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Oscillation of fourth-order delay dynamic equations

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Abstract This paper is concerned with oscillatory behavior of a class of fourth-order delay dynamic equations on a time scale. In the general time scales case, four oscillation theorems are presented that can be used in cases where known results fail to apply. The results obtained can be applied to an equation which is referred to as Swift-Hohenberg delay equation on a time scale. These criteria improve a number of related contributions to the subject. Some illustrative examples are provided.

Keywords Swift-Hohenberg equation, oscillation, delay dynamic equation, fourth-order, time scale

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1 Introduction

This article is concerned with the problem of oscillation of a fourth-order delay dynamic equation

$$(c(t)(b(t)(a(t)x^{\Delta}(t))^{\Delta})^{\Delta})^{\Delta} + p(t)x(\tau(t)) = 0 \tag{1.1}$$

on a time scale $\mathbb{T}$ unbounded above. Here a, b, c, and p are positive real-valued rd-continuous functions defined on $\mathbb{T}$, $\tau \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leqslant t$, and $\tau(t) \to \infty$ as $t \to \infty$.

Since we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ with $t_0 \in \mathbb{T}$. By a solution to (1.1), we mean a nontrivial real-valued function $x \in \mathrm{C}^1_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$, $T_x \in [t_0, \infty)_{\mathbb{T}}$ which satisfies (1.1) for $t \in [T_x, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution x to (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. (1.1) is called oscillatory if all its solutions are oscillatory.

The theory of time scales, was introduced by Hilger [25] in his PhD thesis in order to unify continuous and discrete analysis. The study of dynamic equations on time scales is a new area of mathematics, and work in this topic is rapidly growing. During the past few years, there has been increasing interest in obtaining sufficient conditions for oscillation and nonoscillation of solutions to various classes of dynamic

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equations on time scales, we refer the reader to [1–3, 5–7, 10–14, 17–20, 24, 29, 30, 32–35, 37] and the references cited therein. Thereinto, Agarwal and Bohner [2] studied oscillation of a first-order dynamic equation

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0.$$

Agarwal et al. [3], Erbe et al. [14], and Şahiner [34] considered oscillatory behavior of

$$x^{\Delta^2}(t) + p(t)f(x(\tau(t))) = 0.$$

Regarding oscillation of third-order dynamic equations on time scales, Erbe et al. [13] studied a third-order dynamic equation

$$x^{\Delta^3}(t) + p(t)x(t) = 0.$$

Hassan [24] and Li et al. [29] considered a third-order nonlinear delay dynamic equation

$$(a(t)\{[r(t)x^{\Delta}(t)]^{\Delta}\}^{\gamma})^{\Delta} + f(t, x(\tau(t))) = 0.$$

In the following, we present some background details that motivate the investigation of (1.1). The fourth-order equations have some applications in the real world; see [1,4,8,9,15,16,18–23,26–28,30,35–39]. Agarwal et al. [1] studied a fourth-order dynamic equation

$$(p(t)(x^{\Delta^2}(t))^{\alpha})^{\Delta^2} + q(t)f(x^{\sigma}(t)) = 0, \tag{1.2}$$

where α is the quotient of two odd positive integers, p and q are real-valued positive and rd-continuous functions on a time scale $\mathbb{T}$. As a special case when $\alpha = 1$ and $f(u) = u$, the authors obtained the following result.

Theorem 1.1 (See [1, Theorems 3.1–3.3]). *Assume $\int_{t_0}^{\infty} p^{-1}(t)\Delta t = \infty$, $\alpha = 1$, $f(u) = u$, and there exists a strictly increasing function $\tau \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ such that $\tau(t) > t$ and $\tau \circ \sigma = \sigma \circ \tau$. Suppose further that there exists a positive function $r \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[r(s)q(\tau(s))\tau^{\Delta}(s) - \frac{(r^{\Delta}(s))^2}{4r(s)h(s,t_0;p)} \right] \Delta s = \infty,$$

for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$, where

$$h(t,t_0;p) := \min\left\{ \int_{t_0}^{t} \frac{s-t_0}{p(s)}\Delta s, \int_{t}^{\beta(t)} \frac{\beta(s)-s}{p(s)}\Delta s \right\},$$

$\beta : \mathbb{T} \to \mathbb{T}$ is an increasing function satisfying $\beta(t) > t$. Then (1.2) *is oscillatory.*

Monotone and oscillatory behavior of solutions to a fourth-order dynamic equation

$$(a(t)(x^{\Delta\Delta}(t))^{\alpha})^{\Delta\Delta} + p(t)(x^{\sigma}(t))^{\beta} = 0$$

with the property that

$$\frac{x(t)}{\int_{t_0}^{t}\int_{t_0}^{s} a^{-1/\alpha}(\tau)\Delta\tau\Delta s} \to 0 \quad \text{as} \quad t \to \infty$$

were established by Grace et al. [19]. Grace et al. [20] examined a fourth-order dynamic equation

$$x^{\Delta^4}(t) + p(t)x^{\gamma}(t) = 0, \tag{1.3}$$

where p is a real-valued positive and rd-continuous function on a time scale $\mathbb{T}$. As a special case when $\gamma = 1$, the authors established the following result.

Theorem 1.2 (See [20, Theorem 2.5]). *Assume $\gamma = 1$. If*

$$\limsup_{t\to\infty}\{h_3(t,t_0)Q(t)\} > 1$$

and

$$\limsup_{t\to\infty}\{h_1(t,t_0)Q^*(t)\} > 1,$$

where

$$h_1(t,s) := t - s, \quad h_2(t,s) := \int_s^t (\tau - s)\Delta\tau, \quad h_3(t,s) := \int_s^t h_2(\tau,s)\Delta\tau \quad \text{with} \ \ t,s \in \mathbb{T},$$

and

$$Q(t) := \int_t^\infty p(s)\Delta s, \quad Q^*(t) := \int_t^\infty \int_s^\infty Q(\tau)\Delta\tau\Delta s \quad \text{for} \ \ t \in [t_0,\infty)_{\mathbb{T}},$$

then (1.3) *is oscillatory.*

Grace et al. [18] studied a fourth-order dynamic equation

$$x^{\Delta^4}(t) + p(t)x^\gamma(\sigma(t)) = 0, \tag{1.4}$$

where p is a real-valued positive and rd-continuous function on a time scale $\mathbb{T}$. As a special case when $\gamma = 1$, the authors presented the following criterion.

Theorem 1.3 (See [18, Theorem 5]). *Assume that there exists an rd-continuous function $g : \mathbb{T} \to \mathbb{T}$ such that $g(t) < t$, $g(t)$ is nondecreasing for $t \in [t_0,\infty)_{\mathbb{T}}$, and $\lim_{t\to\infty} g(t) = \infty$. Let $\phi(t) := t - g(t)$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and assume that*

$$\int_{t_0}^\infty \phi(t)h_2(g(t),t_0)p(t)\Delta t = \infty.$$

Suppose that there exists a positive nondecreasing function $\eta \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that

$$\limsup_{t\to\infty}\int_{t_0}^t \left[\eta(s)Q(s) - \frac{\eta^\Delta(s)}{s}\right]\Delta s = \infty,$$

or

$$\limsup_{t\to\infty}\int_{t_0}^t \left[\eta(s)Q(s) - \frac{(\eta^\Delta(s))^2}{4\eta(s)}\right]\Delta s = \infty,$$

where

$$Q(t) := \int_t^\infty \int_s^\infty p(u)\Delta u\Delta s.$$

Moreover, assume that there exists a positive function $\xi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that

$$\limsup_{t\to\infty}\int_{t_1}^t \left[\frac{\phi(s)}{\sigma(s)}h_2(g(s),t_0)\xi^\sigma(s)p(s) - k\frac{\xi^\Delta(s)}{s}\right]\Delta s = \infty,$$

for every constant $k > 0$ and for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then (1.4) *with $\gamma = 1$ is oscillatory.*

Li et al. [30] considered oscillation of unbounded solutions to a fourth-order delay dynamic equation

$$(r(t)x^{\Delta^3}(t))^\Delta + p(t)x(\tau(t)) = 0.$$

Thandapani et al. [35] studied a fourth-order dynamic equation

$$(b(t)(a(t)x^\Delta(t))^\Delta)^{\Delta^2} + q(t)x^\sigma(t-\delta) = 0, \tag{1.5}$$

where $\delta \geqslant 0$, a, b, and q are real-valued positive and rd-continuous functions on a time scale $\mathbb{T}$. They obtained the following oscillation criterion.

Theorem 1.4 (See [35, Theorem 3.5]). *Assume that* $\int_{t_0}^{\infty} a^{-1}(t)\Delta t = \int_{t_0}^{\infty} b^{-1}(t)\Delta t = \infty$. *Suppose also that there exists a positive function* $\alpha \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\alpha(s)q(s) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha(s)R^{\Delta}(s-\delta,t_0)}\right]\Delta s = \infty,$$

for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$, *where*

$$R(t,t_0) := \int_{t_0}^{t} \frac{1}{a(s)}\left(\int_{t_0}^{s} \frac{u-t_0}{b(u)}\Delta u\right)\Delta s.$$

Then (1.5) *is oscillatory.*

Zhang et al. [37] investigated a fourth-order nonlinear dynamic equation

$$(r(t)x^{\Delta^3}(t))^{\Delta} + p(t)f(x(\sigma(t))) = 0$$

and obtained some sufficient conditions for oscillation of the studied equation.

The questions regarding oscillatory properties of equations

$$x^{(4)}(t) + \frac{q_0}{t^4}x(t) = 0 \tag{1.6}$$

and

$$x^{(4)}(t) + \frac{q_0}{t^4}x\left(\frac{t}{2}\right) = 0 \tag{1.7}$$

have been studied in [4, 15, 16, 21–23, 26–28, 36, 38, 39]. Fite [15] obtained that $q_0 > 0$ is not sufficient to ensure oscillation of solutions to (1.6). Let $t_0 = 1$, $r(t) = (t+1)^3$, $\tau(t) = t+1$, $\beta(t) = 2t$, and $\alpha(t) = t^3$. Applications of Theorem 1.1 or Theorem 1.4 imply that (1.6) is oscillatory when $q_0 > 9/2$. Using Theorem 1.2 or [4, Theorem 2.2] in (1.6), we have that (1.6) is oscillatory if $q_0 > 18$. One can easily see that Theorem 1.3 cannot be applied in (1.6) due to the arbitrariness in the choice of k. Grace and Lalli [21–23] proved that condition

$$q_0 > 13824$$

guarantees oscillation of fourth-order delay differential equation (1.7). Zafer [36] obtained that condition

$$q_0 > \frac{1536}{\mathrm{e}\ln 2}$$

ensures oscillation of (1.7). Grace [16] established that condition

$$q_0 > \frac{48}{1+\ln 2}$$

guarantees oscillation of (1.7). Karpuz et al. [27], Zhang and Yan [38], and Zhang et al. [39] obtained that condition

$$q_0 > \frac{48}{\mathrm{e}\ln 2}$$

ensures oscillation of (1.7).

This study is strongly motivated by the research of [8, 9]. Berchio et al. [9] studied a fourth-order differential equation

$$W''''(s) + kW''(s) + f(W(s)) = 0,$$

which is known as Swift-Hohenberg equation if k is positive; see [31]. Bartušek et al. [8] considered a generalized Swift-Hohenberg differential equation with a deviating argument

$$x^{(4)}(t) + g(t)x^{(2)}(t) + p(t)x(\tau(t)) = 0, \tag{1.8}$$

where $g(t) \geqslant g_0 > 0$. One can find that there is a relationship between (1.1) and a fourth-order dynamic equation of the form

$$x^{\Delta^4}(t) + g(t)x^{\Delta^2}(\sigma(t)) + p(t)x(\tau(t)) = 0,$$

which reduces to (1.8) in the case where $\mathbb{T}=\mathbb{R}$. Assume now that v is a positive solution of a second-order linear dynamic equation

$$v^{\Delta^2}(t)+g(t)v(t)=0.$$

It is not difficult to see that

$$\begin{aligned}
&\frac{1}{v(t)}\bigg(v(t)v^{\sigma}(t)\bigg(\frac{x^{\Delta^2}(t)}{v(t)}\bigg)^{\Delta}\bigg)^{\Delta}\\
&\quad=\frac{1}{v(t)}\bigg(v(t)v^{\sigma}(t)\frac{x^{\Delta^3}(t)v(t)-x^{\Delta^2}(t)v^{\Delta}(t)}{v(t)v^{\sigma}(t)}\bigg)^{\Delta}\\
&\quad=\frac{1}{v(t)}(x^{\Delta^4}(t)v(t)+x^{\Delta^3}(\sigma(t))v^{\Delta}(t)-x^{\Delta^3}(t)v^{\Delta}(t)-x^{\Delta^2}(\sigma(t))v^{\Delta^2}(t))\\
&\quad=x^{\Delta^4}(t)+\frac{(x^{\Delta^3}(\sigma(t))-x^{\Delta^3}(t))v^{\Delta}(t)}{v(t)}-\frac{x^{\Delta^2}(\sigma(t))v^{\Delta^2}(t)}{v(t)}\\
&\quad=x^{\Delta^4}(t)+\frac{(x^{\Delta^3}(t)-x^{\Delta^3}(t)+\mu(t)x^{\Delta^4}(t))v^{\Delta}(t)}{v(t)}-\frac{x^{\Delta^2}(\sigma(t))v^{\Delta^2}(t)}{v(t)}\\
&\quad=\bigg(1+\frac{\mu(t)v^{\Delta}(t)}{v(t)}\bigg)x^{\Delta^4}(t)-\frac{x^{\Delta^2}(\sigma(t))v^{\Delta^2}(t)}{v(t)}\\
&\quad=\bigg(1+\frac{\mu(t)v^{\Delta}(t)}{v(t)}\bigg)x^{\Delta^4}(t)+g(t)x^{\Delta^2}(\sigma(t)),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\bigg(v(t)v^{\sigma}(t)\bigg(\frac{x^{\Delta^2}(t)}{v(t)}\bigg)^{\Delta}\bigg)^{\Delta}+p(t)x(\tau(t))\\
&\quad=v(t)\bigg(1+\frac{\mu(t)v^{\Delta}(t)}{v(t)}\bigg)x^{\Delta^4}(t)+v(t)g(t)x^{\Delta^2}(\sigma(t))+p(t)x(\tau(t))\\
&\quad=v^{\sigma}(t)x^{\Delta^4}(t)+v(t)g(t)x^{\Delta^2}(\sigma(t))+p(t)x(\tau(t)).
\end{aligned}$$

Hence,

$$\bigg(v(t)v^{\sigma}(t)\bigg(\frac{x^{\Delta^2}(t)}{v(t)}\bigg)^{\Delta}\bigg)^{\Delta}+p(t)x(\tau(t))=0$$

and

$$x^{\Delta^4}(t)+\frac{v(t)g(t)}{v^{\sigma}(t)}x^{\Delta^2}(\sigma(t))+\frac{p(t)}{v^{\sigma}(t)}x(\tau(t))=0$$

are equivalent, where v is defined as the above statements, and so it is interesting to study (1.1).

So far, there are few results dealing with oscillatory behavior of solutions of fourth-order delay dynamic equations on time scales. Therefore, the purpose of this paper is to derive some oscillation theorems for (1.1). The organization of this paper is as follows: In Section 2, we present the basic definitions and the theory of calculus on time scales. In Section 3, we establish some oscillation criteria for (1.1) by employing Riccati technique. The results obtained improve Theorems 1.1–1.4 in the case $\mathbb{T}=\mathbb{R}$. In Section 4, we provide some examples to illustrate the main results. In Section 5, we give some conclusions for the sake of completeness.

In what follows, all occurring functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all sufficiently large t.

2 Some preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} \mid s < t\},$$

where $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$, $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf\mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup\mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

Fix $t \in \mathbb{T}$ and let $f : \mathbb{T} \to \mathbb{R}$. Define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

In this case, $f^{\Delta}(t)$ is called the (delta) derivative of f at t. f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. The derivative and the shift operator σ are related by the formula

$$f^{\sigma}(t) := f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

Let f be a real-valued function defined on an interval $[a,b]_{\mathbb{T}} := \{t \in \mathbb{T} : a \leqslant t \leqslant b\}$. Let f be a differentiable function on $[a,b]_{\mathbb{T}}$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a,b]_{\mathbb{T}}$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geqslant 0$, and $f^{\Delta}(t) \leqslant 0$ for all $t \in [a,b)_{\mathbb{T}} := \{t \in \mathbb{T} : a \leqslant t < b\}$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$
$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

For $a,\ b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

3 Main results

In this section, we will present some sufficient conditions which ensure that (1.1) is oscillatory. We begin with the following lemma.

Lemma 3.1. *Assume x is an eventually positive solution to* (1.1). *If*

$$\int_{t_0}^{\infty}\frac{\Delta t}{a(t)} = \int_{t_0}^{\infty}\frac{\Delta t}{b(t)} = \int_{t_0}^{\infty}\frac{\Delta t}{c(t)} = \infty, \tag{3.1}$$

then there are only the following two possible cases for $t \in [t_1,\infty)_{\mathbb{T}}$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$ sufficiently large

(1) $x(t) > 0$, $x^{\Delta}(t) > 0$, $(ax^{\Delta})^{\Delta}(t) > 0$, $(b(ax^{\Delta})^{\Delta})^{\Delta}(t) > 0$, $(c(b(ax^{\Delta})^{\Delta})^{\Delta})^{\Delta}(t) < 0$, *or*

(2) $x(t) > 0$, $x^{\Delta}(t) > 0$, $(ax^{\Delta})^{\Delta}(t) < 0$, $(b(ax^{\Delta})^{\Delta})^{\Delta}(t) > 0$, $(c(b(ax^{\Delta})^{\Delta})^{\Delta})^{\Delta}(t) < 0$.

Proof. Let x be an eventually positive solution to (1.1). Then there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. From (1.1), we have

$$(c(b(ax^{\Delta})^{\Delta})^{\Delta})^{\Delta}(t) = -p(t)x(\tau(t)) < 0 \quad \text{for} \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.2}$$

Thus, $c(b(ax^{\Delta})^{\Delta})^{\Delta}$ is decreasing. Then x^{Δ}, $(ax^{\Delta})^{\Delta}$, and $(b(ax^{\Delta})^{\Delta})^{\Delta}$ are of constant sign eventually. We claim that $(b(ax^{\Delta})^{\Delta})^{\Delta} > 0$. If not, there exist a constant $M_1 > 0$ and $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$(b(ax^{\Delta})^{\Delta})^{\Delta}(t) \leqslant -\frac{M_1}{c(t)} < 0 \quad \text{for} \quad t \in [t_2,\infty)_{\mathbb{T}}.$$

Thus, there exist a constant $M_2 > 0$ and $t_3 \in [t_2,\infty)_{\mathbb{T}}$ such that

$$(ax^{\Delta})^{\Delta}(t) \leqslant -\frac{M_2}{b(t)} < 0 \quad \text{for} \quad t \in [t_3,\infty)_{\mathbb{T}}.$$

Hence, there exist a constant $M_3 > 0$ and $t_4 \in [t_3,\infty)_{\mathbb{T}}$ such that

$$x^{\Delta}(t) \leqslant -\frac{M_3}{a(t)} < 0 \quad \text{for} \quad t \in [t_4,\infty)_{\mathbb{T}},$$

which yields $\lim_{t\to\infty} x(t) = -\infty$. This is a contradiction. If $(ax^{\Delta})^{\Delta} < 0$, then $x^{\Delta} > 0$ due to

$$\int_{t_0}^{\infty} \frac{\Delta t}{a(t)} = \infty.$$

If $(ax^{\Delta})^{\Delta} > 0$, then $x^{\Delta} > 0$ due to $(b(ax^{\Delta})^{\Delta})^{\Delta} > 0$. This completes the proof. □

Lemma 3.2. *Assume x is a solution to* (1.1) *which satisfies Case* (1) *of Lemma* 3.1. *Then*

$$(b(ax^{\Delta})^{\Delta})(t) \geqslant c(t)\int_{t_1}^{t} \frac{\Delta s}{c(s)} (b(ax^{\Delta})^{\Delta})^{\Delta}(t). \tag{3.3}$$

If there exist a function $\phi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and a $t_* \in [t_1,\infty)_{\mathbb{T}}$ such that

$$\frac{\phi(t)}{c(t)\int_{t_1}^{t}\frac{\Delta s}{c(s)}} - \phi^{\Delta}(t) \leqslant 0 \quad \textit{for} \quad t \in [t_*,\infty)_{\mathbb{T}}, \tag{3.4}$$

then $b(ax^{\Delta})^{\Delta}/\phi$ is a nonincreasing function on $t \in [t_*,\infty)_{\mathbb{T}}$ and

$$(ax^{\Delta})(t) \geqslant \left(\frac{b(t)}{\phi(t)}\int_{t_*}^{t}\frac{\phi(s)}{b(s)}\Delta s\right)(ax^{\Delta})^{\Delta}(t) \quad \textit{for} \quad t \in [t_*,\infty)_{\mathbb{T}}. \tag{3.5}$$

Furthermore, if there exist a function $\varphi \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and a $t_{**} \in [t_*,\infty)_{\mathbb{T}}$ such that

$$\frac{\varphi(t)}{\frac{b(t)}{\phi(t)}\int_{t_*}^{t}\frac{\phi(s)}{b(s)}\Delta s} - \varphi^{\Delta}(t) \leqslant 0 \quad \textit{for} \quad t \in [t_{**},\infty)_{\mathbb{T}}, \tag{3.6}$$

then ax^{Δ}/φ is a nonincreasing function on $t \in [t_{**},\infty)_{\mathbb{T}}$ and

$$x(t) \geqslant \left(\frac{a(t)}{\varphi(t)}\int_{t_{**}}^{t}\frac{\varphi(s)}{a(s)}\Delta s\right)x^{\Delta}(t) \quad \textit{for} \quad t \in [t_{**},\infty)_{\mathbb{T}}. \tag{3.7}$$

Suppose also that there exist a function $\delta \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and a $t_{***} \in [t_{**},\infty)_{\mathbb{T}}$ such that

$$\frac{\delta(t)}{\frac{a(t)}{\varphi(t)}\int_{t_{**}}^{t}\frac{\varphi(s)}{a(s)}\Delta s} - \delta^{\Delta}(t) \leqslant 0 \quad \textit{for} \quad t \in [t_{***},\infty)_{\mathbb{T}}. \tag{3.8}$$

Then x/δ is a nonincreasing function on $t \in [t_{***},\infty)_{\mathbb{T}}$.

Proof. From $(ax^{\Delta})^{\Delta} > 0$ and $(c(b(ax^{\Delta})^{\Delta})^{\Delta})^{\Delta} < 0$, we have

$$(b(ax^{\Delta})^{\Delta})(t) = (b(ax^{\Delta})^{\Delta})(t_1) + \int_{t_1}^{t} \frac{c(s)(b(ax^{\Delta})^{\Delta})^{\Delta}(s)}{c(s)}\Delta s \geqslant c(t)\int_{t_1}^{t}\frac{\Delta s}{c(s)}(b(ax^{\Delta})^{\Delta})^{\Delta}(t).$$

Thus,

$$\begin{aligned}\left(\frac{b(ax^{\Delta})^{\Delta}}{\phi}\right)^{\Delta}(t) &= \frac{(b(ax^{\Delta})^{\Delta})^{\Delta}(t)\phi(t) - (b(ax^{\Delta})^{\Delta})(t)\phi^{\Delta}(t)}{\phi(t)\phi^{\sigma}(t)}\\ &\leqslant \frac{(b(ax^{\Delta})^{\Delta})(t)}{\phi(t)\phi^{\sigma}(t)}\left(\frac{\phi(t)}{c(t)\int_{t_1}^{t}\frac{\Delta s}{c(s)}} - \phi^{\Delta}(t)\right) \leqslant 0.\end{aligned}$$

Therefore, $b(ax^{\Delta})^{\Delta}/\phi$ is a nonincreasing function on $[t_*, \infty)_{\mathbb{T}}$. Then, we obtain

$$(ax^{\Delta})(t) = (ax^{\Delta})(t_*) + \int_{t_*}^{t}\frac{b(s)(ax^{\Delta})^{\Delta}(s)}{\phi(s)}\frac{\phi(s)}{b(s)}\Delta s \geqslant \left(\frac{b(t)}{\phi(t)}\int_{t_*}^{t}\frac{\phi(s)}{b(s)}\Delta s\right)(ax^{\Delta})^{\Delta}(t).$$

Hence,

$$\left(\frac{ax^{\Delta}}{\varphi}\right)^{\Delta}(t) = \frac{(ax^{\Delta})^{\Delta}(t)\varphi(t) - (ax^{\Delta})(t)\varphi^{\Delta}(t)}{\varphi(t)\varphi^{\sigma}(t)} \leqslant \frac{(ax^{\Delta})(t)}{\varphi(t)\varphi^{\sigma}(t)}\left(\frac{\varphi(t)}{\frac{b(t)}{\phi(t)}\int_{t_*}^{t}\frac{\phi(s)}{b(s)}\Delta s} - \varphi^{\Delta}(t)\right) \leqslant 0.$$

Thus ax^{Δ}/φ is a nonincreasing function on $[t_{**}, \infty)_{\mathbb{T}}$. So we have

$$x(t) = x(t_{**}) + \int_{t_{**}}^{t}\frac{a(s)x^{\Delta}(s)}{\varphi(s)}\frac{\varphi(s)}{a(s)}\Delta s \geqslant \left(\frac{a(t)}{\varphi(t)}\int_{t_{**}}^{t}\frac{\varphi(s)}{a(s)}\Delta s\right)x^{\Delta}(t).$$

Then

$$\left(\frac{x}{\delta}\right)^{\Delta}(t) = \frac{x^{\Delta}(t)\delta(t) - x(t)\delta^{\Delta}(t)}{\delta(t)\delta^{\sigma}(t)} \leqslant \frac{x(t)}{\delta(t)\delta^{\sigma}(t)}\left(\frac{\delta(t)}{\frac{a(t)}{\varphi(t)}\int_{t_{**}}^{t}\frac{\varphi(s)}{a(s)}\Delta s} - \delta^{\Delta}(t)\right) \leqslant 0.$$

So x/δ is a nonincreasing function on $[t_{***}, \infty)_{\mathbb{T}}$. The proof is complete. □

Remark 3.3. The functions ϕ, φ, and δ are existent, e.g., by letting

$$\phi(t) := \int_{t_1}^{t}\frac{\Delta s}{c(s)}, \quad \varphi(t) := \int_{t_*}^{t}\frac{\phi(s)}{b(s)}\Delta s \quad \text{and} \quad \delta(t) := \int_{t_{**}}^{t}\frac{\varphi(s)}{a(s)}\Delta s.$$

In the following, we give the main results. For simplification, we use the notation

$$(\alpha^{\Delta}(t))_+ := \max\{0, \alpha^{\Delta}(t)\} \quad \text{and} \quad (\beta^{\Delta}(t))_+ := \max\{0, \beta^{\Delta}(t)\}.$$

Theorem 3.4. *Let* (3.1) *hold. Assume there exists a positive function* $\alpha \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *such that for all sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$*, for some* $t_* \in [t_1, \infty)_{\mathbb{T}}$*,* $t_{**} \in [t_*, \infty)_{\mathbb{T}}$*, and* $t_4 \in [t_{**}, \infty)_{\mathbb{T}}$*,*

$$\limsup_{t\to\infty}\int_{t_4}^{t}\left[\alpha^{\sigma}(s)p(s)f(s, t_*, t_{**}) - \frac{c(s)\phi^{\sigma}(s)((\alpha^{\Delta}(s))_+)^2}{4\alpha^{\sigma}(s)\phi(s)}\right]\Delta s = \infty, \tag{3.9}$$

where ϕ *and* φ *are defined as in Lemma* 3.2*, and*

$$f(t, t_*, t_{**}) := \frac{1}{\phi^{\sigma}(t)\varphi(\tau(t))}\int_{t_{**}}^{\tau(t)}\frac{\varphi(s)}{a(s)}\Delta s\int_{t_*}^{\tau(t)}\frac{\phi(s)}{b(s)}\Delta s.$$

If there exist positive functions $\beta, \varsigma \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ *such that*

$$\frac{\varsigma(t)}{a(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}} - \varsigma^{\Delta}(t) \leqslant 0 \quad \textit{for all } t \textit{ large enough}, \tag{3.10}$$

and for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left[\beta^{\sigma}(\xi) \frac{\varsigma(\xi)}{\varsigma(\sigma(\xi))} \frac{1}{b(\xi)} g(\xi) - \frac{a(\xi)\varsigma(\sigma(\xi))((\beta^{\Delta}(\xi))_+)^2}{4\beta^{\sigma}(\xi)\varsigma(\xi)} \right] \Delta\xi = \infty, \tag{3.11}$$

where

$$g(\xi) := \int_{\xi}^{\infty} \left[\frac{1}{c(s)} \int_{s}^{\infty} p(v) \frac{\varsigma(\tau(v))}{\varsigma(v)} \Delta v \right] \Delta s,$$

then (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 3.1, we get (3.2) and then x satisfies either Case (1) or Case (2).

Assume Case (1) holds. Define the function ω by

$$\omega(t) := \alpha(t) \frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)}{(b(ax^{\Delta})^{\Delta})(t)} \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.12}$$

Then $\omega(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and

$$\omega^{\Delta}(t) = \alpha^{\Delta}(t) \frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)}{(b(ax^{\Delta})^{\Delta})(t)} + \alpha^{\sigma}(t) \left(\frac{c(b(ax^{\Delta})^{\Delta})^{\Delta}}{b(ax^{\Delta})^{\Delta}} \right)^{\Delta} (t),$$

which implies that

$$\begin{aligned} \omega^{\Delta}(t) &= \alpha^{\Delta}(t) \frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)}{(b(ax^{\Delta})^{\Delta})(t)} + \alpha^{\sigma}(t) \frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})^{\Delta}(t)}{(b(ax^{\Delta})^{\Delta})^{\sigma}(t)} \\ &\quad - \alpha^{\sigma}(t) \frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)(b(ax^{\Delta})^{\Delta})^{\Delta}(t)}{(b(ax^{\Delta})^{\Delta})^{\sigma}(t)(b(ax^{\Delta})^{\Delta})(t)}. \end{aligned} \tag{3.13}$$

Since $b(ax^{\Delta})^{\Delta}/\phi$ is a nonincreasing function on $t \in [t_*, \infty)_{\mathbb{T}}$, we have

$$(b(ax^{\Delta})^{\Delta})^{\sigma}(t) \leqslant \frac{\phi^{\sigma}(t)}{\phi(t)} (b(ax^{\Delta})^{\Delta})(t). \tag{3.14}$$

It follows from Lemma 3.2 that

$$\begin{aligned} \frac{x(\tau(t))}{(b(ax^{\Delta})^{\Delta})^{\sigma}(t)} &= \frac{1}{b^{\sigma}(t)} \frac{x(\tau(t))}{x^{\Delta}(\tau(t))} \frac{x^{\Delta}(\tau(t))}{(ax^{\Delta})^{\Delta}(\tau(t))} \frac{(ax^{\Delta})^{\Delta}(\tau(t))}{((ax^{\Delta})^{\Delta})^{\sigma}(t)} \\ &\geqslant \frac{1}{b^{\sigma}(t)} \left(\frac{a(\tau(t))}{\varphi(\tau(t))} \int_{t_{**}}^{\tau(t)} \frac{\varphi(s)}{a(s)} \Delta s \right) \frac{1}{a(\tau(t))} \\ &\quad \times \left(\frac{b(\tau(t))}{\phi(\tau(t))} \int_{t_*}^{\tau(t)} \frac{\phi(s)}{b(s)} \Delta s \right) \frac{\phi(\tau(t)) b^{\sigma}(t)}{\phi^{\sigma}(t) b(\tau(t))} \\ &= \frac{1}{\varphi(\tau(t))\phi^{\sigma}(t)} \int_{t_{**}}^{\tau(t)} \frac{\varphi(s)}{a(s)} \Delta s \int_{t_*}^{\tau(t)} \frac{\phi(s)}{b(s)} \Delta s. \end{aligned} \tag{3.15}$$

Hence by (3.2) and (3.12)–(3.15), we obtain

$$\omega^{\Delta}(t) \leqslant -\alpha^{\sigma}(t) p(t) f(t, t_*, t_{**}) + \frac{(\alpha^{\Delta}(t))_+}{\alpha(t)} \omega(t) - \frac{1}{c(t)} \frac{\alpha^{\sigma}(t)}{\alpha^2(t)} \frac{\phi(t)}{\phi^{\sigma}(t)} \omega^2(t).$$

Thus,

$$\omega^{\Delta}(t) \leqslant -\alpha^{\sigma}(t) p(t) f(t, t_*, t_{**}) + \frac{c(t)\phi^{\sigma}(t)((\alpha^{\Delta}(t))_+)^2}{4\alpha^{\sigma}(t)\phi(t)}.$$

Integrating the above inequality from t_4 ($t_4 \in [t_{**}, \infty)_{\mathbb{T}}$) to t, we get

$$\int_{t_4}^{t} \left[\alpha^{\sigma}(s) p(s) f(s, t_*, t_{**}) - \frac{c(s)\phi^{\sigma}(s)((\alpha^{\Delta}(s))_+)^2}{4\alpha^{\sigma}(s)\phi(s)} \right] \Delta s \leqslant \omega(t_4) - \omega(t) \leqslant \omega(t_4),$$

which contradicts (3.9).

Assume Case (2) holds. Define the function ν by

$$\nu(t) := \beta(t)\frac{(ax^{\Delta})(t)}{x(t)} \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.16}$$

Then $\nu(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and

$$\nu^{\Delta}(t) = \beta^{\Delta}(t)\frac{(ax^{\Delta})(t)}{x(t)} + \beta^{\sigma}(t)\frac{(ax^{\Delta})^{\Delta}(t)}{x(\sigma(t))} - \beta^{\sigma}(t)\frac{(ax^{\Delta})(t)x^{\Delta}(t)}{x(t)x(\sigma(t))}. \tag{3.17}$$

Hence by (3.16) and (3.17), we have

$$\nu^{\Delta}(t) = \frac{\beta^{\Delta}(t)}{\beta(t)}\nu(t) + \beta^{\sigma}(t)\frac{(ax^{\Delta})^{\Delta}(t)}{x(\sigma(t))} - \frac{1}{a(t)}\frac{\beta^{\sigma}(t)}{\beta^2(t)}\frac{x(t)}{x(\sigma(t))}\nu^2(t). \tag{3.18}$$

Since $x(t) > 0$ and $(ax^{\Delta})^{\Delta}(t) < 0$, we obtain

$$x(t) = x(t_1) + \int_{t_1}^{t} \frac{a(s)x^{\Delta}(s)}{a(s)}\Delta s \geqslant \left(a(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}\right)x^{\Delta}(t).$$

Thus,

$$\left(\frac{x}{\varsigma}\right)^{\Delta}(t) = \frac{x^{\Delta}(t)\varsigma(t) - x(t)\varsigma^{\Delta}(t)}{\varsigma(t)\varsigma^{\sigma}(t)} \leqslant \frac{x(t)}{\varsigma(t)\varsigma^{\sigma}(t)}\left(\frac{\varsigma(t)}{a(t)\int_{t_1}^{t}\frac{\Delta s}{a(s)}} - \varsigma^{\Delta}(t)\right) \leqslant 0.$$

Hence, x/ς is nonincreasing eventually and

$$\frac{x(t)}{x(\sigma(t))} \geqslant \frac{\varsigma(t)}{\varsigma(\sigma(t))}, \quad \frac{x(\tau(t))}{x(t)} \geqslant \frac{\varsigma(\tau(t))}{\varsigma(t)}. \tag{3.19}$$

Hence by (3.18) and (3.19), we see that

$$\nu^{\Delta}(t) \leqslant \frac{\beta^{\Delta}(t)}{\beta(t)}\nu(t) + \beta^{\sigma}(t)\frac{(ax^{\Delta})^{\Delta}(t)}{x(\sigma(t))} - \frac{1}{a(t)}\frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{\beta^{\sigma}(t)}{\beta^2(t)}\nu^2(t). \tag{3.20}$$

On the other hand, by (1.1), we get

$$(c(b(ax^{\Delta})^{\Delta})^{\Delta})(z) - (c(b(ax^{\Delta})^{\Delta})^{\Delta})(t) + \int_t^z p(s)x(\tau(s))\Delta s = 0.$$

It follows from $x^{\Delta}(t) > 0$ and (3.19) that

$$(c(b(ax^{\Delta})^{\Delta})^{\Delta})(z) - (c(b(ax^{\Delta})^{\Delta})^{\Delta})(t) + x(t)\int_t^z p(s)\frac{\varsigma(\tau(s))}{\varsigma(s)}\Delta s \leqslant 0.$$

Letting $z \to \infty$ in the above inequality, we obtain

$$-(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t) + x(t)\int_t^{\infty} p(s)\frac{\varsigma(\tau(s))}{\varsigma(s)}\Delta s \leqslant 0$$

due to $\lim_{z\to\infty}(c(b(ax^{\Delta})^{\Delta})^{\Delta})(z) = l_1 \geqslant 0$, i.e.,

$$(b(ax^{\Delta})^{\Delta})^{\Delta}(t) \geqslant x(t)\left[\frac{1}{c(t)}\int_t^{\infty} p(s)\frac{\varsigma(\tau(s))}{\varsigma(s)}\Delta s\right].$$

Therefore,

$$-(b(ax^{\Delta})^{\Delta})(z) + (b(ax^{\Delta})^{\Delta})(t) + x(t)\int_t^z \left[\frac{1}{c(s)}\int_s^{\infty} p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v\right]\Delta s \leqslant 0.$$

Letting $z \to \infty$ in the latter inequality, from $\lim_{z\to\infty} -(b(ax^\Delta)^\Delta)(z) = l_2 \geqslant 0$, we have

$$(ax^\Delta)^\Delta(t) + x(t)\frac{1}{b(t)}\int_t^\infty \left[\frac{1}{c(s)}\int_s^\infty p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v\right]\Delta s \leqslant 0.$$

Thus, by (3.19), we get

$$\begin{aligned}\frac{(ax^\Delta)^\Delta(t)}{x(\sigma(t))} &\leqslant -\frac{x(t)}{x(\sigma(t))}\frac{1}{b(t)}\int_t^\infty \left[\frac{1}{c(s)}\int_s^\infty p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v\right]\Delta s\\ &\leqslant -\frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{1}{b(t)}\int_t^\infty \left[\frac{1}{c(s)}\int_s^\infty p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v\right]\Delta s. \end{aligned}\tag{3.21}$$

Substituting (3.21) into (3.20), we obtain

$$\begin{aligned}\nu^\Delta(t) \leqslant{}& -\beta^\sigma(t)\frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{1}{b(t)}\int_t^\infty \left[\frac{1}{c(s)}\int_s^\infty p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v\right]\Delta s\\ &+\frac{(\beta^\Delta(t))_+}{\beta(t)}\nu(t) - \frac{1}{a(t)}\frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{\beta^\sigma(t)}{\beta^2(t)}\nu^2(t),\end{aligned}$$

which yields

$$\nu^\Delta(t) \leqslant -\beta^\sigma(t)\frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{1}{b(t)}\int_t^\infty \left[\frac{1}{c(s)}\int_s^\infty p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v\right]\Delta s + \frac{a(t)\varsigma(\sigma(t))((\beta^\Delta(t))_+)^2}{4\beta^\sigma(t)\varsigma(t)}.$$

Integrating the above inequality from t_2 ($t_2 \in [t_1,\infty)_\mathbb{T}$) to t, we have

$$\begin{aligned}&\int_{t_2}^t\left\{\beta^\sigma(\xi)\frac{\varsigma(\xi)}{\varsigma(\sigma(\xi))}\frac{1}{b(\xi)}\int_\xi^\infty \left[\frac{1}{c(s)}\int_s^\infty p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v\right]\Delta s - \frac{a(\xi)\varsigma(\sigma(\xi))((\beta^\Delta(\xi))_+)^2}{4\beta^\sigma(\xi)\varsigma(\xi)}\right\}\Delta\xi\\ &\quad\leqslant \nu(t_2) - \nu(t) \leqslant \nu(t_2),\end{aligned}$$

which contradicts (3.11). The proof is complete. □

Remark 3.5. The function ς is existent, e.g., by letting $\varsigma(t) := \int_{t_1}^t \frac{\Delta s}{a(s)}$.

Motivated by Theorem 3.4, we can obtain the following result.

Theorem 3.6. *Let* (3.1) *hold. Assume for all sufficiently large* $t_1 \in [t_0,\infty)_\mathbb{T}$*, for some* $t_* \in [t_1,\infty)_\mathbb{T}$*, and* $t_{**} \in [t_*,\infty)_\mathbb{T}$*, the second-order dynamic equation*

$$(c(t)u^\Delta(t))^\Delta + p(t)f(t,t_*,t_{**})u^\sigma(t) = 0 \tag{3.22}$$

is oscillatory, where ϕ *and* φ *are defined as in Lemma* 3.2 *and* f *is as in Theorem* 3.4*. If there exists a positive function* $\varsigma \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_\mathbb{T},\mathbb{R})$ *such that* (3.10) *holds and the second-order dynamic equation*

$$(a(t)u^\Delta(t))^\Delta + \frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{g(t)}{b(t)}u^\sigma(t) = 0 \tag{3.23}$$

is oscillatory, where g *is defined as in Theorem* 3.4*, then* (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_\mathbb{T}$. We may assume without loss of generality that there exists a $t_1 \in [t_0,\infty)_\mathbb{T}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_\mathbb{T}$. Proceeding as in the proof of Lemma 3.1, we get (3.2) and then x satisfies either Case (1) or Case (2).

Assume Case (1) holds. From the proof of Theorem 3.4, we obtain (3.15). Define the function ω by

$$\omega(t) := \frac{(c(b(ax^\Delta)^\Delta)^\Delta)(t)}{(b(ax^\Delta)^\Delta)(t)} \quad \text{for} \quad t \in [t_1,\infty)_\mathbb{T}. \tag{3.24}$$

Then $\omega(t) > 0$ for $t \in [t_1,\infty)_\mathbb{T}$ and

$$\omega^\Delta(t) = -\frac{p(t)x(\tau(t))}{(b(ax^\Delta)^\Delta)^\sigma(t)} - \frac{(c(b(ax^\Delta)^\Delta)^\Delta)(t)(b(ax^\Delta)^\Delta)^\Delta(t)}{(b(ax^\Delta)^\Delta)^\sigma(t)(b(ax^\Delta)^\Delta)(t)} \tag{3.25}$$

due to (3.2). Hence by (3.15), (3.24), and (3.25), we obtain

$$\omega^{\Delta}(t)+p(t)f(t,t_*,t_{**})+\frac{\omega^2(t)}{c(t)+\mu(t)\omega(t)}\leqslant 0.$$

It follows from a result of [12] that (3.22) is nonoscillatory for t large enough.

Assume Case (2) holds. From the proof of Theorem 3.4, we obtain (3.21). Define the function ν by

$$\nu(t):=\frac{(ax^{\Delta})(t)}{x(t)}\quad\text{for}\quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.26}$$

Then $\nu(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$ and

$$\nu^{\Delta}(t)=\frac{(ax^{\Delta})^{\Delta}(t)}{x(\sigma(t))}-\frac{(ax^{\Delta})(t)x^{\Delta}(t)}{x(t)x(\sigma(t))}. \tag{3.27}$$

Hence by (3.21), (3.26), and (3.27), we have

$$\nu^{\Delta}(t)+\frac{\varsigma(t)}{\varsigma(\sigma(t))}\frac{1}{b(t)}g(t)+\frac{\nu^2(t)}{a(t)+\mu(t)\nu(t)}\leqslant 0.$$

It follows from a result of [12] that (3.23) is nonoscillatory for t large enough. The proof is complete. □

Remark 3.7. Theorem 3.6 provides a comparison criterion for oscillation of (1.1). One can use some known results in Theorem 3.6 to obtain various classes of oscillation criteria for (1.1). For example, one can easily establish Hille and Nehari type criteria for (1.1) when using the results reported in [32]. The details are left to the reader.

In what follows, we show that assumption (3.9) can be replaced with other conditions by defining Riccati substitutions which differ from (3.12).

Theorem 3.8. *Let (3.1) hold. Assume there exists a positive function $\alpha\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that for all sufficiently large $t_1\in[t_0,\infty)_{\mathbb{T}}$, for some $t_*\in[t_1,\infty)_{\mathbb{T}}$, $t_{**}\in[t_*,\infty)_{\mathbb{T}}$, and $t_4\in[t_{**},\infty)_{\mathbb{T}}$,*

$$\limsup_{t\to\infty}\int_{t_4}^{t}\left[\alpha^{\sigma}(s)p(s)F(s,t_*,t_{**})-\frac{b(s)\varphi^{\sigma}(s)((\alpha^{\Delta}(s))_+)^2}{4\int_{t_1}^{s}\frac{\Delta u}{c(u)}\alpha^{\sigma}(s)\varphi(s)}\right]\Delta s=\infty, \tag{3.28}$$

where ϕ and φ are defined as in Lemma 3.2, and

$$F(t,t_*,t_{**}):=\frac{1}{\varphi^{\sigma}(t)}\int_{t_{**}}^{\tau(t)}\frac{\varphi(s)}{a(s)}\Delta s.$$

If there exist positive functions $\beta,\varsigma\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that (3.10) and (3.11) hold for some $t_2\in[t_1,\infty)_{\mathbb{T}}$, where g is defined as in Theorem 3.4, then (1.1) is oscillatory.

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ and $x(\tau(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 3.1, we get (3.2) and then x satisfies either Case (1) or Case (2).

Assume Case (1) holds. Define the function ω by

$$\omega(t):=\alpha(t)\frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)}{(ax^{\Delta})(t)}\quad\text{for}\quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.29}$$

Then $\omega(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$ and

$$\omega^{\Delta}(t)=\alpha^{\Delta}(t)\frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)}{(ax^{\Delta})(t)}+\alpha^{\sigma}(t)\left(\frac{c(b(ax^{\Delta})^{\Delta})^{\Delta}}{ax^{\Delta}}\right)^{\Delta}(t),$$

which implies that

$$\omega^{\Delta}(t)=\alpha^{\Delta}(t)\frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)}{(ax^{\Delta})(t)}+\alpha^{\sigma}(t)\frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})^{\Delta}(t)}{(ax^{\Delta})^{\sigma}(t)}$$

$$-\alpha^{\sigma}(t)\frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)(ax^{\Delta})^{\Delta}(t)}{(ax^{\Delta})^{\sigma}(t)(ax^{\Delta})(t)}. \tag{3.30}$$

By Lemma 3.2, we have

$$(ax^{\Delta})^{\Delta}(t)\geqslant\frac{\int_{t_1}^{t}\frac{\Delta s}{c(s)}}{b(t)}(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t),\quad \frac{(ax^{\Delta})^{\sigma}(t)}{(ax^{\Delta})(t)}\leqslant\frac{\varphi^{\sigma}(t)}{\varphi(t)}, \tag{3.31}$$

and

$$\begin{aligned}\frac{x(\tau(t))}{(ax^{\Delta})^{\sigma}(t)}&=\frac{1}{a^{\sigma}(t)}\frac{x(\tau(t))}{x^{\Delta}(\tau(t))}\frac{x^{\Delta}(\tau(t))}{x^{\Delta}(\sigma(t))}\\&\geqslant\frac{1}{a^{\sigma}(t)}\left(\frac{a(\tau(t))}{\varphi(\tau(t))}\int_{t_{**}}^{\tau(t)}\frac{\varphi(s)}{a(s)}\Delta s\right)\frac{\varphi(\tau(t))}{\varphi^{\sigma}(t)}\frac{a^{\sigma}(t)}{a(\tau(t))}\\&=\frac{1}{\varphi^{\sigma}(t)}\int_{t_{**}}^{\tau(t)}\frac{\varphi(s)}{a(s)}\Delta s.\end{aligned} \tag{3.32}$$

Hence by (3.2) and (3.29)–(3.32), we obtain

$$\omega^{\Delta}(t)\leqslant-\alpha^{\sigma}(t)p(t)F(t,t_*,t_{**})+\frac{(\alpha^{\Delta}(t))_+}{\alpha(t)}\omega(t)-\frac{\int_{t_1}^{t}\frac{\Delta s}{c(s)}}{b(t)}\frac{\alpha^{\sigma}(t)}{\alpha^2(t)}\frac{\varphi(t)}{\varphi^{\sigma}(t)}\omega^2(t).$$

Thus,

$$\omega^{\Delta}(t)\leqslant-\alpha^{\sigma}(t)p(t)F(t,t_*,t_{**})+\frac{b(t)\varphi^{\sigma}(t)((\alpha^{\Delta}(t))_+)^2}{4\int_{t_1}^{t}\frac{\Delta s}{c(s)}\alpha^{\sigma}(t)\varphi(t)}.$$

Integrating the above inequality from t_4 ($t_4\in[t_{**},\infty)_{\mathbb{T}}$) to t, we have

$$\int_{t_4}^{t}\left[\alpha^{\sigma}(s)p(s)F(s,t_*,t_{**})-\frac{b(s)\varphi^{\sigma}(s)((\alpha^{\Delta}(s))_+)^2}{4\int_{t_1}^{s}\frac{\Delta u}{c(u)}\alpha^{\sigma}(s)\varphi(s)}\right]\Delta s\leqslant\omega(t_4)-\omega(t)\leqslant\omega(t_4),$$

which contradicts (3.28).

The proof of Case (2) is the same as that of Theorem 3.4, and hence is omitted. This completes the proof. □

Theorem 3.9. *Let* (3.1) *hold. Assume there exists a positive function* $\alpha\in\mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that for all sufficiently large* $t_1\in[t_0,\infty)_{\mathbb{T}}$*, for some* $t_*\in[t_1,\infty)_{\mathbb{T}}$*,* $t_{**}\in[t_*,\infty)_{\mathbb{T}}$*, and* $t_4\in[t_{**},\infty)_{\mathbb{T}}$*,*

$$\limsup_{t\to\infty}\int_{t_4}^{t}\left[\alpha^{\sigma}(s)p(s)\frac{\delta(\tau(s))}{\delta^{\sigma}(s)}-\frac{\delta^{\sigma}(s)((\alpha^{\Delta}(s))_+)^2}{4G(s,t_1,t_*)\alpha^{\sigma}(s)\delta(s)}\right]\Delta s=\infty, \tag{3.33}$$

where ϕ *and* φ *are defined as in Lemma* 3.2*, and*

$$G(t,t_1,t_*):=\frac{\int_{t_1}^{t}\frac{\Delta s}{c(s)}}{a(t)\phi(t)}\int_{t_*}^{t}\frac{\phi(s)}{b(s)}\Delta s.$$

If there exist positive functions $\beta,\varsigma\in\mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* (3.10) *and* (3.11) *hold for some* $t_2\in[t_1,\infty)_{\mathbb{T}}$*, where* g *is defined as in Theorem* 3.4*, then* (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume without loss of generality that there exists a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ and $x(\tau(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Lemma 3.1, we get (3.2) and then x satisfies either Case (1) or Case (2).

Assume Case (1) holds. Define the function ω by

$$\omega(t):=\alpha(t)\frac{(c(b(ax^{\Delta})^{\Delta})^{\Delta})(t)}{x(t)}\quad\text{for}\quad t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.34}$$

Then $\omega(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and

$$\omega^\Delta(t) = \alpha^\Delta(t)\frac{(c(b(ax^\Delta)^\Delta)^\Delta)(t)}{x(t)} + \alpha^\sigma(t)\left(\frac{c(b(ax^\Delta)^\Delta)^\Delta}{x}\right)^\Delta(t),$$

which yields

$$\omega^\Delta(t) = \alpha^\Delta(t)\frac{(c(b(ax^\Delta)^\Delta)^\Delta)(t)}{x(t)} + \alpha^\sigma(t)\frac{(c(b(ax^\Delta)^\Delta)^\Delta)^\Delta(t)}{x^\sigma(t)} - \alpha^\sigma(t)\frac{(c(b(ax^\Delta)^\Delta)^\Delta)(t)x^\Delta(t)}{x^\sigma(t)x(t)}. \tag{3.35}$$

By Lemma 3.2, we have

$$x^\Delta(t) \geqslant \frac{1}{a(t)}\left(\frac{b(t)}{\phi(t)}\int_{t_*}^t \frac{\phi(s)}{b(s)}\Delta s\right)\frac{\int_{t_1}^t \frac{\Delta s}{c(s)}}{b(t)}(c(b(ax^\Delta)^\Delta)^\Delta)(t) \tag{3.36}$$

and

$$\frac{x(\tau(t))}{x^\sigma(t)} \geqslant \frac{\delta(\tau(t))}{\delta^\sigma(t)}, \quad \frac{x(t)}{x^\sigma(t)} \geqslant \frac{\delta(t)}{\delta^\sigma(t)}. \tag{3.37}$$

Hence by (3.2) and (3.34)–(3.37), we obtain

$$\omega^\Delta(t) \leqslant -\alpha^\sigma(t)p(t)\frac{\delta(\tau(t))}{\delta^\sigma(t)} + \frac{(\alpha^\Delta(t))_+}{\alpha(t)}\omega(t) - G(t,t_1,t_*)\frac{\alpha^\sigma(t)}{\alpha^2(t)}\frac{\delta(t)}{\delta^\sigma(t)}\omega^2(t).$$

Thus,

$$\omega^\Delta(t) \leqslant -\alpha^\sigma(t)p(t)\frac{\delta(\tau(t))}{\delta^\sigma(t)} + \frac{\delta^\sigma(t)((\alpha^\Delta(t))_+)^2}{4G(t,t_1,t_*)\alpha^\sigma(t)\delta(t)}.$$

Integrating the above inequality from t_4 ($t_4 \in [t_{**}, \infty)_{\mathbb{T}}$) to t, we get

$$\int_{t_4}^t \left[\alpha^\sigma(s)p(s)\frac{\delta(\tau(s))}{\delta^\sigma(s)} - \frac{\delta^\sigma(s)((\alpha^\Delta(s))_+)^2}{4G(s,t_1,t_*)\alpha^\sigma(s)\delta(s)}\right]\Delta s \leqslant \omega(t_4) - \omega(t) \leqslant \omega(t_4),$$

which contradicts (3.33).

The proof of Case (2) is the same as that of Theorem 3.4, and hence is omitted. This completes the proof. □

Remark 3.10. Condition (3.11) can be replaced by

$$\int_{t_0}^\infty p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v = \infty, \quad \text{or} \quad g(t_0) = \infty.$$

4 Examples and discussion

In the following, we present some examples to show applications of the main results in the previous section.

Example 4.1. For $t \geqslant 1$, consider a fourth-order delay differential equation

$$\left(\frac{1}{t}\left(\frac{1}{t}\left(\frac{1}{t}x'(t)\right)'\right)'\right)' + \frac{\lambda}{t^6}x\left(\frac{t}{2}\right) = 0, \tag{4.1}$$

where $\lambda > 0$ is a constant, $a(t) = b(t) = c(t) = 1/t$, $p(t) = \lambda/t^6$, and $\tau(t) = t/2$. Set $\phi(t) = \int_{t_1}^t sds = (t^2 - {t_1}^2)/2$, $\varphi(t) = \int_{t_*}^t s(s^2 - {t_1}^2)/2ds = t^4/8 - {t_*}^4/8 - t^2{t_1}^2/4 + {t_1}^2{t_*}^2/4$, $\varsigma(t) = \int_{t_1}^t sds = (t^2 - {t_1}^2)/2$, $\alpha(t) = t$, and $\beta(t) = t$. Then

$$\frac{t^2}{3} \leqslant \phi(t) \leqslant \frac{t^2}{2}, \quad \frac{t^4}{9} \leqslant \varphi(t) \leqslant \frac{t^4}{8},$$

and

$$f(t,t_*,t_{**}) := \frac{1}{\phi^\sigma(t)\varphi(\tau(t))}\int_{t_{**}}^{\tau(t)}\frac{\varphi(s)}{a(s)}ds\int_{t_*}^{\tau(t)}\frac{\phi(s)}{b(s)}ds \geqslant \frac{t^4}{2600},$$

for t large enough. Thus, (3.9) and (3.11) hold. Using Theorem 3.4, (4.1) is oscillatory.

Example 4.2. For $t \geqslant 1$, consider a fourth-order differential equation (1.6), where $q_0 > 0$ is a constant, $a(t) = b(t) = c(t) = 1$, $p(t) = q_0/t^4$, and $\tau(t) = t$. Set

$$\phi(t) = \varsigma(t) = \int_{t_1}^{t} ds, \quad \varphi(t) = \int_{t_*}^{t}\phi(s)ds, \quad \alpha(t) = t \quad \text{and} \quad \beta(t) = t.$$

Then, for every $k \in (0,1)$,

$$kt \leqslant \phi(t) \leqslant t, \quad \frac{k^2t^2}{2} \leqslant \varphi(t) \leqslant \frac{t^2}{2},$$

and

$$f(t,t_*,t_{**}) := \frac{1}{\phi^\sigma(t)\varphi(\tau(t))}\int_{t_{**}}^{\tau(t)}\frac{\varphi(s)}{a(s)}ds\int_{t_*}^{\tau(t)}\frac{\phi(s)}{b(s)}ds \geqslant \frac{k^5}{6}t^2,$$

for t large enough. Thus, (3.9) and (3.11) are satisfied if $q_0 > 3/(2k^5)$. Using Theorem 3.4, (1.6) is oscillatory when $q_0 > 3/(2{k_0}^5)$ for some constant $k_0 \in (0,1)$. For example, one can take $q_0 > 5/3$ (let $k_0 = (9/10)^{1/5}$). This result shows that Theorem 3.4 improves Theorems 1.1–1.4 in the case where $\mathbb{T} = \mathbb{R}$; see the details in Section 1. It is well known (see [28, Theorem 2.15]) that (1.6) is oscillatory if $q_0 > 1$. How to extend this sharp criterion to fourth-order dynamic equations on time scales remains open at the moment.

Example 4.3. For $t \geqslant 1$, consider a fourth-order delay differential equation (1.7), where $q_0 > 0$ is a constant. Similar to the statement of Example 4.2, one can find that (1.7) is oscillatory if $q_0 > 12/k_0^5$ for some constant $k_0 \in (0,1)$. For example, one can take $q_0 > 13$ (let $k_0 = (12/13)^{1/5}$).

Example 4.4. For $t \in [1,\infty)_{\mathbb{T}}$, consider a fourth-order delay dynamic equation

$$x^{\Delta^4}(t) + p(t)x(\tau(t)) = 0 \tag{4.2}$$

with

$$p(t) \geqslant \frac{\gamma}{t\int_{t_{**}}^{\tau(t)}\int_{t_*}^{u}(s-t_1)\Delta s\Delta u}$$

eventually, where $t_{**} \in [t_*,\infty)_{\mathbb{T}} \subseteq [t_1,\infty)_{\mathbb{T}} \subseteq [1,\infty)_{\mathbb{T}}$, $\gamma > 0$ is a constant. We assume there exists a constant $k_0 \geqslant 1$ such that $\sigma(t) \leqslant k_0 t$. Let

$$a(t) = b(t) = c(t) = 1, \quad \phi(t) = t - t_1, \quad \varphi(t) = \int_{t_*}^{t}(s-t_1)\Delta s, \quad \varsigma(t) = t - t_1 \quad \text{and} \quad \alpha(t) = \beta(t) = t.$$

Then, for each constant $k_1 \in (0,1)$,

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_4}^{t}\left[\alpha^\sigma(s)p(s)f(s,t_*,t_{**}) - \frac{c(s)\phi^\sigma(s)((\alpha^\Delta(s))_+)^2}{4\alpha^\sigma(s)\phi(s)}\right]\Delta s \\
&\quad\geqslant \limsup_{t\to\infty}\int_{t_4}^{t}\left[\frac{\gamma}{s} - \frac{\sigma(s)-t_1}{4\sigma(s)(s-t_1)}\right]\Delta s \geqslant \limsup_{t\to\infty}\int_{t_4}^{t}\left[\frac{\gamma}{s} - \frac{1}{4(s-t_1)}\right]\Delta s \\
&\quad\geqslant \left[\gamma - \frac{1}{4k_1}\right]\limsup_{t\to\infty}\int_{t_4}^{t}\frac{\Delta s}{s} = \infty, \quad \text{if } \gamma > 1/(4k_1),
\end{aligned}$$

and, for each constant $k_2 \in (0,1)$,

$$\begin{aligned}
g(\xi) &= \int_{\xi}^{\infty}\left[\frac{1}{c(s)}\int_{s}^{\infty}p(v)\frac{\varsigma(\tau(v))}{\varsigma(v)}\Delta v\right]\Delta s \\
&\geqslant \int_{\xi}^{\infty}\int_{s}^{\infty}\frac{\gamma}{v\int_{t_{**}}^{\tau(v)}\int_{t_*}^{u}(s-t_1)\Delta s\Delta u}\frac{\tau(v)-t_1}{v-t_1}\Delta v\Delta s
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \gamma k_2 \int_{\xi}^{\infty}\int_{s}^{\infty} \frac{1}{v\int_{t_{**}}^{\tau(v)}\int_{t_*}^{u}(s-t_1)\Delta s\Delta u}\frac{\tau(v)}{v}\Delta v\Delta s\\
&\geqslant \gamma k_2 \int_{\xi}^{\infty}\int_{s}^{\infty} \frac{1}{v\int_{t_{**}}^{\tau(v)}\int_{t_*}^{u}s\Delta s\Delta u}\frac{\tau(v)}{v}\Delta v\Delta s\\
&\geqslant 6\gamma k_2 \int_{\xi}^{\infty}\int_{s}^{\infty} v^{-4}\Delta v\Delta s \geqslant \frac{6\gamma k_2}{(1+k_0+{k_0}^2)(1+k_0)}\frac{1}{\xi^2},
\end{aligned}
$$

and hence, for each constant $k_3 \in (0,1)$ and for each constant $k_4 \in (0,1)$,

$$
\begin{aligned}
&\limsup_{t\to\infty}\int_{t_2}^{t}\left[\beta^{\sigma}(\xi)\frac{\varsigma(\xi)}{\varsigma(\sigma(\xi))}\frac{1}{b(\xi)}g(\xi)-\frac{a(\xi)\varsigma(\sigma(\xi))((\beta^{\Delta}(\xi))_+)^2}{4\beta^{\sigma}(\xi)\varsigma(\xi)}\right]\Delta\xi\\
&\quad\geqslant \limsup_{t\to\infty}\int_{t_2}^{t}\left[\frac{6\gamma k_2}{(1+k_0+{k_0}^2)(1+k_0)}\sigma(\xi)\frac{\xi-t_1}{\sigma(\xi)-t_1}\frac{1}{\xi^2}-\frac{\sigma(\xi)-t_1}{4\sigma(\xi)(\xi-t_1)}\right]\Delta\xi\\
&\quad\geqslant \left[\frac{6\gamma k_2k_3}{(1+k_0+{k_0}^2)(1+k_0)}-\frac{1}{4k_4}\right]\limsup_{t\to\infty}\int_{t_2}^{t}\frac{\Delta\xi}{\xi}=\infty,
\end{aligned}
$$

if

$$
\gamma > \frac{(1+k_0+{k_0}^2)(1+k_0)}{24k_2k_3k_4}.
$$

Therefore, we have by Theorem 3.4 that (4.2) is oscillatory when

$$
\gamma > \max\left\{\frac{1}{4k_1},\frac{(1+k_0+{k_0}^2)(1+k_0)}{24k_2k_3k_4}\right\},
$$

for some constants $k_1,k_2,k_3,k_4 \in (0,1)$. For example, one can take

$$
\gamma > \max\left\{\frac{5}{18},\frac{5(1+k_0+{k_0}^2)(1+k_0)}{108}\right\}
$$

(let $k_1 = 9/10$ and $k_2 = k_3 = k_4 = (9/10)^{1/3}$).

5 Conclusions

In this paper, we suggest some classes of oscillation results for a generalized fourth-order delay dynamic equation (1.1) with a canonical form (3.1). With the help of the methods given in this paper, one can derive some Philos-type oscillation criteria for (1.1). The details are left to the reader.

Three examples provided are differential equations and a direct comparison is made between the results obtained for these examples and the conditions contained in the existing literature; see, for example, Example 4.3 shows that the true constant for the separation of types of solutions (oscillatory and nonoscillatory) of fourth-order equation (1.7) is smaller than those presented in [16, 21–23, 27, 36, 38, 39].

On one hand, results reported in this paper can be applied to fourth-order dynamic equations with delayed arguments. On the other hand, we point out that, contrary to [1, 18–20, 35, 37], we do not need impose restrictive assumptions on the coefficients a, b and c in our oscillation theorems which, in certain sense, is a significant improvement compared to the results in the cited papers. Note that Theorems 1.1–1.4 cannot be applied to (4.2) due to the effect of the delayed argument τ. The methods used in this paper are different from those reported in [1, 18–20, 35, 37]. From the results on oscillation of fourth-order Euler differential equation (1.6) (see Example 4.2 and the details introduced in Section 1), one can see that this paper provides an improved universal method for the study of oscillatory properties of fourth-order dynamic equations on a time scale.

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