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Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations

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We establish some new criteria for the oscillation of second-order nonlinear dynamic equations on a time scale. We study the case of strongly superlinear and the case of strongly sublinear equations subject to various conditions.
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1. Introduction

This paper is concerned with the oscillatory behavior of solutions of second-order nonlinear dynamic equations of the form

$$(ax^{\Delta})^{\Delta}(t) + f(t, x^{\sigma}(t)) = 0, \quad t \geqslant t_0 \tag{1.1}$$

subject to the following hypotheses:

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(i) a is a positive real-valued rd-continuous function satisfying either

$$\int_{t_0}^{\infty} \frac{\Delta s}{a(s)} = \infty \tag{1.2}$$

or

$$\int_{t_0}^{\infty} \frac{\Delta s}{a(s)} < \infty. \tag{1.3}$$

(ii) $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous satisfying

$$\operatorname{sgn} f(t,x) = \operatorname{sgn} x \quad \text{and} \quad f(t,x) \leqslant f(t,y), \quad x \leqslant y, \ t \geqslant t_0. \tag{1.4}$$

By a solution of Eq. (1.1), we mean a nontrivial real-valued function x satisfying Eq. (1.1) for $t \geqslant t_x \geqslant t_0$. A solution x of Eq. (1.1) is called *oscillatory* if it is neither eventually positive nor eventually negative; otherwise it is called *nonoscillatory*. Eq. (1.1) is called *oscillatory* if all its solutions are oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [19]. Several authors have expounded on various aspects of this new theory, see [5,11,12]. In recent years, there has been much research activity concerning the oscillation and nonoscillation of various equations on time scales, we refer the reader to [2,3,10,13,14,16,17]. Most of the results are obtained for special cases of Eq. (1.1), e.g. when $a = 1$ and $f(t,x) = q(t)x$ or $a = 1$ and $f(t,x) = q(t)f(x)$, where f satisfies the condition $|f(x)/x| \geqslant k > 0$ for $x \neq 0$, or $f'(x) \geqslant f(x)/x > 0$ for $x \neq 0$, see [15–17]. In [18], the authors considered the second-order Emden–Fowler dynamic equation on time scales $x^{\Delta\Delta} + q(t)x^{\alpha} = 0$, where α is the ratio of odd integers, and they used the Riccati transformation technique to obtain several oscillation criteria for this equation. For the continuous case of (1.1), i.e.,

$$(ax')'(t) + f(t, x(t)) = 0,$$

numerous oscillation and nonoscillation criteria have been established, see e.g. [1,4,6–9]. The main goal of this paper is to establish new oscillation criteria for (1.1) (without employing the Riccati transformation technique). The paper is organized as follows: In Section 2, we present some basic preliminaries concerning calculus on time scales and prove some auxiliary results that will be used in the remainder of this paper. In Section 3, we present some oscillation criteria when condition (1.2) holds, and Section 4 is devoted to the study of oscillation of Eq. (1.1) when condition (1.3) holds. In Section 5, some applications are discussed.

The results of this paper are presented in a form which is essentially new and of a high degree of generality. The obtained results unify and improve many known oscillation criteria which appeared in the literature.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is unbounded above. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. For a function $f : \mathbb{T} \to \mathbb{R}$ we write $f^{\sigma} = f \circ \sigma$, and f^{Δ} represents the delta derivative of the function f as defined for example in [11]. The reader unfamiliar with time scales may think of f^{Δ} as the usual derivative f' if $\mathbb{T} = \mathbb{R}$ and as the usual forward difference Δf if $\mathbb{T} = \mathbb{Z}$. For the definition of rd-continuity and further details we refer to [11]. Besides the usual properties of the time scales integral, we require in this paper only the use of the chain rule [11, Theorem 1.90]

$$\frac{(x^{1-\alpha})^{\Delta}}{1-\alpha} = x^{\Delta} \int_0^1 [hx^{\sigma} + (1-h)x]^{-\alpha} \mathrm{d}h, \tag{2.1}$$

where $\alpha > 0$ and x is such that the right-hand side of (2.1) is well defined. Integration by parts, the product rule, and the quotient rule are not needed in this paper. One consequence of (2.1), that will be used in the proof of four results throughout this paper, is as follows.

Lemma 2.1. *Suppose $|y|^{\Delta}$ is of one sign on $[t_0, \infty)$ and $\alpha > 0$. Then*

$$\frac{|y|^{\Delta}}{(|y|^{\sigma})^{\alpha}} \leqslant \frac{(|y|^{1-\alpha})^{\Delta}}{1-\alpha} \leqslant \frac{|y|^{\Delta}}{|y|^{\alpha}} \quad \text{on } [t_0, \infty).$$

Proof. Replacing x by $|y|$ in (2.1), we see that either $|y|^{\Delta} > 0$ so that $|y|$ is increasing and hence $|y| \leqslant |y|^{\sigma}$ and thus $|y| \leqslant h|y|^{\sigma} + (1-h)|y| \leqslant |y|^{\sigma}$ for all $h \in [0,1]$, or otherwise $|y|^{\Delta} < 0$ so that $|y|$ is decreasing and hence $|y| \geqslant |y|^{\sigma}$ and thus $|y| \geqslant h|y|^{\sigma} + (1-h)|y| \geqslant |y|^{\sigma}$ for all $h \in [0,1]$. □

We shall obtain results for strongly superlinear and strongly sublinear equations according to the following classification.

Definition 2.2. Eq. (1.1) (or the function f) is said to be *strongly superlinear* if there exists a constant $\beta > 1$ such that

$$\frac{|f(t,x)|}{|x|^{\beta}} \leqslant \frac{|f(t,y)|}{|y|^{\beta}} \quad \text{for } |x| \leqslant |y|, \ xy > 0, \ t \geqslant t_0, \tag{2.2}$$

and it is said to be *strongly sublinear* if there exists a constant $\gamma \in (0,1)$ such that

$$\frac{|f(t,x)|}{|x|^{\gamma}} \geqslant \frac{|f(t,y)|}{|y|^{\gamma}} \quad \text{for } |x| \leqslant |y|, \ xy > 0, \ t \geqslant t_0. \tag{2.3}$$

If (2.2) holds with $\beta = 1$, then (1.1) is called *superlinear* and if (2.3) holds with $\gamma = 1$, then (1.1) is called *sublinear*.

Lemma 2.3. *Condition (1.4) implies that*

$$|f(t,x)| \leqslant |f(t,y)| \quad \text{for } |x| \leqslant |y|, \ xy \geqslant 0, \ t \geqslant t_0. \tag{2.4}$$

Proof. Assume (1.4). Suppose $|x| \leqslant |y|$ and $xy \geqslant 0$. Then either $0 \leqslant x \leqslant y$ and thus $0 \leqslant f(t,x) \leqslant f(t,y)$ so that $|f(t,x)| \leqslant |f(t,y)|$ for all $t \geqslant t_0$, or $y \leqslant x \leqslant 0$ and thus $f(t,y) \leqslant f(t,x) \leqslant 0$ so that again $|f(t,x)| \leqslant |f(t,y)|$ for all $t \geqslant t_0$. Thus (2.4) holds. □

The following lemma will be used throughout.

Lemma 2.4. *Suppose x solves (1.1) and is of one sign on $[t_0,\infty)$. Let $u, v, t \geqslant t_0$. Then*

$$|x|^{\Delta}(v) = x^{\Delta}(v)\operatorname{sgn} x(v) \tag{2.5}$$

and

$$|x|(t) = |x|(u) + a(v)|x|^{\Delta}(v)\int_u^t \frac{\Delta s}{a(s)} - \int_u^t \frac{1}{a(s)}\int_v^s |f(\tau, x(\sigma(\tau)))|\Delta\tau\Delta s. \tag{2.6}$$

Proof. Let $u, v, t \in \mathbb{T}$ with $u, v, t \geqslant t_0$. Let $s \in \mathbb{T}$ with $s \geqslant t_0$. Integrate (1.1) from v to s and divide the resulting equation by $a(s)$. Now integrate the resulting equation from u to t and multiply the resulting equation with $\operatorname{sgn} x(v)$. Then observe $\operatorname{sgn} x(v) = \operatorname{sgn} x(t)$, $|x(v)| = x(v)\operatorname{sgn} x(v)$, $|x(t)| = x(t)\operatorname{sgn} x(t)$, and so (1.4) gives

$$|x|(t) = |x|(u) + a(v)x^{\Delta}(v)\operatorname{sgn} x(v)\int_u^t \frac{\Delta s}{a(s)} - \int_u^t \frac{1}{a(s)}\int_v^s |f(\tau, x(\sigma(\tau)))|\Delta\tau\Delta s. \tag{2.7}$$

Now differentiate (2.7) with respect to t and then plug in $t = v$ to obtain (2.5). Finally use (2.5) in (2.7) to arrive at (2.6). □

3. Criteria under condition (1.2)

In this section, we give some new oscillation criteria for Eq. (1.1) when condition (1.2) holds. We let

$$A(t) := \int_{t_0}^t \frac{\Delta s}{a(s)} \quad \text{for } t \geqslant t_0.$$

The following simple consequence of Lemma 2.4 will be used throughout this section.

Lemma 3.1. *Assume (1.2). Suppose x solves (1.1) and is of one sign on $[t_0,\infty)$. Then on $[t_0,\infty)$,*

$$|x|^{\Delta} \geqslant 0, \quad \textit{hence } |x| \textit{ is increasing}. \tag{3.1}$$

Moreover, pick any $t_1 > t_0$ and let

$$\tilde{c} = x(t_0) \quad \textit{and} \quad c^* = \left\{\frac{|x(t_0)|}{A(t_1)} + a(t_0)|x|^{\Delta}(t_0)\right\}\operatorname{sgn} x(t_0).$$

Then

$$|x| \geqslant |\tilde{c}| \quad \textit{on } [t_0,\infty), \quad \textit{where } \tilde{c}x > 0 \tag{3.2}$$

and

$$|x| \leqslant |c^* A| \quad \textit{on } [t_1,\infty), \quad \textit{where } c^* A x > 0. \tag{3.3}$$

Proof. Using (2.6) with $t \geqslant v = u \geqslant t_0$, we find

$$|x(t)| \leqslant |x(u)| + a(u)|x|^{\Delta}(u)\int_u^t \frac{\Delta s}{a(s)} \quad \text{for all } t \geqslant t_0,$$

which is a contradiction to (1.2) when $|x|^{\Delta}(u) < 0$. This completes the proof of (3.1). Next, (3.2) follows from (3.1). Finally, using (2.6) with $t \geqslant t_1$ and $u = v = t_0$, we find

$$|x(t)| \leqslant |x(t_0)| + a(t_0)|x|^{\Delta}(t_0)A(t) = \left\{\frac{|x(t_0)|}{A(t)} + a(t_0)|x|^{\Delta}(t_0)\right\}A(t) \leqslant |c^*A(t)|$$

so that (3.3) follows. □

The first oscillation result about (1.1) is immediate.

Theorem 3.2. *Assume (1.2). If*

$$\int_{t_0}^{\infty} |f(\tau, c)|\Delta\tau = \infty \quad \text{for all } c \neq 0, \tag{3.4}$$

then Eq. (1.1) is oscillatory.

Proof. Differentiating (2.6) with respect to t and then letting $t = t_0$ and using (3.2) and (2.4), we find, for all $v \geqslant t_0$,

$$|x|^{\Delta}(t_0) \geqslant \frac{1}{a(t_0)}\int_{t_0}^{v} |f(\tau, x(\sigma(\tau)))|\Delta\tau \geqslant \frac{1}{a(t_0)}\int_{t_0}^{v} |f(\tau, \tilde{c})|\Delta\tau,$$

which contradicts (3.4) and completes the proof. □

The next result deals with the oscillation of all bounded solutions of Eq. (1.1).

Theorem 3.3. *Assume (1.2). If*

$$\int_{t_0}^{\infty} \frac{1}{a(s)}\int_{s}^{\infty} |f(\tau, c)|\Delta\tau\Delta s = \infty \quad \text{for all } c \neq 0, \tag{3.5}$$

then all bounded solutions of Eq. (1.1) are oscillatory.

Proof. Let x be a bounded nonoscillatory solution of (1.1) such that x is of one sign on $[t_0, \infty)$. Using (3.2) and (2.4), we obtain

$$|f(\tau, x(\sigma(\tau)))| \geqslant |f(\tau, \tilde{c})| \quad \text{for all } \tau \geqslant t_0.$$

Thus, using (2.6) with $v \geqslant t \geqslant t_0 = u$, together with (3.1), we find

$$|x(t)| \geqslant \int_{t_0}^{t} \frac{1}{a(s)}\int_{s}^{v} |f(\tau, x(\sigma(\tau)))|\Delta\tau\Delta s \geqslant \int_{t_0}^{t} \frac{1}{a(s)}\int_{s}^{v} |f(\tau, \tilde{c})|\Delta\tau\Delta s,$$

which, since x is bounded, contradicts (3.5) and completes the proof. □

Theorem 3.4. *Assume (1.2). Suppose (1.1) is superlinear. If*

$$\limsup_{t\to\infty}\left\{A(t)\int_{t}^{\infty} |f(\tau, c)|\Delta\tau\right\} > |c| \quad \text{for all } c \neq 0, \tag{3.6}$$

then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1) such that x is of one sign on $[t_0, \infty)$. Using (3.2) and (2.2) (with $\beta = 1$), we obtain

$$\frac{|f(\tau, x(\sigma(\tau)))|}{|x(\sigma(\tau))|} \geqslant \frac{|f(\tau, \tilde{c})|}{|\tilde{c}|} \quad \text{for all } \tau \geqslant t_0.$$

Thus, using (2.6) with $v \geqslant t \geqslant t_0 = u$, along with (3.1), we find

$$\begin{aligned}|x(t)| &\geqslant \int_{t_0}^{t} \frac{1}{a(s)}\int_{s}^{v} |f(\tau, x(\sigma(\tau)))|\Delta\tau\Delta s \geqslant \int_{t_0}^{t} \frac{1}{a(s)}\int_{s}^{v} \frac{|f(\tau, \tilde{c})|}{|\tilde{c}|}|x(\sigma(\tau))|\Delta\tau\Delta s \\ &\geqslant \int_{t_0}^{t} \frac{1}{a(s)}\int_{t}^{v} \frac{|f(\tau, \tilde{c})|}{|\tilde{c}|}|x(\sigma(\tau))|\Delta\tau\Delta s \geqslant \int_{t_0}^{t} \frac{1}{a(s)}\int_{t}^{v} \frac{|f(\tau, \tilde{c})|}{|\tilde{c}|}|x(t)|\Delta\tau\Delta s\end{aligned}$$

and hence

$$|\tilde{c}| \geqslant A(t)\int_{t}^{v} |f(\tau, \tilde{c})|\Delta\tau,$$

which contradicts (3.6) and completes the proof. □

Next, we present the following result for the strongly superlinear Eq. (1.1).

Theorem 3.5. *Assume (1.2). Suppose (1.1) is strongly superlinear. If*

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \int_s^{\infty} |f(\tau, c)| \Delta\tau \Delta s = \infty \quad \text{for all } c \neq 0, \tag{3.7}$$

then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1) such that x is of one sign on $[t_0, \infty)$. Using (3.2) and (2.2) (with $\beta > 1$), we obtain

$$\frac{|f(\tau, x(\sigma(\tau)))|}{|x(\sigma(\tau))|^{\beta}} \geqslant \frac{|f(\tau, \tilde{c})|}{|\tilde{c}|^{\beta}} \quad \text{for all } \tau \geqslant t_0.$$

Thus, differentiating (2.6) with respect to t and using (3.1) and Lemma 2.1 (the inequality on the left-hand side), we find, for all $v \geqslant t$,

$$\begin{aligned} |x|^{\Delta}(t) &\geqslant \frac{1}{a(t)} \int_t^{v} |f(\tau, x(\sigma(\tau)))| \Delta\tau \geqslant \frac{1}{a(t)} \int_t^{v} \frac{|f(\tau, \tilde{c})|}{|\tilde{c}|^{\beta}} |x(\sigma(\tau))|^{\beta} \Delta\tau \\ &\geqslant \frac{1}{a(t)} \int_t^{v} \frac{|f(\tau, \tilde{c})|}{|\tilde{c}|^{\beta}} \Delta\tau |x(\sigma(t))|^{\beta} \geqslant \frac{1}{a(t)} \int_t^{v} \frac{|f(\tau, \tilde{c})|}{|\tilde{c}|^{\beta}} \Delta\tau \frac{(\beta - 1)|x|^{\Delta}(t)}{-(|x|^{1-\beta})^{\Delta}(t)} \end{aligned}$$

and hence

$$-(|x|^{1-\beta})^{\Delta}(t) \geqslant \frac{\beta - 1}{|\tilde{c}|^{\beta} a(t)} \int_t^{v} |f(\tau, \tilde{c})| \Delta\tau.$$

Integrating this inequality from t_0 to $t \geqslant t_0$, we obtain

$$|x(t_0)|^{1-\beta} \geqslant |x(t)|^{1-\beta} + \frac{\beta - 1}{|\tilde{c}|^{\beta}} \int_{t_0}^{t} \frac{1}{a(s)} \int_s^{v} |f(\tau, \tilde{c})| \Delta\tau \Delta s \geqslant \frac{\beta - 1}{|\tilde{c}|^{\beta}} \int_{t_0}^{t} \frac{1}{a(s)} \int_s^{v} |f(\tau, \tilde{c})| \Delta\tau \Delta s,$$

which contradicts (3.7) and completes the proof. □

Finally, for the strongly sublinear Eq. (1.1), we have the following result.

Theorem 3.6. *Assume (1.2). Suppose (1.1) is strongly sublinear. If*

$$\int_{t_0}^{\infty} |f(\tau, cA(\tau))| \Delta\tau = \infty \quad \text{for all } c \neq 0, \tag{3.8}$$

then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1) such that x is of one sign on $[t_0, \infty)$. Using (3.3) and (2.3) (with $0 < \gamma < 1$), we obtain

$$\frac{|f(\tau, x(\tau))|}{|x(\tau)|^{\gamma}} \geqslant \frac{|f(\tau, c^* A(\tau))|}{|c^* A(\tau)|^{\gamma}} \quad \text{for all } \tau \geqslant t_1.$$

Thus, using (2.6) with $u = t_0$ and $v \geqslant t \geqslant t_1$, we find

$$\begin{aligned} |x(t)| &\geqslant \int_{t_0}^{t} \frac{1}{a(s)} \int_s^{v} |f(\tau, x(\sigma(\tau)))| \Delta\tau \Delta s \geqslant \int_{t_0}^{t} \frac{1}{a(s)} \int_t^{v} |f(\tau, x(\sigma(\tau)))| \Delta\tau \Delta s \\ &= A(t) \int_t^{v} |f(\tau, x(\sigma(\tau)))| \Delta\tau \geqslant A(t) \int_t^{v} |f(\tau, x(\tau))| \Delta\tau \\ &\geqslant A(t) \int_t^{v} \frac{|f(\tau, c^* A(\tau))|}{|c^* A(\tau)|^{\gamma}} |x(\tau)|^{\gamma} \Delta\tau \end{aligned}$$

(where we have used (3.1) and (2.4) in the second last inequality) and hence

$$\left|\frac{x(t)}{A(t)}\right| \geqslant z(t), \quad \text{where } z(t) := |c^*|^{-\gamma} \int_t^{v} |f(\tau, c^* A(\tau))| \left|\frac{x(\tau)}{A(\tau)}\right|^{\gamma} \Delta\tau.$$

Thus, using Lemma 2.1 (the inequality on the right-hand side),

$$\begin{aligned} -|z|^{\Delta}(\tau) = -z^{\Delta}(\tau) &= |c^*|^{-\gamma} |f(\tau, c^* A(\tau))| \left|\frac{x(\tau)}{A(\tau)}\right|^{\gamma} \\ &\geqslant |c^*|^{-\gamma} |f(\tau, c^* A(\tau))| |z(\tau)|^{\gamma} \geqslant |c^*|^{-\gamma} |f(\tau, c^* A(\tau))| \frac{(1 - \gamma)(-|z|^{\Delta}(\tau))}{-(|z|^{1-\gamma})^{\Delta}(\tau)} \end{aligned}$$

and hence

$$-(|z|^{1-\gamma})^{\Delta}(\tau) \geqslant \frac{1-\gamma}{|c^*|^{\gamma}}|f(\tau, c^*A(\tau))|.$$

Integrating this inequality from t_1 to $t \geqslant t_1$, we obtain

$$|z(t_1)|^{1-\gamma} \geqslant |z(t)|^{1-\gamma} + \frac{1-\gamma}{|c^*|^{\gamma}} \int_{t_1}^{t} |f(\tau, c^*A(\tau))|\Delta\tau \geqslant \frac{1-\gamma}{|c^*|^{\gamma}} \int_{t_1}^{t} |f(\tau, c^*A(\tau))|\Delta\tau,$$

which contradicts (3.8) and completes the proof. □

4. Criteria under condition (1.3)

The purpose of this section is to present criteria for the oscillation of Eq. (1.1) when f is either strongly superlinear or strongly sublinear and when (1.3) holds. We let

$$\widetilde{A}(t) := \int_t^{\infty} \frac{\Delta s}{a(s)} \quad \text{for } t \geqslant t_0.$$

The following consequence of Lemma 2.4 will be used throughout this section.

Lemma 4.1. *Assume (1.3). Suppose x solves (1.1) and is of one sign on $[t_0, \infty)$. Then either on $[t_0, \infty)$*

$$|x|^{\Delta} \geqslant 0, \quad \textit{hence } |x| \textit{ is increasing} \tag{4.1}$$

or there exists $t_2 > t_0$ such that on $[t_2, \infty)$

$$|x|^{\Delta} \leqslant 0, \quad \textit{hence } |x| \textit{ is decreasing}. \tag{4.2}$$

Moreover, let

$$\bar{c} = \left\{|x(t_0)| + a(t_0)|x^{\Delta}(t_0)|\widetilde{A}(t_0)\right\}\operatorname{sgn} x(t_0) \quad \textit{and} \quad \hat{c} = \begin{cases} \frac{x(t_0)}{\widetilde{A}(t_0)} & \textit{if } (4.1) \textit{ holds}, \\ a(t_2)|x^{\Delta}(t_2)|\operatorname{sgn} x(t_0) & \textit{if } (4.2) \textit{ holds}. \end{cases}$$

Then

$$|x| \leqslant |\bar{c}| \quad \textit{on } [t_0, \infty), \quad \textit{where } \bar{c}x > 0 \tag{4.3}$$

and

$$|x| \geqslant |\hat{c}\widetilde{A}| \quad \textit{on } [t_2, \infty), \quad \textit{where } \hat{c}\widetilde{A}x > 0. \tag{4.4}$$

Proof. Assume that (4.1) does not hold. Then there exists $t_2 > t_0$ such that $|x|^{\Delta}(t_2) < 0$. Now, differentiating (2.6) and using $v = t_2$, we find

$$|x|^{\Delta}(t) = \frac{a(t_2)|x|^{\Delta}(t_2)}{a(t)} - \frac{1}{a(t)} \int_{t_2}^{t} |f(\tau, x(\sigma(\tau)))|\Delta\tau \leqslant \frac{a(t_2)}{a(t)}|x|^{\Delta}(t_2) \leqslant 0$$

for all $t \geqslant t_2$, which proves (4.2). Next, using (2.6) with $v = u = t_0 \leqslant t$ and taking into account (2.5), we find

$$|x(t)| \leqslant |x(t_0)| + a(t_0)|x|^{\Delta}(t_0) \int_{t_0}^{t} \frac{\Delta s}{a(s)} \leqslant |x(t_0)| + a(t_0)|x^{\Delta}(t_0)| \int_{t_0}^{t} \frac{\Delta s}{a(s)} \leqslant |\bar{c}| \quad \text{for all } t \geqslant t_0,$$

which proves (4.3). Finally, to prove (4.4), we consider two cases: If (4.1) holds, then

$$|x(t)| \geqslant |x(t_0)| = \left|\frac{x(t_0)}{\widetilde{A}(t_0)}\right|\widetilde{A}(t_0) = |\hat{c}|\widetilde{A}(t_0) \geqslant |\hat{c}|\widetilde{A}(t).$$

If (4.2) holds, then, using (2.6) with $t \geqslant u \geqslant t_2 = v$, we find

$$|x(u)| = |x(t)| - a(t_2)|x|^{\Delta}(t_2) \int_u^t \frac{\Delta s}{a(s)} + \int_u^t \frac{1}{a(s)} \int_{t_2}^{s} |f(\tau, x(\sigma(\tau)))|\Delta\tau\Delta s \geqslant |\hat{c}| \int_u^t \frac{\Delta s}{a(s)},$$

and letting $t \to \infty$ shows $|x(u)| \geqslant |\hat{c}|\widetilde{A}(u)$ for all $u \geqslant t_2$. □

Theorem 4.2. *Assume* (1.3). *If*

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \int_{t_0}^{s} |f(\tau, c\widetilde{A}(\sigma(\tau)))| \Delta\tau \Delta s = \infty \quad \text{for all } c \neq 0, \tag{4.5}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1) such that x is of one sign on $[t_0, \infty)$. Using (4.4) and (2.4), we obtain

$$|f(\tau, x(\sigma(\tau)))| \geqslant |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))| \quad \text{for all } \tau \geqslant t_2.$$

Thus, using (2.6) with $u = v = t_2 \leqslant t$, we find

$$\begin{aligned} |x(t)| &= |x(t_2)| + a(t_2)|x|^{\Delta}(t_2) \int_{t_2}^{t} \frac{\Delta s}{a(s)} - \int_{t_2}^{t} \frac{1}{a(s)} \int_{t_2}^{s} |f(\tau, x(\sigma(\tau)))| \Delta\tau \Delta s \\ &\leqslant |x(t_2)| + a(t_2)|x|^{\Delta}(t_2) \int_{t_2}^{t} \frac{\Delta s}{a(s)} - \int_{t_2}^{t} \frac{1}{a(s)} \int_{t_2}^{s} |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))| \Delta\tau \Delta s, \end{aligned}$$

which contradicts (4.5) and completes the proof. □

For the strongly superlinear Eq. (1.1), we present the following result.

Theorem 4.3. *Assume* (1.3). *Suppose* (1.1) *is strongly superlinear. If*

$$\int_{t_0}^{\infty} |f(\tau, c\widetilde{A}(\sigma(\tau)))| \Delta\tau = \infty \quad \text{for all } c \neq 0, \tag{4.6}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1) such that x is of one sign on $[t_0, \infty)$. Using (4.4) and (2.2) (with $\beta > 1$), we obtain

$$\frac{|f(\tau, x(\sigma(\tau)))|}{|x(\sigma(\tau))|^{\beta}} \geqslant \frac{|f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))|}{|\hat{c}\widetilde{A}(\sigma(\tau))|^{\beta}} \quad \text{for all } \tau \geqslant t_2.$$

Thus, using (2.6) with $t \geqslant u \geqslant t_2 = v$, we find

$$\begin{aligned} |x(u)| &= |x(t)| - a(t_2)|x|^{\Delta}(t_2) \int_{u}^{t} \frac{\Delta s}{a(s)} + \int_{u}^{t} \frac{1}{a(s)} \int_{t_2}^{s} |f(\tau, x(\sigma(\tau)))| \Delta\tau \Delta s \\ &\geqslant -a(t_2)|x|^{\Delta}(t_2) \int_{u}^{t} \frac{\Delta s}{a(s)} + \int_{u}^{t} \frac{1}{a(s)} \int_{t_2}^{u} |f(\tau, x(\sigma(\tau)))| \Delta\tau \Delta s \end{aligned}$$

so that, with (2.4) and $b = a(t_2)|x^{\Delta}(t_2)|$,

$$|x(u)| \geqslant b\widetilde{A}(u) + \widetilde{A}(u) \int_{t_2}^{u} |f(\tau, x(\sigma(\tau)))| \Delta\tau \geqslant b\widetilde{A}(u) + \widetilde{A}(u) \int_{t_2}^{u} \frac{|f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))|}{|\hat{c}\widetilde{A}(\sigma(\tau))|^{\beta}} |x(\sigma(\tau))|^{\beta} \Delta\tau$$

and hence

$$\left|\frac{x(u)}{\widetilde{A}(u)}\right| \geqslant w(u), \quad \text{where } w(u) := b + |\hat{c}|^{-\beta} \int_{t_2}^{u} |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))| \left|\frac{x(\sigma(\tau))}{\widetilde{A}(\sigma(\tau))}\right|^{\beta} \Delta\tau.$$

Thus, using Lemma 2.1 (the inequality on the left-hand side),

$$\begin{aligned} |w|^{\Delta}(\tau) = w^{\Delta}(\tau) &= |\hat{c}|^{-\beta} |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))| \left|\frac{x(\sigma(\tau))}{\widetilde{A}(\sigma(\tau))}\right|^{\beta} \\ &\geqslant |\hat{c}|^{-\beta} |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))| |w(\sigma(\tau))|^{\beta} \geqslant |\hat{c}|^{-\beta} |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))| \frac{(\beta - 1)|w|^{\Delta}(\tau)}{-(|w|^{1-\beta})^{\Delta}(\tau)} \end{aligned}$$

and hence

$$-(|w|^{1-\beta})^{\Delta}(\tau) \geqslant \frac{\beta - 1}{|\hat{c}|^{\beta}} |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))|.$$

Integrating this inequality from t_2 to $t \geqslant t_2$, we obtain

$$|w(t_2)|^{1-\beta} \geqslant |w(t)|^{1-\beta} + \frac{\beta-1}{|\hat{c}|^{\beta}} \int_{t_2}^{t} |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))|\Delta\tau \geqslant \frac{\beta-1}{|\hat{c}|^{\beta}} \int_{t_2}^{t} |f(\tau, \hat{c}\widetilde{A}(\sigma(\tau)))|\Delta\tau,$$

which contradicts (4.6) and completes the proof. □

Finally, for the strongly sublinear Eq. (1.1), we have the following result.

Theorem 4.4. *Assume (1.3). Suppose (1.1) is strongly sublinear. If*

$$\int_{t_0}^{\infty} \frac{1}{a(s)} \int_{t_0}^{s} |f(\tau, c)|\Delta\tau\Delta s = \infty \quad \text{for all } c \neq 0, \tag{4.7}$$

then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1) such that x is of one sign on $[t_0, \infty)$. By Lemma 4.1, either (4.1) or (4.2) holds. In the case of (4.1), we have $|x(t)| \geqslant |x(t_0)|$ for all $t \geqslant t_0$ and thus, by (2.6) with $u = v = t_0 \leqslant t$, together with (2.4),

$$\begin{aligned} |x(t)| &= |x(t_0)| + a(t_0)|x|^{\Delta}(t_0) \int_{t_0}^{t} \frac{\Delta s}{a(s)} - \int_{t_0}^{t} \frac{1}{a(s)} \int_{t_0}^{s} |f(\tau, x(\sigma(\tau)))|\Delta\tau\Delta s \\ &\leqslant |x(t_0)| + a(t_0)|x|^{\Delta}(t_0) \int_{t_0}^{t} \frac{\Delta s}{a(s)} - \int_{t_0}^{t} \frac{1}{a(s)} \int_{t_0}^{s} |f(\tau, x(t_0))|\Delta\tau\Delta s, \end{aligned}$$

a contradiction to (4.7). In the case of (4.2), using (4.3) and (2.3) (with $0 < \gamma < 1$), we obtain

$$\frac{|f(\tau, x(\sigma(\tau)))|}{|x(\sigma(\tau))|^{\gamma}} \geqslant \frac{|f(\tau, \bar{c})|}{|\bar{c}|^{\gamma}} \quad \text{for all } \tau \geqslant t_2.$$

Thus, differentiating (2.6) with respect to t and letting $v = t_2 \leqslant t$, we find

$$\begin{aligned} |x|^{\Delta}(t) &= \frac{a(t_2)|x|^{\Delta}(t_2)}{a(t)} - \frac{1}{a(t)} \int_{t_2}^{t} |f(\tau, x(\sigma(\tau)))|\Delta\tau \\ &\leqslant -\frac{|\bar{c}|^{-\gamma}}{a(t)} \int_{t_2}^{t} |f(\tau, \bar{c})||x(\sigma(\tau))|^{\gamma}\Delta\tau \\ &\leqslant -\frac{|\bar{c}|^{-\gamma}}{a(t)} \int_{t_2}^{t} |f(\tau, \bar{c})|\Delta\tau|x(t)|^{\gamma}, \end{aligned}$$

where we have used again (4.2) in the last inequality. Now, using (2.1) (the inequality on the right-hand side), we obtain

$$\frac{|\bar{c}|^{-\gamma}}{a(t)} \int_{t_2}^{t} |f(\tau, \bar{c})|\Delta\tau \leqslant \frac{-|x|^{\Delta}(t)}{|x(t)|^{\gamma}} \leqslant \frac{-(|x|^{1-\gamma})^{\Delta}(t)}{1-\gamma}.$$

Integrating this inequality from t_2 to $t \geqslant t_2$, we obtain

$$|x(t_2)|^{1-\gamma} \geqslant |x(t)|^{1-\gamma} + \frac{1-\gamma}{|\bar{c}|^{\gamma}} \int_{t_2}^{t} \frac{1}{a(s)} \int_{t_2}^{s} |f(\tau, \bar{c})|\Delta\tau\Delta s \geqslant \frac{1-\gamma}{|\bar{c}|^{\gamma}} \int_{t_2}^{t} \frac{1}{a(s)} \int_{t_2}^{s} |f(\tau, \bar{c})|\Delta\tau\Delta s,$$

which contradicts (4.7) and completes the proof. □

5. Some applications

We shall apply the obtained results for Eq. (1.1) to the second-order Emden–Fowler dynamic equation on time scales

$$(ax^{\Delta})^{\Delta} + q(x^{\sigma})^{\alpha} = 0, \tag{5.1}$$

where a and q are nonnegative rd-continuous functions and α is the ratio of positive odd integers. By Theorems 3.2, 3.4 ,3.5 and 3.6, respectively, we obtain the following.

Theorem 5.1. *Let condition (1.2) hold and define $A(t) = \int_{t_0}^{t} \Delta s/a(s)$. Eq. (5.1) is oscillatory if one of the following conditions holds:*

- $\int_{t_0}^{\infty} q(\tau)\Delta\tau = \infty$, *if* $\alpha > 0$;
- $\limsup_{t\to\infty}\{A(t) \int_{t}^{\infty} q(\tau)\Delta\tau\} > c$ *for any* $c > 0$, *if* $\alpha \geqslant 1$;
- $\int_{t_0}^{\infty} \frac{1}{a(s)} \int_{s}^{\infty} q(\tau)\Delta\tau\Delta s = \infty$, *if* $\alpha > 1$;
- $\int_{t_0}^{\infty} (A(\tau))^{\alpha} q(\tau)\Delta\tau = \infty$, *if* $0 < \alpha < 1$.

By Theorems 4.2, 4.3 and 4.4, respectively, we obtain the following.

Theorem 5.2. *Let condition (1.3) hold and define* $\widetilde{A}(t) = \int_t^\infty \Delta s/a(s)$. *Eq. (5.1) is oscillatory if one of the following conditions holds:*

- $\int_{t_0}^\infty \frac{1}{a(s)} \int_{t_0}^s q(\tau)(\widetilde{A}(\sigma(\tau)))^\alpha \Delta\tau\Delta s = \infty$, *if* $\alpha > 0$;
- $\int_{t_0}^\infty (\widetilde{A}(\sigma(\tau)))^\alpha q(\tau)\Delta\tau = \infty$, *if* $\alpha > 1$;
- $\int_{t_0}^\infty \frac{1}{a(s)} \int_{t_0}^s q(\tau)\Delta\tau\Delta s = \infty$, *if* $0 < \alpha < 1$.

Remark 5.3. From the results of this paper, we can obtain some oscillation criteria for Eq. (1.1) on different types of time scales. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $x^\Delta = x'$. In this case, the results of this paper are the same as those in [20]. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $x^\Delta(t) = \Delta x(t) = x(t+1) - x(t)$. In this case, the results of this paper are the discrete analogues of those in [20]. If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then $\sigma(t) = t + h$ and $x^\Delta(t) = \Delta_h x(t) = (x(t+h) - x(t))/h$. The reformulation of our results are easy and left to the reader. We may employ other types of time scales, e.g. $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2$, and others, see [11,12]. The details are left to the reader.

Remark 5.4. The results of this paper can be extended to dynamic equations of type (1.1) with deviating arguments, e.g.

$$(ax^\Delta)^\Delta + f(t, x(\tau(t))) = 0,$$

where $\tau : \mathbb{T} \to \mathbb{T}$ with $\lim_{t\to\infty}\tau(t) = \infty$. The details are left to the reader.

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