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OSCILLATION OF SECOND ORDER DELAY DYNAMIC EQUATIONS

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ABSTRACT. In this paper we establish some sufficient conditions for oscillation of second order delay dynamic equations on time scales. Our results not only unify the oscillation of second order delay differential and difference equations but also are new for q-difference equations and can be applied on any time scale. We illustrate our results with many examples.

1 Introduction The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [**19**] in order to unify continuous and discrete analysis. Not only can this theory of so-called dynamic equations **unify** the theories of differential equations and difference equations, but also it is able to **extend** these classical cases to cases "in between", e.g., to so-called q-difference equations. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models (see [**9**]). A book on the subject of time scales by Bohner and Peterson [**9**] summarizes and organizes much of the time scale calculus (see also [**1, 10**]). For the notions used below we refer to [**9**] and to the next section, where we recall some of the main tools used in the subsequent sections of this paper.

The problem of obtaining sufficient conditions to ensure that all solutions of certain classes of second order dynamic equations are oscillatory has been studied by a number of authors [**2, 3, 4, 5, 7, 11, 12, 13, 14, 15, 16, 21**]. A large portion of these results has been for the nonlinear

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dynamic equation of the form

$$(\alpha x^{\Delta})^{\Delta}(t) + p(t)f(x(\sigma(t))) = 0 \quad \text{for } t \in \mathbb{T}, \tag{1.1}$$

where $p(t) \geq 0$.

Recently Bohner [**6**] (see also Zhang and Deng [**22**]) considered the linear delay dynamic inequality

$$y^{\Delta}(t) + p(t)y(\tau(t)) \leq 0 \quad \text{for } t \in \mathbb{T} \tag{1.2}$$

and unified oscillation criteria of delay differential and difference equations.

In this paper, we consider the second order linear delay dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0 \quad \text{for } t \in \mathbb{T} \tag{1.3}$$

on a time scale, where the function p is rd-continuous such that $p(t) > 0$ for all $t \in \mathbb{T}$, and where $\tau : \mathbb{T} \to \mathbb{T}$ such that $\tau(t) \leq t$ for all $t \in \mathbb{T}$ is the *delay* function. We note that, if $\mathbb{T} = \mathbb{R}$, then $x^{\Delta} = x'$ (the usual derivative), and (1.3) becomes the second order delay differential equation

$$x''(t) + p(t)x(\tau(t)) = 0 \quad \text{for } t \in \mathbb{R}. \tag{1.4}$$

If $\mathbb{T} = \mathbb{Z}$, then $x^{\Delta} = \Delta x$ (the forward difference operator), and (1.3) becomes the second order delay difference equation

$$x(t+2) - 2x(t+1) + x(t) + p(t)x(\tau(t)) = 0 \quad \text{for } t \in \mathbb{Z}. \tag{1.5}$$

If $\mathbb{T} = h\mathbb{Z} := \{hk : \ k \in \mathbb{Z}\}$ with $h > 0$, then (1.3) becomes the more general second order delay difference equation

$$x(t+2h) - 2x(t+h) + x(t) + h^2 p(t)x(\tau(t)) = 0 \quad \text{for } t \in h\mathbb{Z}. \tag{1.6}$$

If $\mathbb{T} = q^{\mathbb{N}_0} := \left\{q^k : \ k \in \mathbb{N}_0\right\}$ with $q > 1$, then (1.3) becomes the second order delay q-difference equation

$$\begin{aligned} x(q^2 t) - (q+1)x(qt) + qx(t)& \\ + q(q-1)^2 t^2 p(t)x(\tau(t)) &= 0 \quad \text{for } t \in q^{\mathbb{N}_0}. \end{aligned} \tag{1.7}$$

The paper is organized as follows. In Section 2, we present some basic definitions concerning the calculus on time scales. In Section 3, we reduce (1.3) to an inequality of the form (1.2) and then apply the results from [**6**] to derive sufficient conditions for oscillation of all solutions of (1.3). Finally, in Section 4 we present further oscillation criteria, among them some that we derive by the Riccati transformation technique. We illustrate the presented theory with many examples. Our results not only unify oscillation criteria for (1.4) and (1.5) but also are new for equations (1.6) and (1.7), and corresponding equations on arbitrary other time scales. Some of the arguments in the first part of the paper are similar to techniques for delay differential equations which appear in the monographs [**17, 18, 20**].

2 Some preliminaries on time scales A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. As we are interested in oscillatory behavior, we assume throughout that the given time scale $\mathbb{T}$ is unbounded above, i.e., it is a time scale interval of the form $[a, \infty)$ with $a \in \mathbb{T}$. On $\mathbb{T}$ we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function μ is defined by

$$\mu(t) := \sigma(t) - t \quad \text{for all } t \in \mathbb{T}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at each point and if there exists a finite left limit in all left-dense points. The (delta) *derivative* of f at $t \in \mathbb{T}$ is defined by

$$f^{\Delta}(t) = \lim_{\substack{s \to t \\ s \in U(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \quad \text{where } U(t) = \mathbb{T} \setminus \{\sigma(t)\},$$

provided that limit exists. The derivative and the shift operator σ are related by the useful formula

$$f^{\sigma} = f + \mu f^{\Delta} \quad \text{where } f^{\sigma} := f \circ \sigma. \tag{2.1}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable function f and g:

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta \qquad \text{and} \qquad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{2.2}$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\,\Delta t = f(b) - f(a). \tag{2.3}$$

The integration by parts formula follows from (2.2) and reads

$$\int_a^b f(t)g^\Delta(t)\,\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t, \tag{2.4}$$

and infinite integrals are defined as

$$\int_a^\infty f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t.$$

Four examples of time scales together with their derivative operators are given in Section 1 above. Numerous other examples can be found in [**9, 10**].

In Section 3 below we shall also make use of the exponential function $e_p(\cdot, t_0)$, which is defined as the unique solution of the initial value problem

$$x^\Delta = p(t)x, \qquad x(t_0) = 1. \tag{2.5}$$

It is known [**9**] that (2.5) has a unique solution provided p is rd-continuous and regressive (we write $p \in \mathcal{R}$), i.e.,

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

It is also known [**9**] that this unique solution of (2.5) is always positive if p is in addition positively regressive (we write $p \in \mathcal{R}^+$), i.e.,

$$1 + \mu(t)p(t) > 0 \quad \text{for all } t \in \mathbb{T}.$$

Examples of $e_p(t,s)$ for $t \geq s$ on the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ are

$$\exp\left(\int_s^t p(u)\mathrm{d}u\right) \qquad \text{and} \qquad \prod_{u=s}^{t-1}(1+p(u)),$$

respectively.

The following auxiliary result concerning equations of the form (1.3) will be used throughout this paper.

Lemma 2.1. *If x is an eventually positive solution of* (1.3)*, then there exists $t_0 \in \mathbb{T}$ with*

$$x(t) \geq x(\tau(t)) > 0, \quad x^{\Delta}(t) > 0, \quad \text{and} \quad x^{\Delta\Delta}(t) < 0 \tag{2.6}$$

for all $t \geq t_0 \geq a$.

Proof. Let x be an eventually positive solution of (1.3). Hence there exists $t_1 \in \mathbb{T}$ with $t_1 \geq a$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \geq t_1$. In view of (3.1) we have

$$x^{\Delta\Delta}(t) = -p(t)x(\tau(t)) < 0 \quad \text{for all } t \geq t_1, \tag{2.7}$$

and so x^{Δ} is eventually decreasing and hence eventually of one sign. We first show that x^{Δ} is eventually positive. Indeed, since p is a positive function, the decreasing function x^{Δ} is either eventually positive or eventually negative. Suppose there exists $t_2 \geq t_1$ such that $x^{\Delta}(t_2) < 0$. Then from (2.7) we have

$$\begin{aligned} x(t) &\overset{(2.3)}{=} x(t_2) + \int_{t_2}^t x^{\Delta}(s)\Delta s \\ &\overset{(2.7)}{\leq} x(t_2) + (t-t_2)x^{\Delta}(t_2) \\ &\to -\infty \quad \text{as } t \to \infty, \end{aligned}$$

a contradiction. Hence

$$x^{\Delta}(t) \geq 0 \quad \text{for all } t \geq t_1.$$

Now pick any $t_0 \in \mathbb{T}$ with $t_0 > t_1$. Then (2.6) holds. ☐

Another result that we will use frequently is taken from [**8**] and reads as follows.

Lemma 2.2. *Let $a \in \mathbb{T}$. If $\mathbb{T}$ is a time scale that is unbounded above, then*

$$\int_a^\infty \frac{\Delta t}{t} = \infty. \tag{2.8}$$

3 Reduction to first order delay dynamic inequalities In this section we give some sufficient conditions for oscillation of (1.3) by reducing this equation to a first order delay dynamic inequality of the form (1.2) and by applying the main result of [**6**] (see also [**22**]) to the resulting first order inequality. We begin by stating the main result from [**6**] for easy reference. Throughout this section we assume that

$$p(t) > 0, \quad \tau(t) < t \quad \text{for all } t \in \mathbb{T}, \quad \text{and } \lim_{t\to\infty} \tau(t) = \infty. \tag{3.1}$$

Theorem 3.1. *Assume* (3.1). *If* (1.2) *possesses an eventually positive solution, then*

$$\alpha \geq 1, \quad \textit{where } \alpha := \limsup_{t\to\infty} \sup_{\substack{\lambda>0 \\ -\lambda p \in \mathcal{R}^+}} \left\{\lambda e_{-\lambda p}(t, \tau(t))\right\}.$$

The main result in this section now is a consequence of Theorem 3.1 and reads as follows.

Theorem 3.2. *Assume* (3.1). *If* (1.3) *possesses an eventually positive solution, then*

$$\alpha(c) \geq 1 \quad \textit{for all } c \in (0,1)$$

where

$$\alpha(c) := \limsup_{t\to\infty} \sup_{\substack{\lambda>0 \\ -\lambda cp\tau \in \mathcal{R}^+}} \left\{\lambda e_{-\lambda cp\tau}(t, \tau(t))\right\}.$$

Proof. Let x be an eventually positive solution of (1.3). Hence there exists $t_0 \in \mathbb{T}$ such that (2.6) holds. Define $y := x^\Delta$. Then

$$\begin{aligned} x(t) &\overset{(2.3)}{=} x(t_0) + \int_{t_0}^t y(s)\Delta s \\ &\overset{(2.6)}{\geq} x(t_0) + (t - t_0)y(t) \\ &\overset{(2.6)}{\geq} (t - t_0)y(t). \end{aligned}$$

Let $c \in (0,1)$. Then for $t \geq t_0/(1-c) =: t^* \geq t_0$ we have $t - t_0 \geq ct$ and hence $x(t) \geq cty(t)$. Thus

$$x(\tau(t)) \geq c\tau(t)y(\tau(t)) \quad \text{for all } t \geq t^*. \tag{3.2}$$

Substituting (3.2) into (1.3) provides for $t \geq t^*$

$$y^\Delta(t) = x^{\Delta\Delta}(t) = -p(t)x(\tau(t)) \leq -cp(t)\tau(t)y(\tau(t)).$$

Therefore y is an eventually positive solution of

$$y^\Delta(t) + cp(t)\tau(t)y(\tau(t)) \leq 0. \tag{3.3}$$

Clearly, (3.3) is an inequality of the form (1.2), and therefore the claim follows by applying Theorem 3.1. □

From Theorem 3.2 we have the following corollary.

Corollary 3.3. *Assume* (3.1). *If* $\alpha(c) < 1$ *for some* $c \in (0,1)$, *then all solutions of* (1.3) *are oscillatory on* $[a, \infty)$.

Example 3.4. If $\mathbb{T} = \mathbb{R}$, then

$$f(\lambda) := \lambda e_{-\lambda cp\tau}(t, \tau(t)) = \lambda e^{-\lambda M} \quad \text{with } M = \int_{\tau(t)}^t cp(s)\tau(s)\mathrm{d}s.$$

For $\tilde{\lambda} := 1/M > 0$ we have $f'(\tilde{\lambda}) = 0$ and $f''(\tilde{\lambda}) < 0$. Thus

$$\sup_{\substack{\lambda > 0 \\ -\lambda cp\tau \in \mathcal{R}^+}} f(\lambda) = f(\tilde{\lambda}) = \frac{\tilde{\lambda}}{e} = \frac{1}{Me}$$

so that $\alpha(c) < 1$ iff

$$\liminf_{t \to \infty} \int_{\tau(t)}^t p(s)\tau(s)\mathrm{d}s > \frac{1}{ce}. \tag{3.4}$$

Hence (3.4) for some $c \in (0,1)$ implies that (1.4) is oscillatory.

Example 3.5. Let $\mathbb{T} = \{t_n : n \in \mathbb{Z}\}$ be a time scale such that $\sigma(t_n) = t_{n+1}$. Let $k \in \mathbb{N}$ and $\tau(t_n) = t_{n-k}$. Then

$$f(\lambda) := \lambda e_{-\lambda cp\tau}(t, \tau(t)) \leq \lambda(1 - \lambda S)^k \quad \text{with } S = \frac{1}{k}\int_{\tau(t)}^{t} cp(s)\tau(s)\Delta s,$$

where we used the arithmetic-geometric inequality. For

$$\tilde{\lambda} := \frac{1}{(k+1)S} > 0$$

we have $f'(\tilde{\lambda}) = 0$ and $f''(\tilde{\lambda}) < 0$. Thus

$$\sup_{\substack{\lambda>0 \\ -\lambda cp\tau \in \mathcal{R}^+}} f(\lambda) \leq f(\tilde{\lambda}) = \frac{1}{(k+1)S}\left(1 - \frac{1}{k+1}\right)^k = \frac{k^k}{S(k+1)^{k+1}}$$

so that $\alpha(c) < 1$ if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\tau(s)\Delta s > \frac{1}{c}\left(\frac{k}{k+1}\right)^{k+1}. \tag{3.5}$$

Hence (3.5) for some $c \in (0, 1)$ implies that the corresponding equation (1.3) is oscillatory. Note that all three dynamic equations (1.5)–(1.7) can be accommodated within this example.

In the following theorem we establish new oscillation criteria for (1.3) which are different from the above condition and can be verified easily.

Theorem 3.6. *Assume* (3.1). *If*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\tau(s)\Delta s > \frac{1}{c} \quad \textit{for some } c \in (0, 1), \tag{3.6}$$

then every solution of (1.3) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that x is an eventually positive solution of (1.3) (we shall consider only this case, since the substitution $\tilde{x} = -x$ transforms (1.3) into an equation of the same form). Hence there exists $t_0 \in \mathbb{T}$ such that (2.6) holds. Proceeding as in Theorem 3.2 we get (3.3). Integrating (3.3) from $\tau(t)$ to sufficiently large t provides

$$\begin{aligned} 0 &\overset{(3.3)}{\geq} \int_{\tau(t)}^{t} \left\{ y^{\Delta}(s) + cp(s)\tau(s)y(\tau(s)) \right\} \Delta s \\ &\overset{(2.3)}{=} y(t) - y(\tau(t)) + \int_{\tau(t)}^{t} cp(s)\tau(s)y(\tau(s))\Delta s \\ &\overset{(2.6)}{\geq} y(t) - y(\tau(t)) + y(\tau(t)) \int_{\tau(t)}^{t} cp(s)\tau(s)\Delta s \\ &= y(t) + y(\tau(t)) \left[\int_{\tau(t)}^{t} cp(s)\tau(s)\Delta s - 1 \right] \\ &\overset{(3.6)}{>} 0. \end{aligned}$$

This is a contradiction and the proof is complete. □

Remark 3.7. Note that Theorems 3.2 and 3.6 are not applicable to equations of type (1.3) with $\tau(t) = t$. So the delay appearing in (1.3) plays a crucial rôle in the qualitative behavior. In the following section we establish some new oscillation criteria for (1.3) which can be applied even in the case without delay.

4 Riccati transformation technique Throughout this section we assume

$$p(t) > 0, \quad \tau(t) \leq t \quad \text{for all } t \in \mathbb{T} \qquad \text{and} \qquad \lim_{t\to\infty} \tau(t) = \infty. \tag{4.1}$$

We also assume throughout that

$$\int_a^{\infty} \sigma(t)\, p(t)\Delta t = \infty. \tag{4.2}$$

Theorem 4.1. *Assume* (3.1). *If* (4.2) *holds, then every bounded solution of* (1.3) *is oscillatory on* $[a, \infty)$.

Proof. Suppose that there exists an eventually positive and bounded solution x. Then there exists $t_0 \in \mathbb{T}$ such that (2.6) holds, and without loss of generality, there exists $\alpha, \beta \in \mathbb{R}$ such that

$$0 < \alpha < x(\tau(t)) < \beta \quad \text{for all } t \geq t_0. \tag{4.3}$$

Let $X(t) = tx^{\Delta}(t)$. Then

$$
\begin{aligned}
X(t) &\overset{(2.3)}{=} X(t_0) + \int_{t_0}^{t} X^{\Delta}(s)\Delta s \\
&\overset{(2.2)}{=} X(t_0) + \int_{t_0}^{t} \left\{x^{\Delta}(s) + \sigma(s)x^{\Delta\Delta}(s)\right\} \Delta s \\
&\overset{(1.3)}{=} X(t_0) + \int_{t_0}^{t} \left\{x^{\Delta}(s) - \sigma(s)p(s)x(\tau(s))\right\} \Delta s \\
&\overset{(2.3)}{=} X(t_0) + x(t) - x(t_0) - \int_{t_0}^{t} \sigma(s)p(s)x(\tau(s))\Delta s \\
&\overset{(4.3)}{\leq} X(t_0) + \beta - x(t_0) - \alpha \int_{t_0}^{t} \sigma(s)p(s)\Delta s \\
&\overset{(4.2)}{\to} -\infty \quad \text{as } t \to \infty,
\end{aligned}
$$

i.e., there exists a constant $M > 0$ such that

$$x^{\Delta}(t) \leq -M/t \quad \text{for } t \geq T$$

for some $T \geq t_0$, and this implies by (2.8) that $\lim_{t\to\infty} x(t) = -\infty$, contradicting $x(t) > 0$ for all $t \geq t_0$. Thus every bounded solution is oscillatory. □

The following example is illustrative.

Example 4.2. Consider the Euler delay dynamic equation

$$x^{\Delta\Delta}(t) + \frac{1}{t\sigma(t)}x(\tau(t)) = 0 \quad \text{for } t \geq a, \tag{4.4}$$

where τ satisfies (4.1). From (2.8) we have

$$\int_{a}^{t} \sigma(s)\, p(s)\Delta s = \int_{a}^{t} \frac{\Delta s}{s} \to \infty \quad \text{as } t \to \infty.$$

By Theorem 3.2, every bounded solution of (4.4) oscillates.

The next theorem improves Theorem 4.1 and gives a condition under which every solution of (1.3) oscillates. We first need the following auxiliary result.

Lemma 4.3. *If* (4.1) *and* (4.2) *hold, then an eventually positive solution x of* (1.3) *satisfies eventually*

$$x(t) \geq t x^{\Delta}(t) \quad \text{and} \quad \frac{x(t)}{t} \quad \text{is nonincreasing.} \tag{4.5}$$

Proof. First note that there exists $t_0 \in \mathbb{T}$ such that (2.6) holds. Let $X(t) = x(t) - t x^{\Delta}(t)$. Then

$$\begin{aligned} X(t) &\overset{(2.3)}{=} X(t_0) + \int_{t_0}^{t} X^{\Delta}(s)\Delta s \\ &\overset{(2.2)}{=} X(t_0) + \int_{t_0}^{t} \left\{ x^{\Delta}(s) - \sigma(s) x^{\Delta\Delta}(s) - x^{\Delta}(s) \right\} \Delta s \\ &\overset{(1.3)}{=} X(t_0) + \int_{t_0}^{t} \sigma(s) p(s) x(\tau(s))\Delta s \\ &\overset{(2.6)}{\geq} X(t_0) + x(\tau(t_0)) \int_{t_0}^{t} \sigma(s) p(s) \Delta s \\ &\overset{(4.2)}{\to} \infty \quad \text{as } t \to \infty. \end{aligned}$$

Hence the first part of (4.5) follows, and because of

$$Y^{\Delta}(t) \overset{(2.2)}{=} \frac{t x^{\Delta}(t) - x(t)}{t \sigma(t)} \quad \text{where } Y(t) = \frac{x(t)}{t},$$

the second part of (4.5) is true as well. □

Theorem 4.4. *Assume that* (4.1) *and* (4.2) *hold. If*

$$\lim_{t \to \infty} \left\{ t \int_{t}^{\infty} p(s) \frac{\tau(s)}{s} \Delta s \right\} = \infty, \tag{4.6}$$

then every solution of (1.3) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.3). Then there exists $t_0 \in \mathbb{T}$ such that (2.6) holds. From (1.3), (2.3) and (2.6) we have for $T \geq t \geq t_0$

$$\int_{t}^{T} p(s) x(\tau(s))\Delta s = -\int_{t}^{T} x^{\Delta\Delta}(s)\Delta s = x^{\Delta}(t) - x^{\Delta}(T) \leq x^{\Delta}(t)$$

and hence

$$\int_t^\infty p(s)x(\tau(s))\Delta s \le x^\Delta(t).$$

This and Lemma 4.3 provides for sufficiently large $t \in \mathbb{T}$

$$\begin{aligned} x(t) &\ge tx^\Delta(t) \\ &\ge t\int_t^\infty p(s)x(\tau(s))\Delta s \\ &\overset{(4.5)}{\ge} t\int_t^\infty p(s)\frac{\tau(s)}{s}x(s)\Delta s \\ &\ge x(t)\left\{t\int_t^\infty p(s)\frac{\tau(s)}{s}\Delta s\right\} \end{aligned}$$

so that for sufficiently large $t \in \mathbb{T}$

$$1 \ge t\int_t^\infty p(s)\frac{\tau(s)}{s}\Delta s.$$

This contradicts (4.6) and completes the proof. □

Note that from the above results we can derive some oscillation criteria for equations (1.4)–(1.7). For example from Theorems 4.1 and 4.4 we have the following results.

Example 4.5. Let $h > 0$. Assume that (4.1) holds and

$$\lim_{n\to\infty} \sum_{\nu=a/h}^{n} (\nu+1)p(\nu h) = \infty.$$

Then every bounded solution of (1.6) oscillates on $[a,\infty)$. If in addition

$$\lim_{n\to\infty} \left\{n\sum_{\nu=n}^{\infty} p(\nu h)\frac{\tau(\nu h)}{\nu}\right\} = \infty,$$

then every solution of (1.6) is oscillatory on $[a,\infty)$.

Example 4.6. Let $q > 1$. Assume that (4.1) holds and

$$\lim_{n\to\infty} \sum_{\nu=\log_q a}^{n} q^{2\nu}p(q^\nu) = \infty.$$

Then every bounded solution of (1.7) oscillates on $[a,\infty)$. If in addition

$$\lim_{n\to\infty}\left\{q^n\sum_{\nu=n}^{\infty}p(q^\nu)\tau(q^\nu)\right\}=\infty,$$

then every solution of (1.7) is oscillatory on $[a,\infty)$.

Next we present the main result of this section using the Riccati substitution. We first need the following auxiliary result which is proved in [**3**].

Lemma 4.7. *If x and z are differentiable functions such that $x(t)\neq 0$ for all $t\in\mathbb{T}$, then*

$$(z^\Delta)^2 = x^\Delta\left(\frac{z^2}{x}\right)^\Delta + xx^\sigma\left[\left(\frac{z}{x}\right)^\Delta\right]^2. \tag{4.7}$$

Theorem 4.8. *Assume that* (4.1) *and* (4.2) *hold. If there exists a differentiable function z such that*

$$\lim_{t\to\infty}\int_a^t\left\{\frac{z^2(\sigma(s))}{\sigma(s)}p(s)\tau(s)-\left(z^\Delta(s)\right)^2\right\}\Delta s=\infty, \tag{4.8}$$

then every solution of (1.3) *is oscillatory on* $[a,\infty)$.

Proof. Again we suppose that x is an eventually positive solution of (1.3). Then there exists $t_0\in\mathbb{T}$ such that (2.6) holds. We introduce the Riccati substitution

$$w=\frac{z^2x^\Delta}{x}$$

and observe that we have eventually

$$\begin{aligned}
-w^\Delta &\overset{(2.2)}{=} \frac{z^2(x^\Delta)^2-\left[(z^\sigma)^2x^{\Delta\Delta}+(z^2)^\Delta x^\Delta\right]x}{xx^\sigma}\\
&\overset{(1.3)}{=} \frac{(z^\sigma)^2p(x\circ\tau)}{x^\sigma}-x^\Delta\left(\frac{z^2}{x}\right)^\Delta\\
&\overset{(4.5)}{\geq} \frac{(z^\sigma)^2p\tau}{\sigma}-x^\Delta\left(\frac{z^2}{x}\right)^\Delta\\
&\overset{(4.7)}{=} \frac{(z^\sigma)^2p\tau}{\sigma}+xx^\sigma\left[\left(\frac{z}{x}\right)^\Delta\right]^2-(z^\Delta)^2\\
&\overset{(2.6)}{\geq} \frac{(z^\sigma)^2p\tau}{\sigma}-(z^\Delta)^2.
\end{aligned}$$

Thus we have for sufficiently large $t \geq t_1 \geq a$

$$\begin{aligned} w(t_1) &\overset{(2.6)}{\geq} w(t_1) - w(t) \\ &\overset{(2.3)}{=} -\int_{t_1}^{t} w^{\Delta}(s)\Delta s \\ &\geq \int_{t_1}^{t} \left\{ \frac{z^2(\sigma(s))}{\sigma(s)} p(s)\tau(s) - \left(z^{\Delta}(s)\right)^2 \right\} \Delta s \\ &\overset{(4.8)}{\to} \infty \quad \text{as } t \to \infty. \end{aligned}$$

This contradiction completes the proof. ∎

The following two corollaries are immediate, where in Theorem 4.8 we choose $z(t) \equiv 1$ and $z(t) = \sqrt{t}$, respectively.

Corollary 4.9. *Assume that* (4.1) *and* (4.2) *hold. If*

$$\lim_{t \to \infty} \int_a^t p(s) \frac{\tau(s)}{\sigma(s)} \Delta s = \infty,$$

then every solution of (1.3) *is oscillatory on* $[a, \infty)$.

Corollary 4.10. *Assume that* (4.1) *and* (4.2) *hold. If*

$$\lim_{t \to \infty} \int_a^t \left\{ p(s)\tau(s) - \frac{1}{\left(\sqrt{s} + \sqrt{\sigma(s)}\right)^2} \right\} \Delta s = \infty,$$

then every solution of (1.3) *is oscillatory on* $[a, \infty)$.

Sometimes the following criterion is easier to check than the one given in Corollary 4.10, but it follows easily from Corollary 4.10 as we always have $\sigma(t) \geq t$ for all $t \in \mathbb{T}$.

Corollary 4.11. *Assume that* (4.1) *and* (4.2) *hold. If*

$$\lim_{t \to \infty} \int_a^t \left\{ p(s)\tau(s) - \frac{1}{4s} \right\} \Delta s = \infty,$$

then every solution of (1.3) *is oscillatory on* $[a, \infty)$.

The following example illustrates Corollary 4.11.

Example 4.12. Consider the delay dynamic equation

$$x^{\Delta\Delta}(t) + \frac{\gamma}{t\tau(t)}x(\tau(t)) = 0 \quad \text{for } t \geq 1, \tag{4.9}$$

where τ satisfies (4.1). By (2.8) we have

$$\int_1^t \sigma(s)p(s)\Delta s = \int_1^t \frac{\gamma\sigma(s)}{s\tau(s)}\Delta s \geq \int_1^t \frac{\gamma}{s}\Delta s \to \infty \quad \text{as } t \to \infty.$$

Also

$$\lim_{t\to\infty}\int_a^t \left\{\tau(s)p(s) - \frac{1}{4s}\right\}\Delta s = \lim_{t\to\infty}\int_a^t \left\{\frac{\gamma}{s} - \frac{1}{4s}\right\}\Delta s = \infty$$

provided that $\gamma > 1/4$. By Corollary 4.11, every solution of (4.9) oscillates if $\gamma > 1/4$.

Now we give the Kamenev-type oscillation criteria for (1.3). Since the proof is similar to that of the proof of [**21,** Theorem 3.2], we will omit it.

Theorem 4.13. *Assume* (4.1) *and* (4.2). *Furthermore assume that for every odd* $n \in \mathbb{N}$

$$\lim_{t\to\infty}\frac{1}{t^n}\int_a^t (t-s)^n\left\{\tau(s)p(s) - \frac{1}{4s}\right\}\Delta s = \infty.$$

Then every solution of (1.3) *is oscillatory on* $[a,\infty)$.

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