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On the oscillation of second-order half-linear dynamic equations[1]

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We obtain some oscillation criteria for solutions to the second-order half-linear dynamic equation

$$\left(a(x^{\Delta})^{\alpha}\right)^{\Delta}(t)+q(t)x^{\alpha}(t)=0,$$

when $\int^{\infty}a^{-1/\alpha}(s)\Delta s=\infty$ or $\int^{\infty}a^{-1/\alpha}(s)\Delta s<\infty$. These criteria unify and extend known criteria for corresponding half-linear differential and difference equations. Some of our results are new even in the continuous and the discrete cases.

Keywords: dynamic equation; half-linear; oscillation; second-order

AMS Subject Classification: 34C10; 39A10

1. Introduction

Consider the second-order half-linear dynamic equation

$$\left(a(x^{\Delta})^{\alpha}\right)^{\Delta}(t)+q(t)x^{\alpha}(t)=0, \tag{1.1}$$

where a and q are real-valued, positive and rd-continuous functions on a time scale $\mathbb{T}\subset\mathbb{R}$ with $\sup\mathbb{T}=\infty$ and $\alpha\geq 1$ is the ratio of two positive odd integers.

For completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and since oscillation of solutions is our primary concern, we make the assumption that $\sup\mathbb{T}=\infty$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward and backward jump operators are defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}\quad\text{and}\quad\rho(t)=\sup\{s\in\mathbb{T}:s<t\},$$

where $\inf\emptyset=\sup\mathbb{T}$ and $\sup\emptyset=\inf\mathbb{T}$ and $\emptyset$ denotes the empty set. A point $t\in\mathbb{T}$, $t>\inf\mathbb{T}$ is said to be left-dense if $\rho(t)=t$, right-dense if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and right-scattered if $\sigma(t)>t$. A function $g:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous provided g is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The graininess function μ for a time scale $\mathbb{T}$ is defined

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by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, the notation f^σ denotes the composition $f \circ \sigma$.

We recall that a solution of equation (1.1) is said to be oscillatory on $[t_0, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory. Since $a(t) > 0$, we shall consider both the cases

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s < \infty. \tag{1.3}$$

The purpose of this paper is to obtain some time scale analogues of the results for the continuous case $\mathbb{T} = \mathbb{R}$ and the discrete case $\mathbb{T} = \mathbb{Z}$ due to Agarwal et al. [5,6]. For related results in the continuous and the discrete cases, see Refs. [1,2,4,7] and in the time scales case [3,8,11–14].

2. Preliminary results

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined to be the number (if it exists) such that for all $\varepsilon > 0$ there is a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

If the (delta) derivative $f^\Delta(t)$ exists for all $t \in \mathbb{T}$, then we say that f is (delta) differentiable on $\mathbb{T}$.

We will make use of the product and quotient rules [9, Theorem 1.20] for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two (delta) differentiable functions f and g

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}, \tag{2.1}$$

as well as of the chain rule [9, Theorem 1.90] for the derivative of the composite function $f \circ g$ for a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ and a (delta) differentiable function $g : \mathbb{T} \to \mathbb{R}$

$$(f \circ g)^\Delta = \left\{ \int_0^1 f'(g + h\mu g^\Delta)\mathrm{d}h \right\} g^\Delta. \tag{2.2}$$

For $b, c \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_b^c f^\Delta(t)\Delta t = f(c) - f(b),$$

and infinite integrals are defined as

$$\int_b^\infty f(t)\Delta t = \lim_{c\to\infty}\int_b^c f(t)\Delta t.$$

Note that in the case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) = 0, \quad f^\Delta(t) = f'(t), \quad \int_b^c f(t)\Delta t = \int_b^c f(t)\mathrm{d}t,$$

and in the case $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t+1, \quad \rho(t) = t-1, \quad \mu(t) = 1, \quad f^\Delta(t) = \Delta f(t) = f(t+1) - f(t),$$

and (if $b < c$)

$$\int_b^c f(t)\Delta t = \sum_{t=b}^{c-1} f(t).$$

For more discussion on time scales, we refer to Refs. [9,10,16].

Finally, we recall the following lemma from Ref. [15] which will be needed for the proof of one of the results in the next section.

LEMMA 2.1. *If X and Y are nonnegative and $\gamma > 1$, then*

$$X^\gamma - \gamma XY^{\gamma-1} + (\gamma - 1)Y^\gamma \geq 0,$$

where equality holds if and only if $X = Y$.

3. Main results

The following result is concerned with the oscillation of equation (1.1) when condition (1.3) holds.

THEOREM 3.1. *Let condition* (1.3) *hold. If there exists a positive nondecreasing delta differentiable function ξ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$*

$$\limsup_{t\to\infty}\int_{t_1}^t [\xi(s)q(s) - \eta^\alpha(s)\xi^\Delta(s)]\Delta s = \infty \tag{3.1}$$

and

$$\int_{t_1}^\infty \left(\frac{1}{a(s)}\int_{t_1}^s \theta^\alpha(u)q(u)\Delta u\right)^{1/\alpha} \Delta s = \infty, \tag{3.2}$$

where

$$\eta(t) = \left(\int_{t_1}^t a^{-1/\alpha}(s)\Delta s\right)^{-1} \quad \text{and} \quad \theta(t) = \int_t^\infty a^{-1/\alpha}(s)\Delta s, \tag{3.3}$$

then equation (1.1) *is oscillatory.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of equation (1.1) on $[t_0, \infty)_{\mathbb{T}}$. It suffices to discuss the case that x is eventually positive (as $-x$ also solves (1.1) if x does), say $x(t) > 0$ for $t \geq t_1 \geq t_0$. Since $a(x^\Delta)^\alpha$ is decreasing, it is eventually of one sign and hence x^Δ is eventually of one sign. Thus, we shall distinguish the following two cases:

(I) $x^\Delta(t) > 0$ for $t > t_1$; and
(II) $x^\Delta(t) < 0$ for $t > t_1$.

Case (*I*). We first note that in this case (2.2) implies

$$(x^\alpha)^\Delta = \alpha x^\Delta \int_0^1 (x + h\mu x^\Delta)^{\alpha-1} \mathrm{d}h > 0 \quad \text{on } [t_1, \infty)_{\mathbb{T}},$$

and hence x^α is increasing on $[t_1, \infty)_{\mathbb{T}}$. Now, let

$$w := \frac{\xi a(x^\Delta)^\alpha}{x^\alpha} \quad \text{on } [t_1, \infty)_{\mathbb{T}}.$$

Then, on $[t_1, \infty)_{\mathbb{T}}$ we have by (2.1) that

$$\begin{aligned} w^\Delta &= \left(\frac{\xi}{x^\alpha}\right)^\Delta \left(a(x^\Delta)^\alpha\right)^\sigma + \frac{\xi}{x^\alpha}\left(a(x^\Delta)^\alpha\right)^\Delta \\ &= -\xi q + \left(a(x^\Delta)^\alpha\right)^\sigma \left[\frac{\xi^\Delta x^\alpha - \xi(x^\alpha)^\Delta}{x^\alpha (x^\sigma)^\alpha}\right] \\ &= -\xi q + \frac{\xi^\Delta \left(a(x^\Delta)^\alpha\right)^\sigma}{(x^\sigma)^\alpha} - \frac{\xi (x^\alpha)^\Delta \left(a(x^\Delta)^\alpha\right)^\sigma}{x^\alpha (x^\sigma)^\alpha} \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\leq -\xi q + \frac{\xi^\Delta \left(a(x^\Delta)^\alpha\right)^\sigma}{(x^\sigma)^\alpha} \quad \leq -\xi q + \frac{\xi^\Delta a(x^\Delta)^\alpha}{x^\alpha} \\ &= -\xi q + a\xi^\Delta \left(\frac{x^\Delta}{x}\right)^\alpha. \end{aligned} \tag{3.5}$$

Now

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^t x^\Delta(s)\Delta s \\ &= x(t_1) + \int_{t_1}^t a^{-1/\alpha}(s)\left(a(s)(x^\Delta(s))^\alpha\right)^{1/\alpha} \Delta s \\ &\geq \left(\int_{t_1}^t a^{-1/\alpha}(s)\Delta s\right)(a(t)(x^\Delta(t))^\alpha)^{1/\alpha}, \end{aligned}$$

and thus

$$\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\alpha} \leq \frac{1}{a(t)}\left(\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s\right)^{-\alpha} = \frac{\eta^{\alpha}(t)}{a(t)} \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.6}$$

Using (3.6) in (3.5), we have

$$w^{\Delta} \leq -\xi q + \eta^{\alpha}\xi^{\Delta} \quad \text{on } [t_1, \infty)_{\mathbb{T}}. \tag{3.7}$$

Integrating (3.7) from t_1 to t, we obtain

$$0 < w(t) \leq w(t_1) - \int_{t_1}^{t}\left[\xi(s)q(s) - \eta^{\alpha}(s)\xi^{\Delta}(s)\right]\Delta s,$$

which gives a contradiction using (3.1).

Case (*II*). For $s \geq t \geq t_1$, we have

$$a(s)(-x^{\Delta}(s))^{\alpha} \geq a(t)(-x^{\Delta}(t))^{\alpha},$$

and hence

$$-x^{\Delta}(s) \geq a^{-1/\alpha}(s)a^{1/\alpha}(t)(-x^{\Delta}(t)). \tag{3.8}$$

Integrating (3.8) from $t \geq t_1$ to $u \geq t$ and letting $u \to \infty$ yields

$$x(t) \geq \left(\int_{t}^{\infty} a^{-1/\alpha}(s)\Delta s\right)(-a^{1/\alpha}(t)x^{\Delta}(t)) = -\theta(t)a^{1/\alpha}(t)x^{\Delta}(t) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}},$$

and thus

$$\begin{aligned}(x(t))^{\alpha} &\geq -(\theta(t))^{\alpha}a(t)(x^{\Delta}(t))^{\alpha} \geq -(\theta(t))^{\alpha}a(t_1)(x^{\Delta}(t_1))^{\alpha} = b(\theta(t))^{\alpha} \\ &\text{for } t \in [t_1, \infty)_{\mathbb{T}},\end{aligned} \tag{3.9}$$

with $b := -a(t_1)(x^{\Delta}(t_1))^{\alpha} > 0$. Using (3.9) in equation (1.1), we find

$$-\left(a(x^{\Delta})^{\alpha}\right)^{\Delta}(t) \geq b\theta^{\alpha}(t)q(t) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.10}$$

Integrating (3.10) from t_1 to t, we have

$$-a(t)(x^{\Delta}(t))^{\alpha} \geq -a(t_1)(x^{\Delta}(t_1))^{\alpha} + b\int_{t_1}^{t}\theta^{\alpha}(s)q(s)\Delta s \geq b\int_{t_1}^{t}\theta^{\alpha}(s)q(s)\Delta s,$$

so that

$$-x^{\Delta}(t) \geq \left(\frac{b}{a(t)}\int_{t_1}^{t}\theta^{\alpha}(s)q(s)\Delta s\right)^{1/\alpha}. \tag{3.11}$$

Integrating (3.11) from t_1 to t, we obtain

$$\infty > x(t_1) \geq -x(t) + x(t_1) \geq \int_{t_1}^{t} \left(\frac{b}{a(s)} \int_{t_1}^{s} \theta^{\alpha}(u) q(u) \Delta u \right)^{1/\alpha} \Delta s \to \infty \quad \text{as} \quad t \to \infty$$

by (3.2), a contradiction. This completes the proof. □

We note that the proof of Theorem 3.1 is presented in a form that it contains the case when condition (1.2) holds. From the proof of Theorem 3.1, we can easily see that if condition (1.2) holds, then Case (II) is disregarded and the only case valid is Case (I). For if a solution x of (1.1) satisfies $x(t) > 0$ for all $t \geq t_1 \geq t_0$ and $x^{\Delta}(t_2) < 0$ for some $t_2 \geq t_1$, then $a(t)(x^{\Delta})^{\alpha} \times (t) \leq a(t_2)(x^{\Delta})^{\alpha}(t_2)$ implies $(a(t))^{1/\alpha} x^{\Delta}(t) \leq (a(t_2))^{1/\alpha} x^{\Delta}(t_2) =: c < 0$ for all $t \geq t_2$ and thus

$$0 < x(t) \leq x(t_2) + c \int_{t_2}^{t} (a(s))^{-1/\alpha} \Delta s \to -\infty \quad \text{as} \quad t \to \infty$$

due to (1.2), a contradiction. Thus, we have the following result.

THEOREM 3.2. *Let condition* (1.2) *hold. If there exists a positive nondecreasing delta differentiable function* ξ *such that for every* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *condition* (3.1) *holds, then equation* (1.1) *is oscillatory.*

Next, we present the following result.

THEOREM 3.3. *Let conditions* (1.3) *and* (3.2) *hold. If there exists a nondecreasing positive delta differentiable function* ξ *such that for* $t_1 \in [t_0, \infty)_{\mathbb{T}}$

$$\limsup_{t \to \infty} \int_{t_1}^{t} \left[\xi(s) q(s) - \frac{a(s)}{(\alpha+1)^{\alpha+1}} \left(\frac{(\xi^{\Delta}(s))^{\alpha+1}}{\xi^{\alpha}(s)} \right) \right] \Delta s = \infty, \tag{3.12}$$

then equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_1 \geq t_0$. Proceeding as in the proof of Theorem 3.1, we obtain the Cases (I) and (II). The proof of Case (II) is similar to that of Case (II) in the proof of Theorem 3.1 and hence is omitted. Thus, we only consider Case (I) and define w as in the proof of Theorem 3.1 and obtain (3.4). Now from (2.2)

$$(x^{\alpha})^{\Delta} = \alpha x^{\Delta} \int_0^1 (x + \mu h x^{\Delta})^{\alpha-1} \mathrm{d}h \geq \alpha x^{\Delta} \int_0^1 x^{\alpha-1} \mathrm{d}h = \alpha x^{\alpha-1} x^{\Delta}. \tag{3.13}$$

Using (3.13) in (3.4), we obtain on $[t_1, \infty)_{\mathbb{T}}$ that

$$\begin{aligned} w^{\Delta} &\leq -\xi q + \frac{\xi^{\Delta}}{\xi^{\sigma}} w^{\sigma} - \alpha \xi \left(\frac{x^{\Delta}}{x} \right) \frac{\left(a(x^{\Delta})^{\alpha} \right)^{\sigma}}{(x^{\sigma})^{\alpha}} \\ &= -\xi q + \frac{\xi^{\Delta}}{\xi^{\sigma}} w^{\sigma} - \alpha \frac{\xi}{\xi^{\sigma}} \left(\frac{x^{\Delta}}{x} \right) w^{\sigma}. \end{aligned} \tag{3.14}$$

Since $a(x^{\Delta})^{\alpha}$ is decreasing and x^{α} is increasing, we have

$$\frac{w^{\sigma}}{\xi^{\sigma}} = \frac{\left(a(x^{\Delta})^{\alpha} \right)^{\sigma}}{(x^{\alpha})^{\sigma}} \leq \frac{a(x^{\Delta})^{\alpha}}{x^{\alpha}} = \frac{w}{\xi},$$

and therefore

$$\frac{x^{\Delta}}{x} = \xi^{-1/\alpha} a^{-1/\alpha} w^{1/\alpha} \geq \xi^{-1/\alpha} a^{-1/\alpha} \left(\frac{\xi}{\xi^{\sigma}} \right)^{1/\alpha} (w^{\sigma})^{1/\alpha} \quad \text{on } [t_1, \infty)_{\mathbb{T}}. \tag{3.15}$$

Using (3.15) in (3.14), we find

$$w^{\Delta} \leq -\xi q + \frac{\xi^{\Delta}}{\xi^{\sigma}} w^{\sigma} - \alpha a^{-1/\alpha} \frac{\xi}{(\xi^{\sigma})^{1+1/\alpha}} (w^{\sigma})^{1+1/\alpha} \quad \text{on } [t_1, \infty)_{\mathbb{T}}. \tag{3.16}$$

Now set

$$X = (\alpha \xi)^{\alpha/(\alpha+1)} \frac{a^{-1/(\alpha+1)}}{\xi^{\sigma}} w^{\sigma},$$

and

$$Y = \left(\frac{\alpha}{\alpha+1} \right)^{\alpha} \left(\frac{\xi^{\Delta}}{\xi^{\sigma}} \right)^{\alpha} \left[\alpha^{-\alpha/(\alpha+1)} \xi^{-\alpha/(\alpha+1)} \xi^{\sigma} a^{1/(\alpha+1)} \right]^{\alpha}$$

in Lemma 2.1 with $\gamma = (\alpha + 1)/\alpha > 1$ to conclude that

$$\alpha a^{-1/\alpha} \frac{\xi}{(\xi^{\sigma})^{1+1/\alpha}} (w^{\sigma})^{1+1/\alpha} - \frac{\xi^{\Delta}}{\xi^{\sigma}} w^{\sigma} + \frac{a}{(\alpha+1)^{\alpha+1}} \frac{(\xi^{\Delta})^{\alpha+1}}{\xi^{\alpha}} \geq 0,$$

and therefore by (3.16)

$$w^{\Delta} \leq -\xi q + \frac{a}{(\alpha+1)^{\alpha+1}} \frac{(\xi^{\Delta})^{\alpha+1}}{\xi^{\alpha}} \quad \text{on } [t_1, \infty)_{\mathbb{T}}. \tag{3.17}$$

Integrating (3.17) from t_1 to t, we have

$$w(t) \leq w(t_1) - \int_{t_1}^{t} \left[\xi(s) q(s) - \frac{a(s)}{(\alpha+1)^{\alpha+1}} \frac{(\xi^{\Delta}(s))^{\alpha+1}}{\xi^{\alpha}(s)} \right] \Delta s. \tag{3.18}$$

Taking the lim sup of both sides of (3.18) as $t \to \infty$ and using (3.12), we obtain a contradiction to the fact that $w(t) > 0$ for $t \geq t_1$. This completes the proof. □

When condition (1.2) holds, we have the following result.

THEOREM 3.4. *Let condition* (1.2) *hold. If there exists a nondecreasing positive delta differentiable function* ξ *such that for any* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *condition* (3.12) *holds, then equation* (1.1) *is oscillatory.*

Finally, we present the following interesting result.

THEOREM 3.5. *Let conditions* (1.3) *and* (3.2) *hold. If there exists a positive delta differentiable function* ξ *such that for every* $t_1 \in [t_0, \infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\xi(s)q(s)-\left(\frac{a^{1/\alpha}(s)}{4\alpha}\right)\left(\frac{(\xi^{\Delta}(s))^2}{\xi(s)}\right)(\eta^{\sigma}(s))^{\alpha-1}\right]\Delta s=\infty, \tag{3.19}$$

where η *is as in* (3.3)*, then equation* (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1), say $x(t) > 0$ for $t \geq t_1 \geq t_0 \geq 0$. Proceeding as in the proof of Theorem 3.3, we obtain (3.16) which can be written as

$$w^{\Delta} \leq -\xi q + \frac{\xi^{\Delta}}{\xi^{\sigma}} w^{\sigma} - \alpha a^{-1/\alpha}\frac{\xi}{(\xi^{\sigma})^{1+1/\alpha}}(w^{\sigma})^{1/\alpha-1}(w^{\sigma}2 \quad \text{on } [t_1, \infty)_{\mathbb{T}}. \tag{3.20}$$

Now inequality (3.6), i.e.,

$$\frac{x}{x^{\Delta}} \geq \frac{a^{1/\alpha}}{\eta},$$

implies on $[t_1, \infty)_{\mathbb{T}}$ that

$$w^{1/\alpha-1} = \xi^{1/\alpha-1}a^{1/\alpha-1}\left(\frac{x}{x^{\Delta}}\right)^{\alpha-1} \geq \xi^{1/\alpha-1}a^{1/\alpha-1}\frac{a^{\alpha-1/\alpha}}{\eta^{\alpha-1}} = \xi^{1/\alpha-1}\eta^{1-\alpha}. \tag{3.21}$$

Using (3.21) in (3.20), we have on $[t_1, \infty)_{\mathbb{T}}$ that

$$\begin{aligned} w^{\Delta} &\leq -\xi q + \frac{\xi^{\Delta}}{\xi^{\sigma}} w^{\sigma} - \alpha a^{-1/\alpha}\frac{\xi}{(\xi^{\sigma})^2}(\eta^{\sigma})^{1-\alpha}(w^{\sigma})^2 \\ &= -\xi q + \frac{(\xi^{\Delta})^2 a^{1/\alpha}}{4\alpha\xi(\eta^{\sigma})^{1-\alpha}} - \left(\frac{\sqrt{\alpha a^{-1/\alpha}\,\xi(\eta^{\sigma})^{1-\alpha}}}{\xi^{\sigma}} w^{\sigma} - \frac{\xi^{\Delta}}{2\sqrt{\alpha a^{-1/\alpha}\xi(\eta^{\sigma})^{1-\alpha}}}\right)^2 \\ &\leq -\xi q + \frac{a^{1/\alpha}}{4\alpha}\frac{(\xi^{\Delta})^2}{\xi}(\eta^{\sigma})^{\alpha-1}. \end{aligned}$$

Integrating both sides of this inequality from t_1 to t, taking the lim sup of the resulting inequality as $t \to \infty$ and applying condition (3.19), we obtain a contradiction to the fact that $w(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. This completes the proof. □

When condition (1.2) holds, we have the following result.

THEOREM 3.6. *Let condition* (1.2) *hold. If there exists a positive delta differentiable function* ξ *such that for any* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *condition* (3.19) *holds, then equation* (1.1) *is oscillatory.*

Example 3.7. Here, we shall reformulate the above conditions which are sufficient for the oscillation of equation (1.1) when (1.2) holds on different time scales:

If $\mathbb{T} = \mathbb{R}$, then conditions (3.1), (3.12) and (3.19), respectively, become

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\xi(s)q(s) - \xi'(s)\left(\int_{t_1}^{s} a^{-1/\alpha}(u)\mathrm{d}u\right)^{-\alpha}\right]\mathrm{d}s = \infty, \tag{3.22}$$

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\xi(s)q(s) - \frac{a(s)}{(\alpha+1)^{\alpha+1}}\frac{(\xi'(s))^{\alpha+1}}{\xi^{\alpha}(s)}\right]\mathrm{d}s = \infty, \tag{3.23}$$

and

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\xi(s)q(s) - \frac{a^{1/\alpha}(s)}{4\alpha}\frac{(\xi'(s))^2}{\xi(s)}\left(\int_{t_1}^{s} a^{-1/\alpha}(u)\mathrm{d}u\right)^{1-\alpha}\right]\mathrm{d}s = \infty. \tag{3.24}$$

We remark that while (3.23) is well known, e.g. [5], conditions (3.22) and (3.24) are new.

If $\mathbb{T} = \mathbb{Z}$, then conditions (3.1), (3.12) and (3.19), respectively, become

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t}\left[\xi(s)q(s) - (\Delta\xi(s))\left(\sum_{u=t_1}^{s-1} a^{-1/\alpha}(u)\right)^{-\alpha}\right] = \infty, \tag{3.25}$$

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t}\left[\xi(s)q(s) - \frac{a(s)}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\xi(s))^{\alpha+1}}{(\xi(s))^{\alpha}}\right] = \infty, \tag{3.26}$$

and

$$\limsup_{t\to\infty}\sum_{s=t_1}^{t}\left[\xi(s)q(s) - \frac{a^{1/\alpha}(s)}{4\alpha}\frac{(\Delta\xi(s))^2}{\xi(s)}\left(\sum_{u=t_1}^{s} a^{-1/\alpha}(u)\right)^{1-\alpha}\right] = \infty. \tag{3.27}$$

We remark that condition (3.26) is included in [6, Theorem 2.1], while conditions (3.25) and (3.27) are new.

We may employ other types of time scales, e.g. $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}_0^2$, etc. [9,10]. The details are left to the reader.

Notes

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