$ are normalized conjoined bases of (H_λ) for every $\lambda \in \mathbb{R}$. Moreover, the continuity of $\tilde X_j(\cdot,\cdot)$ and the compactness of $[\tau_j,\tau_{j+1}]$ yield $\varepsilon_j > 0$ such that

$$\tilde X_j(t,\lambda) \quad \text{is invertible on} \quad [\tau_j,\tau_{j+1}] \times [\lambda_0 - \varepsilon_j, \lambda_0 + \varepsilon_j].$$

Put $\varepsilon := \min\{\varepsilon_j,\ j = 0,\dots,k\}$. Then the above construction shows that, for each $j \in \{0,\dots,k\}$, $\tilde Z_j(\cdot,\lambda)$ and $Z(\cdot,\lambda)$ are normalized conjoined bases of (H_λ) for all $\lambda \in \mathbb{R}$ such that, by (3.12) and (2.5), their initial conditions do not depend on λ and

$$\tilde X_j(t,\lambda) \quad \text{is invertible on} \quad [\tau_j,\tau_{j+1}] \times [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon].$$

Let $m(j,\lambda)$ denote the number of proper focal points of $Z(\cdot,\lambda)$ in $(\tau_j,\tau_{j+1}]$, so that

$$m(\lambda) = \sum_{j=0}^{k} m(j,\lambda) \quad \text{for} \quad \lambda \in \mathbb{R} \tag{3.13}$$

is the number of proper focal points of $Z(\cdot,\lambda)$ in $(\alpha,\beta]$. Then, by Lemma 3.6 and assumption (1.3), we have

$$0 \le m(j,\lambda) = \operatorname{ind}\big(\tilde X_j^{-1} X\big)(\tau_j,\lambda) - \operatorname{ind}\big(\tilde X_j^{-1} X\big)(\tau_{j+1},\lambda) \le n \tag{3.14}$$

for every $j = 0,\dots,k$ and $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. Hence, $m(\lambda) \le (k+1)\,n < \infty$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. In particular, $m(\lambda_0)$ is finite. Next, fix an index $j = 0,\dots,k$ and a point $t_0 \in [\tau_j,\tau_{j+1}]$, and put $Q(\lambda) := \big(\tilde X_j^{-1} X\big)(t_0,\lambda)$ for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. Then by (2.6) of Lemma 2.1, the symmetric matrix-valued function Q is nondecreasing and continuous on the interval $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$. Therefore, the eigenvalues $\mu_1(\lambda) \le \cdots \le \mu_n(\lambda)$ of $Q(\lambda)$ are nondecreasing and continuous on $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, too. This implies that $\operatorname{ind} Q$ is nonincreasing and, by the continuity of the eigenvalues $\mu_1,\dots,\mu_n$,

$$\operatorname{ind} Q(\lambda_0) = \operatorname{ind} Q\big(\lambda_0^+\big) \quad \text{and} \quad \operatorname{ind} Q\big(\lambda_0^-\big) = \operatorname{ind} Q(\lambda_0) + \operatorname{def} Q(\lambda_0) - \operatorname{def} Q\big(\lambda_0^-\big), \tag{3.15}$$

because the number of negative eigenvalues of $Q(\lambda_0)$ is the same as the number of negative eigenvalues of $Q(\lambda_0^+)$, and the number of negative eigenvalues of $Q\big(\lambda_0^-\big)$ is equal to the number of negative eigenvalues of $Q(\lambda_0)$ plus the number of those eigenvalues which are zero at λ_0 but were negative before λ_0. Moreover, since $\operatorname{Ker} Q(\cdot) = \operatorname{Ker} X(t_0,\cdot)$ is piecewise constant on $\mathbb{R}$ by Lemma 2.2, it follows from (3.15) that

$$\begin{aligned} \operatorname{ind} Q\big(\lambda_0^-\big) &= \operatorname{ind} Q(\lambda_0) + \operatorname{def} X(t_0,\lambda_0) - \operatorname{def} X\big(t_0,\lambda_0^-\big) \\ &= \operatorname{ind} Q(\lambda_0) + \operatorname{rank} X\big(t_0,\lambda_0^-\big) - \operatorname{rank} X(t_0,\lambda_0). \end{aligned} \tag{3.16}$$

Therefore, from the first formula of (3.15) and (3.16) with $t_0 := \tau_j$, we get

$$\operatorname{ind}\big(\tilde X_j^{-1} X\big)(\tau_j,\lambda_0) = \operatorname{ind}\big(\tilde X_j^{-1} X\big)\big(\tau_j,\lambda_0^+\big), \tag{3.17}$$

$$\operatorname{ind}\big(\tilde X_j^{-1} X\big)\big(\tau_j,\lambda_0^-\big) = \operatorname{ind}\big(\tilde X_j^{-1} X\big)(\tau_j,\lambda_0) + \operatorname{rank} X\big(\tau_j,\lambda_0^-\big) - \operatorname{rank} X(\tau_j,\lambda_0), \tag{3.18}$$

while from the first formula of (3.15) and (3.16) with $t_0 := \tau_{j+1}$, we get

$$\operatorname{ind}\big(\tilde X_j^{-1}X\big)(\tau_{j+1},\lambda_0) = \operatorname{ind}\big(\tilde X_j^{-1}X\big)\big(\tau_{j+1},\lambda_0^+\big), \tag{3.19}$$

$$\operatorname{ind}\big(\tilde X_j^{-1}X\big)\big(\tau_{j+1},\lambda_0^-\big) = \operatorname{ind}\big(\tilde X_j^{-1}X\big)(\tau_{j+1},\lambda_0) + \operatorname{rank} X\big(\tau_{j+1},\lambda_0^-\big) - \operatorname{rank} X(\tau_{j+1},\lambda_0). \tag{3.20}$$

Thus, we obtain

$$\begin{aligned} m(j,\lambda_0^+) &\overset{(3.14)}{=} \operatorname{ind}\big(\tilde X_j^{-1}X\big)\big(\tau_j,\lambda_0^+\big) - \operatorname{ind}\big(\tilde X_j^{-1}X\big)\big(\tau_{j+1},\lambda_0^+\big) \\ &\overset{(3.17),\,(3.19)}{=} \operatorname{ind}\big(\tilde X_j^{-1}X\big)(\tau_j,\lambda_0) - \operatorname{ind}\big(\tilde X_j^{-1}X\big)(\tau_{j+1},\lambda_0) \overset{(3.14)}{=} m(j,\lambda_0), \end{aligned} \tag{3.21}$$

and similarly

$$\begin{aligned} m(j,\lambda_0^+) - m(j,\lambda_0^-) &\overset{(3.21),\,(3.14)}{=} \operatorname{ind}\big(\tilde X_j^{-1}X\big)(\tau_j,\lambda_0) - \operatorname{ind}\big(\tilde X_j^{-1}X\big)(\tau_{j+1},\lambda_0) \\ &\qquad - \operatorname{ind}\big(\tilde X_j^{-1}X\big)\big(\tau_j,\lambda_0^-\big) + \operatorname{ind}\big(\tilde X_j^{-1}X\big)\big(\tau_{j+1},\lambda_0^-\big) \\ &\overset{(3.18),\,(3.20)}{=} \operatorname{rank} X\big(\tau_{j+1},\lambda_0^-\big) - \operatorname{rank} X(\tau_{j+1},\lambda_0) \\ &\qquad - \operatorname{rank} X\big(\tau_j,\lambda_0^-\big) + \operatorname{rank} X(\tau_j,\lambda_0). \end{aligned} \tag{3.22}$$

Consequently, we get from equations (3.13) and (3.21) that $m\big(\lambda_0^+\big) = m(\lambda_0)$, proving equality (3.10), and by telescope summation from equations (3.13) and (3.22) that formula (3.11) holds. The proof is complete. □

The proof of the local oscillation theorem now follows.

P r o o f o f T h e o r e m 3.4. We apply Lemma 3.7 with $\alpha := a$ and $\beta := b$. Then the number $m(\lambda)$ in Lemma 3.7 equals to $n_1(\lambda)$, so that condition (3.10) is equivalent to the first equation of (3.2). Since the conjoined basis $Z(\cdot,\lambda) = (X(\cdot,\lambda), U(\cdot,\lambda))$ satisfies (2.5), we have

$$\operatorname{rank} X(\alpha,\lambda^-) = \operatorname{rank} X(a,\lambda^-) \equiv \operatorname{rank} X(a,\lambda) \quad \text{for all } \lambda \in \mathbb{R}.$$

Hence, formula (3.11) yields

$$n_1(\lambda^+) - n_1(\lambda^-) = m(\lambda^+) - m(\lambda^-) = \operatorname{rank} X(b,\lambda^-) - \operatorname{rank} X(b,\lambda) \geq 0,$$

showing that the second equation of (3.2) holds. The two conditions in (3.2) then imply that the function n_1 is nondecreasing on $\mathbb{R}$. Since n_1 takes only the values in $\mathbb{N}_0$, it follows that the limit $m := \lim_{\lambda\to-\infty} n_1(\lambda)$ exists with $m \in \mathbb{N}_0$. Therefore, $n_1(\lambda) \equiv m$ for λ sufficiently negative, i.e., for all $\lambda \leq \lambda_0$ for some $\lambda_0 < 0$. This in turn implies by the second equation of (3.2) that $n_1(\lambda^+) - n_1(\lambda^-) = m - m = 0$ for all $\lambda \leq \lambda_0$, so that $\operatorname{rank} X(b,\lambda^-) - \operatorname{rank} X(b,\lambda) \equiv 0$ for all $\lambda \leq \lambda_0$. The proof is complete. □

4 Applications

In this section, we present several applications of the global oscillation theorem (Theorem 3.5). For a given $\lambda \in \mathbb{R}$, consider the quadratic functional

$$\mathcal{F}(z,\lambda) := \int_a^b \big\{x^T(t)\,C(t,\lambda)\,x(t) + u^T(t)\,B(t,\lambda)\,u(t)\big\}\,\mathrm{d}t$$

over *admissible* pairs $z = (x,u)$, i.e., $x \in \mathrm{C}_\mathrm{p}^1$ and $B(\cdot,\lambda)\,u \in \mathrm{C}_\mathrm{p}$ satisfy

$$x'(t) = A(t,\lambda)\,x(t) + B(t,\lambda)\,u(t) \quad \text{for all} \quad t \in [a,b],$$

and the Dirichlet boundary conditions (1.1), i.e., $x(a) = 0 = x(b)$. The following definiteness property of the functional $\mathcal{F}(\cdot,\lambda)$ will play an important role in the statements below. We say that $\mathcal{F}(\cdot,\lambda)$ is *positive definite* and write $\mathcal{F}(\cdot,\lambda) > 0$ if $\mathcal{F}(z,\lambda) > 0$ for every admissible $z = (x,u)$ satisfying $x(a) = 0 = x(b)$ and $x \neq 0$ on $[a,b]$. This means that $\mathcal{F}(\cdot,\lambda)$ is not positive definite if there exists an admissible $z = (x,u)$ with $x(a) = 0 = x(b)$ and $x \neq 0$ such that $\mathcal{F}(z,\lambda) \leq 0$. The latter case will be abbreviated by $\mathcal{F}(\cdot,\lambda) \not> 0$.

Proposition 4.1 (Positive definiteness) *Let $\lambda_0 \in \mathbb{R}$ be fixed. The functional $\mathcal{F}(\cdot,\lambda_0)$ is positive definite if and only if $B(\cdot,\lambda_0) \geq 0$ on $[a,b]$ and the principal solution $\hat{Z}(\cdot,\lambda_0) = \big(\hat{X}(\cdot,\lambda_0), \hat{U}(\cdot,\lambda_0)\big)$ of (H_{λ_0}) has no proper focal point in $(a,b]$.*

P r o o f. We refer to [8, Theorem 1]. □

Other conditions equivalent to the positivity and nonnegativity of $\mathcal{F}(\cdot,\lambda)$ are well-known in the literature, such as the solvability of Riccati matrix equations and inequalities or perturbations of the boundary conditions of admissible pairs, see e.g., [4], [6], [14]. Next, we introduce another useful tool from the theory of quadratic functionals.

Theorem 4.2 (Comparison theorem) *Assume* (1.3) *and* (2.1). *If $\mathcal{F}(\cdot,\lambda_0)$ is positive definite for some $\lambda_0 \in \mathbb{R}$, then $\mathcal{F}(\cdot,\lambda)$ is positive definite for all $\lambda \leq \lambda_0$.*

P r o o f. Let $\lambda < \lambda_0$ be fixed. Then our assumption (2.1) yields that

$$\mathcal{H}(t,\lambda_0) \geq \mathcal{H}(t,\lambda) \quad \text{for all} \quad t \in [a,b]. \tag{4.1}$$

Let $z = (x,u)$ be admissible for $\mathcal{F}(\cdot,\lambda)$ with $x(a) = 0 = x(b)$ and $x \neq 0$. Then, by $B(\cdot,\lambda) \geq 0$ on $[a,b]$ and the first part of [7, Proposition 2.1.3], for $x_0 := x$ there exists a corresponding control u_0 on $[a,b]$ such that $z_0 := (x_0,u_0)$ is admissible for the functional $\mathcal{F}(\cdot,\lambda_0)$. Hence, our assumption yields $\mathcal{F}(z_0,\lambda_0) > 0$. On the other hand, by the second part of [7, Proposition 2.1.3], we have

$$x^T(t)\,C(t,\lambda)\,x(t) + u^T(t)\,B(t,\lambda)\,u(t) \geq x_0^T(t)\,C(t,\lambda_0)\,x_0(t) + u_0^T(t)\,B(t,\lambda_0)\,u_0(t)$$

for all $t \in [a,b]$. Consequently, $\mathcal{F}(z,\lambda) \geq \mathcal{F}(z_0,\lambda_0) > 0$, and the positivity of $\mathcal{F}(\cdot,\lambda)$ is established. □

Remark 4.3 The proof of Theorem 4.2 also shows that the set of admissible functions $x(\cdot,\lambda)$ is nondecreasing in λ. More precisely, if we define for $\lambda \in \mathbb{R}$ the set

$$\mathcal{A}(\lambda) := \big\{x \in \mathrm{C}_\mathrm{p}^1 : \text{ there exists } u \text{ with } B(\cdot,\lambda)\,u \in \mathrm{C}_\mathrm{p} \text{ and } z = (x,u) \text{ is admissible for } \mathcal{F}(\cdot,\lambda)\big\},$$

then $\mathcal{A}(\lambda) \subseteq \mathcal{A}(\lambda_0)$ for all $\lambda, \lambda_0 \in \mathbb{R}$ with $\lambda \leq \lambda_0$. In [9], [19], the admissible set $\mathcal{A}(\lambda) \equiv \mathcal{A}$ is constant on $\mathbb{R}$, because in these references the coefficients $A(\cdot,\lambda) \equiv A(\cdot)$ and $B(\cdot,\lambda) \equiv B(\cdot)$ are independent of λ.

The results of Proposition 4.1 and Theorem 4.2 show that under condition (1.3), the equality $n_1(\lambda) \equiv 0$ for all $\lambda \leq \lambda_0$ is equivalent to the positivity of $\mathcal{F}(\cdot,\lambda_0)$, and also to the positivity of $\mathcal{F}(\cdot,\lambda)$ for all $\lambda \leq \lambda_0$. Therefore, the global oscillation theorem yields the following generalization of [9, Theorem 2.9] for our system (H_λ).

Theorem 4.4 (Oscillation theorem) *Assume* (1.3) *and* (2.1). *Then*

$$n_1(\lambda) = n_2(\lambda) \quad \textit{for all} \quad \lambda \in \mathbb{R} \tag{4.2}$$

if and only if there exists $\lambda_0 \in \mathbb{R}$, $\lambda_0 < 0$, such that $\mathcal{F}(\cdot,\lambda_0) > 0$.

P r o o f. If $\mathcal{F}(\cdot,\lambda_0) > 0$, then $\mathcal{F}(\cdot,\lambda) > 0$ for all $\lambda \leq \lambda_0$, by Theorem 4.2. In turn, Proposition 4.1 yields that $n_1(\lambda) \equiv 0$ for all $\lambda \leq \lambda_0$, i.e., $m = 0$ in (3.7). Hence, formula (3.6) of Theorem 3.5 shows that condition (4.2) is satisfied. Conversely, if (4.2) holds, then $m = 0$ in (3.6), so that $n_1(\lambda) \equiv m = 0$ for all $\lambda \leq \lambda_0$, by (3.7). This shows through Proposition 4.1 that $\mathcal{F}(\cdot,\lambda) > 0$ for all $\lambda \leq \lambda_0$, which completes the proof. □

Now we consider the question of existence of finite eigenvalues. Of course, one can easily deduce from Theorem 4.4 that under (1.3) and (2.1), the eigenvalue problem (E) has no finite eigenvalues if and only if $\mathcal{F}(\cdot,\lambda) > 0$ for all $\lambda \in \mathbb{R}$.

Theorem 4.5 (Existence of finite eigenvalues: necessary condition) *Assume* (1.3) *and* (2.1). *If* (E) *has a finite eigenvalue, then there exist $\lambda_0, \lambda_1 \in \mathbb{R}$ with $\lambda_0 < \lambda_1$ and $m \in \mathbb{N}_0$ such that $n_1(\lambda) \equiv m$ for all $\lambda \leq \lambda_0$ and $\mathcal{F}(\cdot,\lambda_1) \not> 0$.*

P r o o f. The existence of a finite eigenvalue of (E) means that there exists $\lambda_1 \in \mathbb{R}$ such that $n_2(\lambda_1) \geq 1$. By Theorem 3.5, we know that equality (3.6) is satisfied for some $m \in \mathbb{N}_0$ and $n_1(\lambda) \equiv m$ for all $\lambda \leq \lambda_0$ for some

$\lambda_0 < 0$. Without loss of generality we may take $\lambda_0 < \lambda_1$, so that the first part of this theorem is proven. Next, it follows from (3.6) with $\lambda = \lambda_1$ that $n_1(\lambda_1) = n_2(\lambda_1) + m \geq n_2(\lambda_1) \geq 1$, showing that the principal solution of (H_{λ_1}) has at least one proper focal point in $(a, b]$. By Proposition 4.1, we then get $\mathcal{F}(\cdot, \lambda_1) \not> 0$. □

If $m = 0$, then the conditions in Theorem 4.5 turn out to be sufficient for the existence of a finite eigenvalue of (E).

Theorem 4.6 (Existence of finite eigenvalues: sufficient condition) *Assume* (1.3) *and* (2.1). *If there exist* $\lambda_0, \lambda_1 \in \mathbb{R}$ *with* $\lambda_0 < \lambda_1$ *such that* $\mathcal{F}(\cdot, \lambda_0) > 0$ *and* $\mathcal{F}(\cdot, \lambda_1) \not> 0$, *then the eigenvalue problem* (E) *has at least one finite eigenvalue.*

Proof. The positivity of $\mathcal{F}(\cdot, \lambda_0)$ implies by Theorem 4.4 that $n_1(\lambda) = n_2(\lambda)$ for all $\lambda \in \mathbb{R}$. If we assume that there is no finite eigenvalue of (E) at all, i.e., if $n_2(\lambda) \equiv 0$ for all $\lambda \in \mathbb{R}$, then $n_1(\lambda) \equiv 0$ for all $\lambda \in \mathbb{R}$ as well. In particular, $n_1(\lambda_1) = 0$. This means by Proposition 4.1 that $\mathcal{F}(\cdot, \lambda_1) > 0$, which contradicts our assumption. Therefore, under the given conditions, the eigenvalue problem (E) must have at least one finite eigenvalue. □

Remark 4.7 The results of Theorems 4.5 and 4.6 are new even for the special case of system (H_λ) in (1.6). Moreover, Theorem 4.6 is a generalization of [7, Theorem 7.6.1] with differentiable Hamiltonian and Dirichlet boundary conditions to possibly abnormal system (H_λ).

Now we characterize the smallest finite eigenvalue of (E) in terms of the positivity of the quadratic functional $\mathcal{F}(\cdot, \lambda)$. Note that in comparison with [7, Theorem 7.6.2], which used a normality assumption, we now need to work with the positive definite functional $\mathcal{F}(\cdot, \lambda)$ rather than with the nonnegative one.

Theorem 4.8 (Characterization of the smallest finite eigenvalue) *Assume* (1.3) *and* (2.1), *and let there exist* $\lambda_0, \lambda_1 \in \mathbb{R}$ *with* $\lambda_0 < \lambda_1$ *such that* $\mathcal{F}(\cdot, \lambda_0) > 0$ *and* $\mathcal{F}(\cdot, \lambda_1) \not> 0$. *Then the eigenvalue problem* (E) *possesses a smallest finite eigenvalue* $\lambda_{\min}$, *which is characterized by any of the following conditions*:

$$\lambda_{\min} = \sup \mathcal{P}, \qquad \mathcal{P} := \{\lambda \in \mathbb{R},\ \mathcal{F}(\cdot, \lambda) > 0\}, \tag{4.3}$$

$$\lambda_{\min} = \min \mathcal{N}, \quad \mathcal{N} := \{\lambda \in \mathbb{R},\ \mathcal{F}(\cdot, \lambda) \not> 0\}. \tag{4.4}$$

Moreover, the algebraic multiplicity of $\lambda_{\min}$ *is then equal to* $n_1(\lambda_{\min})$, *i.e., to the number of proper focal points of the principal solution of* $(\mathrm{H}_{\lambda_{\min}})$ *in* $(a, b]$.

Proof. By Theorem 4.6, the eigenvalue problem (E) has at least one finite eigenvalue. Since $\lambda_0 \in \mathcal{P}$, we know that the set $\mathcal{P}$ is nonempty, which by Theorem 4.2 implies that $(-\infty, \lambda_0) \subseteq \mathcal{P}$ and $n_1(\lambda) \equiv 0$ for all $\lambda \leq \lambda_0$. In addition, $\lambda_1 \notin \mathcal{P}$, so that $\mathcal{P}$ is bounded from above and therefore $\mathcal{P} = (-\infty, \omega)$, where $\omega = \sup \mathcal{P}$ exists. It follows that $n_1(\omega) \geq 1$, because by Theorem 3.4, the function n_1 is right-continuous on $\mathbb{R}$. We will show that $\lambda_{\min} = \omega$ is the smallest finite eigenvalue of (E). From Theorem 4.4 we see that $n_1(\lambda) = n_2(\lambda)$ for all $\lambda \in \mathbb{R}$. Hence, $n_2(\lambda) \equiv 0$ for all $\lambda < \omega$ and $n_2(\omega) = n_1(\omega) \geq 1$, proving that ω is the smallest finite eigenvalue of (E) with algebraic multiplicity $n_1(\omega)$. Concerning (4.4), we note that the set $\mathcal{N}$ is nonempty, because $\lambda_1 \in \mathcal{N}$, and as in the first part the interval $(-\infty, \lambda_0]$ is not contained in $\mathcal{N}$. Therefore, $\mathcal{N}$ is bounded from below. Let $\nu \in \mathcal{N}$, i.e., $n_1(\nu) \geq 1$. From the equality $n_1(\lambda) = n_2(\lambda)$ for all $\lambda \in \mathbb{R}$, it follows that $n_2(\nu) \geq 1$. Since we know from Theorem 3.5 and Corollary 3.3 that the function n_2 is right-continuous on $\mathbb{R}$ and the finite eigenvalues are isolated and bounded from below, it follows that $\kappa := \min\{\nu \in \mathbb{R},\ n_2(\nu) \geq 1\} = \min \mathcal{N}$ exists and satisfies $\lambda_0 < \kappa$. Furthermore, by the definition of κ we have $n_2(\lambda) \equiv 0$ for all $\lambda < \kappa$ and $n_2(\kappa) \geq 1$. This yields that $\lambda_{\min} = \kappa$ is the smallest finite eigenvalue of (E) with multiplicity $n_2(\kappa) = n_1(\kappa)$. □

For completeness we provide the following Sturmian comparison theorem for system (H_λ).

Theorem 4.9 (Sturmian comparison theorem) *Assume* (1.3) *and* (2.1). *If, for some* $\lambda_0 \in \mathbb{R}$, *the principal solution of* (H_{λ_0}) *has* m *proper focal points in* $(a, b]$, *then any conjoined basis of* (H_λ) *with* $\lambda \leq \lambda_0$ *has at most* $m + n$ *proper focal points in* $(a, b]$, *and any conjoined basis of* (H_λ) *with* $\lambda \geq \lambda_0$ *has at least* m *proper focal points in* $(a, b]$.

Proof. This statement is a consequence of [17, Theorems 1.2 and 1.3], in which we observe that inequality (4.1) holds. □

5 Geometric properties of finite eigenvalues

In this section, we show that the algebraic definition of finite eigenvalues (Definition 3.1) can be replaced by the corresponding geometric notion of Definition 5.3 below. The following concept of a "degenerate solution " of system (H_λ) plays the key role in these considerations.

Definition 5.1 (Degenerate solution) Let $\lambda_0 \in \mathbb{R}$ be given. A solution $z(\cdot,\lambda_0)$ of (H_{λ_0}) is said to be *degenerate at* λ_0 (or it is a *degenerate solution at* λ_0), if there exists $\delta > 0$ such that for all $\lambda \in (\lambda_0 - \delta, \lambda_0]$, the solution $z(\cdot,\lambda)$ of system (H_λ) given by the initial condition $z(a,\lambda) = z(a,\lambda_0)$ satisfies

$$\mathcal{H}_\lambda(\cdot,\lambda)\, z(\cdot,\lambda) = 0 \quad \text{on} \quad [a,b]. \tag{5.1}$$

In the opposite case, we say that the solution $z(\cdot,\lambda_0)$ is *nondegenerate at* λ_0.

Unlike in the classical controllable case such as in [7] or in the possibly abnormal case with linear dependence on λ as in [9], [19], rather than with a single solution $z(\cdot,\lambda_0)$ of (H_{λ_0}), a degenerate solution in Definition 5.1 should be identified with the *family of solutions* $\{z(\cdot,\lambda)\}_{\lambda\in(\lambda_0-\delta,\lambda_0]}$ of systems (H_λ) for $\lambda \in (\lambda_0 - \delta, \lambda_0]$, which all start with the same value $z(a,\lambda) \equiv z(a,\lambda_0)$. However, as we shall see in the proof of Theorem 5.5 below, condition (5.1) will imply the independence of the solutions $z(\cdot,\lambda)$ on λ for $\lambda \in (\lambda_0 - \delta, \lambda_0]$, and so it is equally correct to call this family of solutions as *the degenerate solution* $z(\cdot,\lambda_0)$.

Remark 5.2 (i) Condition (5.1) can be written as $\|z(\cdot,\lambda)\|_\lambda = 0$ when we use the semi-norm defined by

$$\|z(\cdot,\lambda)\|_\lambda^2 := \int_a^b z^T(t,\lambda)\, \mathcal{H}_\lambda(t,\lambda)\, z(t,\lambda)\, \mathrm{d}t.$$

(ii) Under (1.6), i.e., for the linear dependence on λ, a degenerate solution at λ_0 is a solution $z(\cdot,\lambda_0)$ of (H_{λ_0}) satisfying

$$W z(\cdot,\lambda_0) = 0 \quad \text{on} \quad [a,b]. \tag{5.2}$$

This condition was used in [9], [19] in this context. Equality (5.2) is indeed equivalent to (5.1) where $\lambda \in (\lambda_0 - \delta, \lambda_0]$, because in the linear case any solution $z(\cdot,\lambda_0)$ of (H_{λ_0}) satisfying (5.2) is at the same time a solution of system (H_λ) for any $\lambda \in \mathbb{R}$. Therefore, the degeneracy condition (5.1) is a local property for the nonlinear dependence on λ, but it is in fact a global property for the linear dependence on λ.

(iii) Condition (5.1) can be considered in Definition 5.1 for λ in the open interval $(\lambda_0 - \delta, \lambda_0)$ only, because the continuity of $\mathcal{H}_\lambda(t,\cdot)$ will then ensure that (5.1) holds also at $\lambda = \lambda_0$.

Let us introduce the linear spaces of all solutions of (E) with $\lambda = \lambda_0$ and of all degenerate solutions at λ_0

$$\begin{aligned} \mathcal{E}(\lambda_0) &:= \big\{ z(\cdot,\lambda_0) \in \mathrm{C}^1_\mathrm{p} : \ \ z(\cdot,\lambda) \text{ is a solution of } (H_{\lambda_0}) \text{ with } (1.1) \big\}, \\ \mathcal{W}(\lambda_0) &:= \big\{ z(\cdot,\lambda_0) \in \mathcal{E}(\lambda_0) : \ \ z(\cdot,\lambda) \text{ is a degenerate solution at } \lambda_0 \big\}. \end{aligned}$$

We are now ready to define finite eigenfunctions of the eigenvalue problem (E) corresponding to the finite eigenvalue λ_0.

Definition 5.3 (Finite eigenfunction) Under (2.1), every nondegenerate solution $z(\cdot,\lambda_0)$ at λ_0 of (E) with $\lambda = \lambda_0$ is called a *finite eigenfunction* corresponding to the finite eigenvalue λ_0, and then the number

$$\omega(\lambda_0) := \dim \mathcal{E}(\lambda_0) - \dim \mathcal{W}(\lambda_0) \tag{5.3}$$

is called the *geometric multiplicity* of λ_0.

Remark 5.4 (i) If the system (H_λ) is strictly normal according to [7, Definition 4.1.2], i.e., under (2.1) if the system (1.4) at $\lambda = \lambda_0$ has only the trivial solution $z(\cdot,\lambda_0) = 0$, then this trivial solution $z(\cdot,\lambda_0)$ is the only degenerate solution at λ_0 of (H_λ). Hence, in this case $\dim \mathcal{W}(\lambda_0) = 0$, and finite eigenfunctions corresponding to the finite eigenvalue λ_0 coincide with nontrivial solutions of (E) with $\lambda = \lambda_0$. Moreover, the geometric multiplicity of λ_0 is then equal to $\dim \mathcal{E}(\lambda_0) = \operatorname{def} \hat{X}(b,\lambda_0)$.

(ii) The statement of part (i) applies especially for the case when $\mathcal{H}_\lambda(t,\lambda) > 0$, i.e., when the Hamiltonian $\mathcal{H}(t,\lambda)$ is strictly monotone in λ.

In the following result, we show that system (H_λ) has the property of self-adjoint differential systems according to its algebraic and geometric multiplicities of finite eigenvalues. This result generalizes [9, Theorem A.2] to the case when (H_λ) depends nonlinearly on λ.

Theorem 5.5 (Geometric characterization of finite eigenvalues) *Assume* (2.1). *A number* $\lambda_0 \in \mathbb{R}$ *is a finite eigenvalue of* (E) *with algebraic multiplicity* $\theta(\lambda_0) \geq 1$ *defined in* (3.1) *if and only if there exists a corresponding finite eigenfunction* $z(\cdot, \lambda_0)$. *In this case, the geometric multiplicity of* λ_0 *defined in* (5.3) *is equal to its algebraic multiplicity, i.e.,*

$$\omega(\lambda_0) = \theta(\lambda_0).$$

Proof. Let λ_0 be a finite eigenvalue of (E) with algebraic multiplicity $\theta(\lambda_0) \geq 1$. By the uniqueness of solutions of system (H_{λ_0}), a function $z(\cdot, \lambda_0) = (x(\cdot, \lambda_0), u(\cdot, \lambda_0))$ solves (H_{λ_0}) with $x(a, \lambda_0) = 0$ if and only if for some vector $c \in \mathbb{R}^n$ we have

$$x(\cdot, \lambda_0) = \hat{X}(\cdot, \lambda_0)\, c \quad \text{and} \quad u(\cdot, \lambda_0) = \hat{U}(\cdot, \lambda_0)\, c \quad \text{on} \quad [a, b].$$

And in this case the condition $x(b, \lambda_0) = 0$ means that $\hat{X}(b, \lambda_0)\, c = 0$, i.e., $c \in \operatorname{Ker} \hat{X}(b, \lambda_0)$. We now analyze the situation of $c \in \operatorname{Ker} \hat{X}(b, \lambda_0^-)$.

First suppose that $z(\cdot, \lambda_0) \in \mathcal{W}(\lambda_0)$, i.e., there is $\delta > 0$ such that condition (5.1) holds for every $\lambda \in (\lambda_0 - \delta, \lambda_0]$ and for the solution $z(\cdot, \lambda)$ of (H_λ) satisfying the initial condition $z(a, \lambda) = z(a, \lambda_0)$. Since $\operatorname{Ker} \hat{X}(b, \cdot)$ is piecewise constant on $\mathbb{R}$ by Lemma 2.2, the number δ may be chosen so that in addition to (5.1) for all $\lambda \in (\lambda_0 - \delta, \lambda_0]$ we also have $\operatorname{Ker} \hat{X}(b, \lambda) \equiv \operatorname{Ker} \hat{X}(b, \lambda_0^-)$ for all $\lambda \in (\lambda_0 - \delta, \lambda_0)$. As the identity $[z'(t, \lambda)]_\lambda = [z_\lambda(t, \lambda)]'$ holds, it follows from the differentiation of system (H_λ) with respect to λ that for every $t \in [a, b]$ and every $\lambda \in (\lambda_0 - \delta, \lambda_0]$ we have

$$[z_\lambda(t, \lambda)]' = \mathcal{H}(t, \lambda)\, z_\lambda(t, \lambda) + \mathcal{H}_\lambda(t, \lambda)\, z(t, \lambda) \overset{(5.1)}{=} \mathcal{H}(t, \lambda)\, z_\lambda(t, \lambda). \tag{5.4}$$

Moreover, since $z(\cdot, \lambda)$ is a multiple of the principal solution $\hat{Z}(\cdot, \lambda)$, the initial condition of $z(\cdot, \lambda)$ does not depend on λ, i.e., $z_\lambda(a, \lambda) = 0$. Hence, by the uniqueness of solutions for system (H_λ), we get from (5.4) with $z_\lambda(a, \lambda) = 0$ that $z_\lambda(\cdot, \lambda) \equiv 0$ on $[a, b]$ for every $\lambda \in (\lambda_0 - \delta, \lambda_0]$. Therefore, we can emphasize the very important observation that

$$\text{the functions } z(\cdot, \lambda) \text{ are independent of } \lambda \text{ for } \lambda \in (\lambda_0 - \delta, \lambda_0]. \tag{5.5}$$

This yields that $z(\cdot, \lambda) \equiv z(\cdot, \lambda_0)$ on $[a, b]$ for every $\lambda \in (\lambda_0 - \delta, \lambda_0]$. Fix any $\lambda_1 \in (\lambda_0 - \delta, \lambda_0)$. Then condition (5.5) shows that the function $z(\cdot, \lambda_0)$ solves the system (H_{λ_1}). Moreover, since $z(a, \lambda_1) = z(a, \lambda_0)$ as we know that the initial conditions of $z(a, \lambda)$ do not depend on λ, the uniqueness of solutions of system (H_{λ_1}) implies the equality

$$\begin{pmatrix} \hat{X}(\cdot, \lambda_0)\, c \\ \hat{U}(\cdot, \lambda_0)\, c \end{pmatrix} = \begin{pmatrix} x(\cdot, \lambda_0) \\ u(\cdot, \lambda_0) \end{pmatrix} = \begin{pmatrix} \hat{X}(\cdot, \lambda_1)\, c \\ \hat{U}(\cdot, \lambda_1)\, c \end{pmatrix} \quad \text{on} \quad [a, b].$$

The endpoint condition $x(b, \lambda_0) = 0$ then yields $\hat{X}(b, \lambda_1)\, c = 0$, i.e., $c \in \operatorname{Ker} \hat{X}(b, \lambda_1)$. Since λ_1 was arbitrary in the interval $(\lambda_0 - \delta, \lambda_0)$ and since $\operatorname{Ker} \hat{X}(b, \cdot)$ is constant on $(\lambda_0 - \delta, \lambda_0)$, it follows that $c \in \operatorname{Ker} \hat{X}(b, \lambda_0^-)$.

Conversely, let $c \in \operatorname{Ker} \hat{X}\big(b, \lambda_0^-\big)$ be given. Then $\hat{X}(b, \lambda)\, c = 0$ for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ for some $\varepsilon > 0$. Let $\tilde{Z}(\cdot, \lambda) = \big(\tilde{X}(\cdot, \lambda), \tilde{U}(\cdot, \lambda)\big)$ be a conjoined basis of (H_λ) such that (2.4) holds, $\tilde{Z}(\cdot, \lambda)$ and $\hat{Z}(\cdot, \lambda)$ are normalized conjoined bases of (H_λ), and such that $\tilde{X}(b, \lambda_0)$ is invertible (similarly as in the proof of Lemma 2.2). Then for some $\delta \in (0, \varepsilon)$, the matrix $\tilde{X}(b, \lambda)$ is invertible for all $\lambda \in (\lambda_0 - \delta, \lambda_0]$. For these values of λ, we define the function $z(\cdot, \lambda) = (x(\cdot, \lambda), u(\cdot, \lambda))$ by $z(\cdot, \lambda) := \zeta(b, \cdot, \lambda)\, c$ on $[a, b]$, where $\zeta(t, \tau, \lambda)$ is given by (2.8). By using $c \in \operatorname{Ker} \hat{X}\big(b, \lambda_0^-\big)$ we then see that $z(\cdot, \lambda) := \hat{X}(\cdot, \lambda)\, c$ on $[a, b]$ for every $\lambda \in (\lambda_0 - \delta, \lambda_0]$. It follows that

$$c^T\, \big(\tilde{X}^{-1} \hat{X}\big)(b, \lambda)\, c = 0 \quad \text{for all} \quad \lambda \in (\lambda_0 - \delta, \lambda_0]. \tag{5.6}$$

By taking the derivative of equation (5.6) with respect to λ at any $\lambda_1 \in (\lambda_0 - \delta, \lambda_0)$ and the left-hand derivative of (5.6) at $\lambda_1 = \lambda_0$ we get from Lemma 2.1

$$0 = c^T\, (\tilde{X}^{-1}\hat{X})_\lambda(b, \lambda_1)\, c \overset{(2.6)}{=} \int_a^b z^T(\tau, \lambda_1)\, \mathcal{H}_\lambda(\tau, \lambda_1)\, z(\tau, \lambda_1)\, \mathrm{d}\tau.$$

Assumption (2.1) then implies that $\mathcal{H}_\lambda(\cdot, \lambda_1)\, z(\cdot, \lambda_1) = 0$ on $[a, b]$.

Thus, we have shown that a nondegenerate solution $z(\cdot, \lambda_0) = (x(\cdot, \lambda_0), u(\cdot, \lambda_0))$ at λ_0 of system (H_{λ_0}) satisfying the boundary conditions $x(a, \lambda_0) = 0 = x(b, \lambda_0)$, i.e., by Definition 5.3 a finite eigenfunction of (E), is of the form $z(\cdot, \lambda_0) = \hat{Z}(\cdot, \lambda_0)\, c$ for some $c \in \operatorname{Ker} \hat{X}(b, \lambda_0) \setminus \operatorname{Ker} \hat{X}\big(b, \lambda_0^-\big)$. It follows that the geometric multiplicity $\omega(\lambda_0)$ of λ_0 is then equal to the number $\operatorname{def} \hat{X}(b, \lambda_0) - \operatorname{def} \hat{X}\big(b, \lambda_0^-\big) = \theta(\lambda_0)$, i.e., to the algebraic multiplicity of λ_0. The proof is now complete. □

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