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POSITIVITY OF BLOCK TRIDIAGONAL MATRICES*

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Abstract. This paper relates disconjugacy of linear Hamiltonian difference systems (LHdS) (and hence positive definiteness of certain discrete quadratic functionals) to positive definiteness of some block tridiagonal matrices associated with these systems and functionals. As a special case of a Hamiltonian system, Sturm–Liouville difference equations are considered, and analogous results are obtained for these important objects.

Key words. linear Hamiltonian difference system, Sturm–Liouville difference equation, block tridiagonal matrix, discrete quadratic functional

AMS subject classifications. 15A09, 15A63, 39A10, 39A12

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1. Introduction.

The aim of this paper is to relate disconjugacy of linear Hamiltonian difference systems (LHdS)

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k \tag{H}$$

and hence positivity of the discrete quadratic functional

$$\mathcal{F}(x,u) = \sum_{k=0}^{N} \left\{ u_k^T B_k u_k + x_{k+1}^T C_k x_{k+1} \right\} \tag{F}$$

to positive definiteness of a certain block tridiagonal symmetric matrix associated with (H) and (F). Here A, B, C are sequences of real $n \times n$ matrices such that

$$I - A_k \text{ are invertible and } B_k, C_k \text{ are symmetric for all } k \in \mathbb{N}_0. \tag{1}$$

To introduce our problem in more detail, we first recall the relation of disconjugacy of linear Hamiltonian systems (both differential and difference) to positivity of corresponding quadratic functionals and to solvability of the associated Riccati matrix equation. A statement of this kind is usually called a Reid roundabout theorem.

Proposition 1.1 ([4, 6, 7]). *Consider the linear Hamiltonian differential system*

$$x' = A(t)x + B(t)u, \qquad u' = C(t)x - A^T(t)u, \tag{$\tilde{\mathrm{H}}$}$$

where $A, B, C : I = [a,b] \to \mathbb{R}^{n\times n}$ are continuous real $n \times n$ matrix valued functions such that

$$B(t), C(t) \textit{ are symmetric and } B(t) \textit{ is positive semidefinite for all } t \in [a,b],$$

and suppose that this system is identically normal *in I, i.e., the only solution (x,u) of ($\tilde{\mathrm{H}}$) for which $x \equiv 0$ on some nondegenerate subinterval of I is the trivial solution $(x,u) \equiv (0,0)$. Then the following statements are equivalent.*

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(i) *System* ($\tilde{\mathrm{H}}$) *is* disconjugate *in I, i.e., the $2n \times n$ matrix solution* $\binom{X}{U}$ *of* ($\tilde{\mathrm{H}}$) *given by the initial condition $X(a) = 0$, $U(a) = I$ has no* focal point *in I, i.e.*, $\det X(t) \neq 0$ *in* $(a, b]$.

(ii) *The quadratic functional*

$$(\tilde{\mathrm{F}}) \qquad \tilde{\mathcal{F}}(x, u) = \int_a^b \left[u^T(t)B(t)u(t) + x^T(t)C(t)x(t)\right] \mathrm{d}t$$

is positive for every $x, u : I \to \mathbb{R}^n$ satisfying $x' = A(t)x + B(t)u$, $x(a) = 0 = x(b)$ and $x \not\equiv 0$ in I.

(iii) *There exists a symmetric solution $Q : I \to \mathbb{R}^{n\times n}$ of the Riccati matrix differential equation*

$$(\tilde{\mathrm{R}}) \qquad Q' + A^T(t)Q + QA(t) + QB(t)Q - C(t) = 0.$$

In the last decade, a considerable effort has been made to find a discrete analogue of this statement; see [1] and the references given therein. Finally, this problem was resolved in [2] and the discrete Roundabout Theorem reads as follows.

Proposition 1.2 ([1, 2]). *Assume* (1). *Then the following statements are equivalent.*

(i) *System* (H) *is* disconjugate *in the discrete interval $J := [0, N] \cap \mathbb{N}_0$, $N \in \mathbb{N}$, i.e., the $2n \times n$ matrix solution* $\binom{X}{U}$ *of* (H) *given by the initial condition $X_0 = 0$, $U_0 = I$ has no* focal point *in $J^* := [0, N+1] \cap \mathbb{N}_0$, i.e.,*

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k \quad \textit{and} \quad D_k = X_k X_{k+1}^{\dagger}(I - A_k)^{-1}B_k \geq 0.$$

(*Here* Ker *and* † *stand for the kernel and the Moore–Penrose generalized inverse of the matrix indicated, respectively, and the matrix inequality $\geq$ stands for nonnegative definiteness.*)

(ii) *The discrete quadratic functional $\mathcal{F}$ is positive for every $(x, u) : J^* \to \mathbb{R}^n$ satisfying $\Delta x_k = A_k x_{k+1} + B_k u_k$, $x_0 = 0 = x_{N+1}$ and $x \not\equiv 0$ in J^*.*

(iii) *There exist symmetric matrices $Q : J^* \to \mathbb{R}^{n\times n}$ such that $(I + BQ)$ are invertible, $(I + BQ)^{-1}B \geq 0$ in J, and solve the discrete Riccati matrix difference equation*

$$(R) \qquad Q_{k+1} = C_k + (I - A_k^T)Q_k(I + B_kQ_k)^{-1}(I - A_k)^{-1}.$$

The main difference between continuous and discrete functionals $\tilde{\mathcal{F}}$ and $\mathcal{F}$ is that the space of x, u appearing in the discrete quadratic functional has finite dimension. This suggests investigating positivity of $\mathcal{F}$ not only via oscillation properties of (H) and solvability of (R) as given in the roundabout theorem (which are typical methods for an "infinite-dimensional" treatment), but also via linear algebra and matrix theory. The main idea of this approach can be illustrated in the case of the Sturm-Liouville equation

$$(2) \qquad -\Delta(r_k\Delta y_k) + p_k y_{k+1} = 0$$

and the corresponding quadratic functional

$$(3) \qquad \mathcal{J}(y) = \sum_{k=0}^{N} \left\{r_k(\Delta y_k)^2 + p_k y_{k+1}^2\right\}$$

as follows.

Expanding the differences in $\mathcal{J}$, for any $y = \{y_k\}_{k=0}^{N+1}$ satisfying

$$y_0 = 0 = y_{N+1}, \tag{4}$$

we have

$$\begin{aligned}
\mathcal{J}(y) &= \sum_{k=0}^{N} \left\{ (r_k + p_k) y_{k+1}^2 - 2 r_k y_k y_{k+1} + r_k y_k^2 \right\} \\
&= \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}^T \begin{pmatrix} \beta_0 & -r_1 & & \\ -r_1 & \beta_1 & \ddots & \\ & \ddots & \ddots & -r_{N-1} \\ & & -r_{N-1} & \beta_{N-1} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix},
\end{aligned}$$

where $\beta_k = r_k + p_k + r_{k+1}$. Hence $\mathcal{J}(y) > 0$ for any nontrivial $y \in \mathbb{R}^{N+2}$ satisfying (4) iff the matrix

$$\mathcal{L} := \begin{pmatrix} \beta_0 & -r_1 & & \\ -r_1 & \beta_1 & \ddots & \\ & \ddots & \ddots & -r_{N-1} \\ & & -r_{N-1} & \beta_{N-1} \end{pmatrix}$$

is positive definite.

From the elementary course of linear algebra it is known that $\mathcal{L}$ is positive definite iff all its principal minors $\Delta_1 = \beta_0$, $\Delta_2 = \beta_0\beta_1 - r_1^2, \ldots, \Delta_N = \det \mathcal{L}$ are positive. On the other hand, by (i) of Proposition 1.2 we have $\mathcal{J}(y) > 0$ for any nontrivial y satisfying (4) iff the solution $\tilde{y}$ of (2) given by the initial condition $\tilde{y}_0 = 0$, $\tilde{y}_1 = \frac{1}{r_0}$ satisfies

$$\tilde{y}_{k+1} \neq 0 \quad \text{and} \quad \delta_k := \frac{\tilde{y}_k}{r_k \tilde{y}_{k+1}} \geq 0, \qquad k = 0, \ldots, N. \tag{5}$$

Using Laplace's rule for computation of determinants, we have the formula

$$\Delta_k = \beta_{k-1}\Delta_{k-1} - r_{k-1}^2 \Delta_{k-2}. \tag{6}$$

Expanding the forward differences in (2) we have

$$y_{k+2} = \frac{1}{r_{k+1}} \left[\beta_k y_{k+1} - r_k y_k \right].$$

This recurrent formula, coupled with (6) and the initial condition $\tilde{y}_0 = 0$, $\tilde{y}_1 = \frac{1}{r_0}$, gives

$$\tilde{y}_2 = \frac{\beta_0}{r_1 r_0} = \frac{\Delta_1}{r_1 r_0}, \qquad \tilde{y}_3 = \frac{1}{r_2 r_1 r_0} \left[\beta_0\beta_1 - r_1^2 \right] = \frac{\Delta_2}{r_2 r_1 r_0},$$

and by induction

$$\tilde{y}_{k+1} = \frac{1}{r_k \ldots r_2 r_1 r_0} \left[\beta_{k-1}\Delta_{k-1} - r_{k-1}^2 \Delta_{k-2} \right] = \frac{\Delta_k}{r_k \ldots r_2 r_1 r_0}.$$

Consequently,

$$\delta_k = \frac{\tilde{y}_k}{r_k \tilde{y}_{k+1}} = \frac{\Delta_{k-1}}{\Delta_k}, \quad \Delta_0 := 1. \tag{7}$$

Now, by the Jacobi diagonalization method, there exists an $N \times N$ triangular matrix $\mathcal{M}$ such that

$$\mathcal{M}^T \mathcal{L} \mathcal{M} = \operatorname{diag}\{\delta_1, \dots, \delta_N\}.$$

From the last identity one may easily see why the quantities δ_k come to play in the definition of disconjugacy of (2).

In this paper we establish a similar identity relating the quadratic functional $\mathcal{F}$ and the matrices D_k from Proposition 1.2. This identity reveals why the matrices D_k appear in the definition of disconjugacy of (H). In particular, we find a block triangular matrix $\mathcal{M}$ such that

$$\mathcal{M}^T \mathcal{L} \mathcal{M} = \operatorname{diag}\{D_1, \dots, D_N\},$$

where $\mathcal{L}$ in this identity is the matrix representing the functional $\mathcal{F}$ (see Theorem 2.2 in the next section) and D_k are given in (i) of Proposition 1.2.

The paper is organized as follows. The next section is devoted to preliminary results. We recall some basic properties of solutions of (H), and we also show the relation between higher order Sturm–Liouville difference equations and LHdS (H). The main results of the paper, the equivalence of positive definiteness of a block tridiagonal matrix to nonnegative definiteness of certain matrices constructed via solutions of (H) (which reduce to δ_k in the scalar case) are given in section 3. In the last section we deal with LHdS (H) which correspond to higher order Sturm–Liouville equations; here the results of the previous section are simplified considerably. The statements of section 4 complement results of [3, 5].

2. Preliminary results. Subject to our general assumption (1) we consider a *linear Hamiltonian difference system* (H) and the corresponding *discrete quadratic functional* $\mathcal{F}$ defined by (F). Here, $x = \{x_k\}_{k \in \mathbb{N}_0}$ and $u = \{u_k\}_{k \in \mathbb{N}_0}$ are sequences of $\mathbb{R}^n$-vectors, and we say that such a pair (x, u) is *admissible* on J provided that

$$\Delta x_k = A_k x_{k+1} + B_k u_k \text{for all} k \in J$$

holds. An x is called admissible (on J) if there exists u such that the pair (x, u) is admissible (on J). The functional $\mathcal{F}$ is then said to be *positive definite* (and we write $\mathcal{F} > 0$) whenever

$$\begin{cases} \mathcal{F}(x,u) > 0 \text{ for all on } J \text{ admissible } (x,u) \\ \text{with } x_0 = x_{N+1} = 0 \text{ and } \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \neq 0 \end{cases}$$

holds. Throughout the paper we denote by (X, U) the *principal solution* of (H) (at 0), i.e., the solution introduced in Proposition 1.2(i). Concerning *Moore–Penrose inverses* we will need the following basic lemma which is proved, e.g., in [2, Remark 2(iii)].

LEMMA 2.1. *For any two matrices V and W we have*

$$\mathrm{Ker}V \subset \mathrm{Ker}W \quad \textit{iff} \quad W = WV^\dagger V \quad \textit{iff} \quad W^\dagger = V^\dagger V W^\dagger.$$

It is the goal of this paper to relate the condition from Proposition 1.2(i) to a condition on certain block tridiagonal $kn \times kn$ matrices of the form

$$\mathcal{L}_k = \begin{pmatrix} T_0 & S_1 & & \\ S_1^T & T_1 & \ddots & \\ & \ddots & \ddots & S_{k-1} \\ & & S_{k-1}^T & T_{k-1} \end{pmatrix}, \qquad k \in \mathbb{N}_0,$$

where we put, for $k \in \mathbb{N}_0$,

$$T_k = C_k + (I - A_k^T)B_k^\dagger(I - A_k) + B_{k+1}^\dagger \quad \text{and} \quad S_k = -B_k^\dagger(I - A_k). \tag{8}$$

Let $\mathcal{L} = \mathcal{L}_N$. Our first result then is the following.

THEOREM 2.2. *Let (x, u) be admissible on J with $x_0 = x_{N+1} = 0$. Then we have*

$$\mathcal{F}(x, u) = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}^T \mathcal{L} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}.$$

Proof. Let (x, u) be admissible on J so that

$$u_k^T B_k u_k = u_k^T B_k B_k^\dagger B_k u_k = \left(x_{k+1}^T(I - A_k^T) - x_k^T\right) B_k^\dagger \left((I - A_k)x_{k+1} - x_k\right)$$

holds for all $k \in J$. Then, if $x_0 = x_{N+1} = 0$, we have

$$\begin{aligned}
\mathcal{F}(x, u) &= \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + \left(x_{k+1}^T(I - A_k^T) - x_k^T\right) B_k^\dagger \left((I - A_k)x_{k+1} - x_k\right) \right\} \\
&= \sum_{k=0}^{N} \left\{ x_{k+1}^T (T_k - B_{k+1}^\dagger) x_{k+1} + 2x_k^T S_k x_{k+1} + x_k^T B_k^\dagger x_k \right\} \\
&= \sum_{k=0}^{N} \left\{ x_{k+1}^T T_k x_{k+1} + 2x_k^T S_k x_{k+1} - \Delta(x_k^T B_k^\dagger x_k) \right\} \\
&= \sum_{k=0}^{N} \left\{ x_{k+1}^T T_k x_{k+1} + 2x_k^T S_k x_{k+1} \right\} - x_{N+1}^T B_{N+1}^\dagger x_{N+1} + x_0^T B_0^\dagger x_0 \\
&= \sum_{k=1}^{N-1} \left\{ x_{k+1}^T T_k x_{k+1} + 2x_k^T S_k x_{k+1} \right\} + x_1^T T_0 x_1 \\
&= \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}^T \mathcal{L} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix},
\end{aligned}$$

and hence our desired result is shown. ∎

By introducing the space

$$\mathcal{A} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} : \; x = \{x_k\}_{k\in\mathbb{N}_0} \text{ is admissible on } J \text{ with } x_0 = x_{N+1} = 0 \right\},$$

we can write an immediate consequence of Theorem 2.2.

COROLLARY 2.3. *$\mathcal{F} > 0$ iff $\mathcal{L} > 0$ on $\mathcal{A}$.*

Note that $\mathcal{L} > 0$ on $\mathcal{A}$, i.e., $\chi^T\mathcal{L}\chi > 0$ for all $\chi \in \mathcal{A} \setminus \{0\}$, is equivalent to

$$\mathcal{M}^T\mathcal{L}\mathcal{M} \geq 0 \quad \text{and} \quad \mathrm{Ker}\mathcal{M}^T\mathcal{L}\mathcal{M} \subset \mathrm{Ker}\mathcal{M}$$

whenever $\mathcal{M}$ is a matrix with $\mathrm{Im}\mathcal{M} = \mathcal{A}$. In the next section we will present such a matrix $\mathcal{M}$ for the general Hamiltonian case, and in the last section we will give this matrix $\mathcal{M}$ and further results for the Sturm–Liouville case. A *Sturm–Liouville difference equation*

$$\text{(SL)} \qquad \sum_{\mu=0}^{n} (-\Delta)^\mu \left\{ r_k^{(\mu)} \Delta^\mu y_{k+n-\mu} \right\} = 0, \qquad k \in \mathbb{N}_0$$

with reals $r_k^{(\mu)}$, $0 \leq \mu \leq n$, such that $r_k^{(n)} \neq 0$ for all $k \in \mathbb{N}_0$ is a special case of a linear Hamiltonian difference system. We define, for all $k \in \mathbb{N}_0$,

$$\text{(9)} \qquad \left\{ \begin{array}{l} A_k = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \quad B_k = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \frac{1}{r_k^{(n)}} \end{pmatrix}, \\[2ex] \text{and} \qquad C_k = \begin{pmatrix} r_k^{(0)} & & & \\ & r_k^{(1)} & & \\ & & \ddots & \\ & & & r_k^{(n-1)} \end{pmatrix}. \end{array} \right.$$

Then assumption (1) is satisfied and the following result from [3, Lemma 4] holds.

LEMMA 2.4. *Suppose* (9). *Then x is admissible on J iff there exists a sequence $y = \{y_k\}_{0\leq k\leq N+n-1}$ of reals such that*

$$x_k = \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \Delta^2 y_{k+n-3} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix} \qquad \text{for all} \quad 0 \leq k \leq N+1$$

holds, and in this case the functional defined by (F) *takes the form*

$$\text{(10)} \qquad \mathcal{F}(x,u) = \sum_{k=0}^{N} \sum_{\nu=0}^{n} r_k^{(\nu)} \left\{ \Delta^\nu y_{k+n-\nu} \right\}^2,$$

where u is such that (x,u) is admissible.

We conclude this section with two auxiliary results where we use the notation introduced in Proposition 1.2.

LEMMA 2.5. *Let $\tilde{A}_k := (I - A_k)^{-1}$ and $D_k = X_k X_{k+1}^\dagger \tilde{A}_k B_k$, $k \in \mathbb{N}_0$.*

(i) *Suppose* (1). *If* $\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k$, *then*

$$D_k \text{ is symmetric} \quad \text{and} \quad \mathrm{Ker}X_{k+1}^T \subset \mathrm{Ker}B_k\tilde{A}_k^T.$$

(ii) *Suppose* (9). *Then we have for all* $0 \le k \le n-1$

$$\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k, \quad \mathrm{rank}X_{k+1} = k+1, \quad \text{and} \quad D_k = 0.$$

Proof. While part (i) is shown in [2, Remark 2 (ii)], (ii) is the contents of [3, Lemma 4]. □

LEMMA 2.6. *We have, for all* $k \in \mathbb{N}_0$,

$$X_{k+1}^T T_k X_{k+1} = \Delta\left\{X_k^T(U_k + B_k^\dagger X_k)\right\} - X_k^T S_k X_{k+1} - X_{k+1}^T S_k^T X_k. \tag{11}$$

Proof. The calculation

$$\begin{aligned}
&X_{k+1}^T T_k X_{k+1} + X_k^T S_k X_{k+1} + X_{k+1}^T S_k^T X_k = X_{k+1}^T\left\{U_{k+1} - (I - A_k^T)U_k\right\}\\
&\quad + X_{k+1}^T(I - A_k^T)B_k^\dagger(I - A_k)X_{k+1} + X_{k+1}^T B_{k+1}^\dagger X_{k+1}\\
&\quad - X_k^T B_k^\dagger(I - A_k)X_{k+1} - X_{k+1}^T(I - A_k^T)B_k^\dagger X_k\\
&= X_{k+1}^T U_{k+1} - (X_k^T + U_k^T B_k)U_k + (X_k^T + U_k^T B_k)B_k^\dagger(X_k + B_k U_k)\\
&\quad + X_{k+1}^T B_{k+1}^\dagger X_{k+1} - X_k^T B_k^\dagger(X_k + B_k U_k) - (X_k^T + U_k^T B_k)B_k^\dagger X_k\\
&= \Delta\left\{X_k^T U_k + X_k^T B_k^\dagger X_k\right\}
\end{aligned}$$

shows that formula (11) holds. □

3. The linear Hamiltonian difference system. In this section we present a matrix $\mathcal{M}$ satisfying $\mathrm{Im}\mathcal{M} = \mathcal{A}$. We then give a proof of the following crucial result.

THEOREM 3.1. *If* $\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k$ *holds on* J, *then we have*

$$\mathcal{M}^T \mathcal{L} \mathcal{M} = \mathrm{diag}\left\{D_1, \dots, D_N\right\}.$$

To further motivate our investigations, first we would like to briefly consider the case that the matrices

$$B_k \text{ are invertible for all } k \in J. \tag{12}$$

Then we have $\mathcal{A} = \mathbb{R}^{Nn}$, and $\mathcal{F} > 0$ iff $\mathcal{L} > 0$. Then, if (X, U) is the principal solution of (H) at 0 such that X_k are invertible for all $1 \le k \le N+1$, and if we put

$$F_{k+1} = \begin{pmatrix} D_1 S_1 D_2 S_2 \dots D_k S_k \\ -D_2 S_2 \dots D_k S_k \\ \vdots \\ (-1)^{k-1} D_k S_k \\ (-1)^k I \end{pmatrix} \quad \text{for all } 0 \le k \le N,$$

it is easy to show that, for all $0 \le k \le N$,

$$D_{k+1}^{-1} = T_k - S_k^T D_k S_k, \qquad \mathcal{L}_{k+1} F_{k+1} D_{k+1} = \begin{pmatrix} 0 \\ (-1)^k I \end{pmatrix} \tag{13}$$

and

$$\mathcal{L}_{k+1}^{-1} = \begin{pmatrix} \mathcal{L}_k^{-1} & 0 \\ 0 & 0 \end{pmatrix} + F_{k+1} D_{k+1} F_{k+1}^T. \tag{14}$$

THEOREM 3.2. *Assume* (12). *We put* $\mathcal{L}_0 := I$ *and* $\tilde{D}_0 := 0$. *Then we have recursively, for* $k = 0, 1, 2, \ldots, N$,

$$\mathcal{L}_{k+1} > 0 \Longleftrightarrow \mathcal{L}_k > 0 \quad \textit{and} \quad \tilde{D}_{k+1} := \left\{ T_k - S_k^T \tilde{D}_k S_k \right\}^{-1} > 0. \tag{15}$$

Proof. Of course (15) is obvious for $k = 0$. However, if (15) holds for some $0 \le k \le N-1$, then (put $R_k = \binom{0}{S_k}$)

$$\mathcal{L}_{k+1} = \begin{pmatrix} \mathcal{L}_k & R_k \\ R_k^T & T_k \end{pmatrix} > 0 \Longleftrightarrow \mathcal{L}_k > 0 \quad \text{and} \quad T_k - R_k^T \mathcal{L}_k^{-1} R_k > 0.$$

However, by (14), $T_k - R_k^T \mathcal{L}_k^{-1} R_k = T_k - S_k^T D_k S_k$, where our D_k here are exactly the $\tilde{D}_k$ because of (13) and $D_0 = 0$. This proves our desired assertion. □

Now we turn our attention to the general case of an LHdS (H), where we assume (1). Let us define D_{ij} for $1 \le i \le j \le N$ matrices by

$$D_{ij} = X_i X_j^\dagger D_j.$$

Then we have

$$D_{ii} = X_i X_i^\dagger D_i = X_i X_i^\dagger X_i X_{i+1}^\dagger \tilde{A}_i B_i = X_i X_{i+1}^\dagger \tilde{A}_i B_i = D_i$$

and, by Lemma 2.1, if $\mathrm{Ker} X_j \subset \mathrm{Ker} X_i$,

$$D_{ij} = X_i X_j^\dagger D_j = X_i X_j^\dagger X_j X_{j+1}^\dagger \tilde{A}_j B_j = X_i X_{j+1}^\dagger \tilde{A}_j B_j.$$

We now define, for any $m \in J$, an $mn \times mn$ matrix $\mathcal{M}_m$ by

$$\mathcal{M}_m = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1m} \\ 0 & D_{22} & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & D_{mm} \end{pmatrix}$$

and put $\mathcal{M} = \mathcal{M}_N$.

THEOREM 3.3. *If* $\mathrm{Ker} X_{k+1} \subset \mathrm{Ker} X_k$ *holds on* J, *then* $\mathrm{Im} \mathcal{M} = \mathcal{A}$.

Proof. We assume $\mathrm{Ker} X_{k+1} \subset \mathrm{Ker} X_k$ on J. First, let $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{A}$ and put

$$c_0 := 0, \quad c_{k+1} := c_k - X_{k+1}^\dagger \tilde{A}_k B_k (U_k c_k - u_k) \text{ for } k \in \mathbb{N}_0,$$

where $u = \{u_k\}_{k \in \mathbb{N}_0}$ is such that (x, u) is admissible on J. Let $d_k := U_k c_k - u_k$ for $k \in \mathbb{N}_0$. We have $X_0 c_0 = 0 = x_0$, and $X_k c_k = x_k$ for some $k \in J$ implies

$$\begin{aligned} X_{k+1} c_{k+1} &= X_{k+1} c_k - X_{k+1} X_{k+1}^\dagger \tilde{A}_k B_k (U_k c_k - u_k) \\ &= (\tilde{A}_k X_k + \tilde{A}_k B_k U_k) c_k - \tilde{A}_k B_k (U_k c_k - u_k) \\ &= \tilde{A}_k X_k c_k + \tilde{A}_k B_k u_k = \tilde{A}_k x_k + \tilde{A}_k B_k u_k = x_{k+1} \end{aligned}$$

because of Lemmas 2.5(i) and 2.1. Hence

$$X_k c_k = x_k \quad \text{for all} \quad 0 \le k \le N+1.$$

Next, we have for $j \in J$,

$$D_j d_j = X_j X_{j+1}^{\dagger} \tilde{A}_j B_j (U_j c_j - u_j) = X_j (c_j - c_{j+1}) = -X_j \Delta c_j$$

so that

$$D_{ij} d_j = X_i X_j^{\dagger} D_j d_j = -X_i X_j^{\dagger} X_j \Delta c_j = -X_i \Delta c_j$$

for all $0 \le i \le j \le N$ because of $\mathrm{Ker} X_j \subset \mathrm{Ker} X_i$ and Lemma 2.1. Therefore

$$\begin{aligned}\sum_{j=i}^{N} D_{ij} d_j &= -X_i \sum_{j=i}^{N} \Delta c_j = -X_i (c_{N+1} - c_i) \\ &= -X_i X_{N+1}^{\dagger} X_{N+1} c_{N+1} + X_i c_i = x_i\end{aligned}$$

for all $0 \le i \le N$ so that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \mathcal{M} \begin{pmatrix} d_1 \\ \vdots \\ d_N \end{pmatrix} \in \mathrm{Im}\mathcal{M}.$$

Conversely, let $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathrm{Im}\mathcal{M}$ and put $x_0 = x_{N+1} = 0$. We pick $d_1, \ldots, d_N \in \mathbb{R}^n$ with

$$x_i = \sum_{j=i}^{N} D_{ij} d_j \quad \text{for all} \quad 1 \le i \le N.$$

Then we have

$$(I - A_0) x_1 - x_0 = \sum_{j=1}^{N} (I - A_0) X_1 X_j^{\dagger} D_j d_j = B_0 \sum_{j=1}^{N} X_j^{\dagger} D_j d_j \in \mathrm{Im} B_0,$$

$$(I - A_N) x_{N+1} - x_N = -x_N = -D_{NN} d_N = -D_{NN}^T d_N \in \mathrm{Im} B_N,$$

and, for $k \in J \setminus \{N\}$,

$$\begin{aligned}(I - A_k) x_{k+1} - x_k &= (I - A_k) \sum_{j=k+1}^{N} X_{k+1} X_j^{\dagger} D_j d_j - \sum_{j=k}^{N} X_k X_j^{\dagger} D_j d_j \\ &= \{(I - A_k) X_{k+1} - X_k\} \sum_{j=k+1}^{N} X_j^{\dagger} D_j d_j - D_k d_k \\ &= B_k U_k \sum_{j=k+1}^{N} X_j^{\dagger} D_j d_j - D_k d_k \in \mathrm{Im} B_k\end{aligned}$$

by Lemma 2.5(i) so that x is admissible on J and hence $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{A}$. $\square$

Our next result directly yields Theorem 3.1.

THEOREM 3.4. *If* $\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k$ *holds on* J*, then*

$$\mathcal{M}_{m+1}^T \mathcal{L}_{m+1} \mathcal{M}_{m+1} = \begin{pmatrix} \mathcal{M}_m^T \mathcal{L}_m \mathcal{M}_m & 0 \\ 0 & D_{m+1} \end{pmatrix}.$$

Proof. Let $k \in J$ and $m \in J \setminus \{N\}$. We define $kn \times n$ matrices P_k and R_k by

$$P_k = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix} \quad \text{and} \quad R_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ S_k \end{pmatrix}.$$

Then we have the recursions

$$\mathcal{M}_{m+1} = \begin{pmatrix} \mathcal{M}_m & P_m X_{m+1}^\dagger D_{m+1} \\ 0 & X_{m+1} X_{m+1}^\dagger D_{m+1} \end{pmatrix} \quad \text{and} \quad \mathcal{L}_{m+1} = \begin{pmatrix} \mathcal{L}_m & R_m \\ R_m^T & T_m \end{pmatrix}.$$

Hence by putting $\tilde{\mathcal{M}}_k = \binom{\mathcal{M}_k}{0}$,

$$\mathcal{M}_{m+1} = \begin{pmatrix} \tilde{\mathcal{M}}_m & P_{m+1} X_{m+1}^\dagger D_{m+1} \end{pmatrix}.$$

Therefore the matrix $\mathcal{M}_{m+1}^T \mathcal{L}_{m+1} \mathcal{M}_{m+1}$ turns out to be

$$\begin{pmatrix} \tilde{\mathcal{M}}_m^T \mathcal{L}_{m+1} \tilde{\mathcal{M}}_m & \tilde{\mathcal{M}}_m^T \mathcal{L}_{m+1} P_{m+1} X_{m+1}^\dagger D_{m+1} \\ D_{m+1} (X_{m+1}^\dagger)^T P_{m+1}^T \mathcal{L}_{m+1} \tilde{\mathcal{M}}_m & D_{m+1} (X_{m+1}^\dagger)^T P_{m+1}^T \mathcal{L}_{m+1} P_{m+1} X_{m+1}^\dagger D_{m+1} \end{pmatrix}$$
$$= \begin{pmatrix} \mathcal{M}_m^T \mathcal{L}_m \mathcal{M}_m & \Omega_m X_{m+1}^\dagger D_{m+1} \\ D_{m+1} (X_{m+1}^\dagger)^T \Omega_m^T & D_{m+1} (X_{m+1}^\dagger)^T \Lambda_m X_{m+1}^\dagger D_{m+1} \end{pmatrix}$$

with

$$\Omega_m = \tilde{\mathcal{M}}_m^T \mathcal{L}_{m+1} P_{m+1} \quad \text{and} \quad \Lambda_m = P_{m+1}^T \mathcal{L}_{m+1} P_{m+1}.$$

Hence our result follows directly from (ii) and (iii) of the following lemma. ∎

LEMMA 3.5. *If* $\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k$ *holds on* J*, then we have, for all* $k \in J \setminus \{N\}$*,*

(i) $\Lambda_k = X_{k+1}^T (U_{k+1} + B_{k+1}^\dagger X_{k+1})$;

(ii) $D_{k+1} (X_{k+1}^\dagger)^T \Lambda_k X_{k+1}^\dagger D_{k+1} = D_{k+1}$;

(iii) $\Omega_k = 0$.

Proof. First of all we have, by formula (11) of Lemma 2.6,

$$\Lambda_0 = X_1^T T_0 X_1 = X_1^T (U_1 + B_1^\dagger X_1)$$

and, if $\Lambda_{k-1} = X_k^T (U_k + B_k^\dagger X_k)$ already holds for some $k \in \{1, \dots, N-1\}$, again by applying formula (11) we have

$$\begin{aligned} \Lambda_k &= \begin{pmatrix} P_k \\ X_{k+1} \end{pmatrix}^T \begin{pmatrix} \mathcal{L}_k & R_k \\ R_k^T & T_k \end{pmatrix} \begin{pmatrix} P_k \\ X_{k+1} \end{pmatrix} \\ &= P_k^T \mathcal{L}_k P_k + P_k^T R_k X_{k+1} + X_{k+1}^T R_k^T P_k + X_{k+1}^T T_k X_{k+1} \\ &= X_{k+1}^T (U_{k+1} + B_{k+1}^\dagger X_{k+1}). \end{aligned}$$

Hence (i) is shown, and by (i), for all $k \in J \setminus \{N\}$,

$$\begin{aligned}
&D_{k+1}(X_{k+1}^{\dagger})^T \Lambda_k X_{k+1}^{\dagger} D_{k+1} = D_{k+1} U_{k+1} X_{k+1}^{\dagger} D_{k+1} + D_{k+1} B_{k+1}^{\dagger} D_{k+1} \\
&= X_{k+1} X_{k+2}^{\dagger} \tilde{A}_{k+1} B_{k+1} U_{k+1} X_{k+1}^{\dagger} D_{k+1} + X_{k+1} X_{k+2}^{\dagger} \tilde{A}_{k+1} X_{k+1} X_{k+1}^{\dagger} D_{k+1} \\
&= X_{k+1} X_{k+2}^{\dagger} X_{k+2} X_{k+1}^{\dagger} D_{k+1} = X_{k+1} X_{k+1}^{\dagger} D_{k+1} = D_{k+1}
\end{aligned}$$

takes care of part (ii). Finally, $\Omega_0 = 0$, and if $\Omega_{k-1} = 0$ already holds for some $k \in \{1, \ldots, N-1\}$, then

$$\begin{aligned}
\Omega_k &= \begin{pmatrix} \mathcal{M}_k^T & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}_k & R_k \\ R_k^T & T_k \end{pmatrix} \begin{pmatrix} P_k \\ X_{k+1} \end{pmatrix} \\
&= \mathcal{M}_k^T (\mathcal{L}_k P_k + R_k X_{k+1}) \\
&= \begin{pmatrix} \tilde{\mathcal{M}}_{k-1}^T \\ D_k (X_k^{\dagger})^T P_k^T \end{pmatrix} \left(\mathcal{L}_k P_k + \begin{pmatrix} 0 \\ S_k X_{k+1} \end{pmatrix} \right) \\
&= \begin{pmatrix} \Omega_{k-1} \\ D_k (X_k^{\dagger})^T \Lambda_{k-1} + D_k S_k X_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ D_k (U_k + B_k^{\dagger} X_k) - D_k B_k^{\dagger} (X_k + B_k U_k) \end{pmatrix} = 0
\end{aligned}$$

because of part (i). Hence (iii) follows, and all our desired results are shown. □

4. The Sturm–Liouville difference equation. In this section we deal with the case where (H) and (F) correspond to a higher order Sturm–Liouville equation (SL) and its corresponding quadratic functional (10). The special structure of the matrices A, B, C enables us to simplify the results of the previous section.

We start with an identity which plays the crucial role in the proof of the main results of this section.

LEMMA 4.1. *Let (X, U) be the principal solution of LHdS corresponding to* (SL). *If X_{k+1} is nonsingular, then*

$$D_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k = \operatorname{diag} \left\{ 0, \ldots, 0, \frac{\det X_k}{r_k^{(n)} \det X_{k+1}} \right\}.$$

Proof. Nonsingularity of X_{k+1} implies that $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$, hence the matrix $D_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k$ is symmetric according to Lemma 2.5(i), and since $B_k = \operatorname{diag}\{0, \ldots, 0, (1/r_k^{(n)})\}$, the only nonzero entry of D_k is in the right lower corner. Using Laplace's rule, we have

$$\begin{aligned}
&\det X_k = \det \left[(I - A_k) X_{k+1} - B_k U_k \right] \\
&= \det \left[(I - A_k) X_{k+1} - \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ \frac{1}{r_k^{(n)}} (U_k)_{n,1} & \ldots & \frac{1}{r_k^{(n)}} (U_k)_{n,n} \end{pmatrix} \right] \\
&= \sum_{\nu=1}^{n} \left[((I - A_k) X_{k+1})_{n,\nu} - \frac{1}{r_k^{(n)}} (U_k)_{n,\nu} \right] (\operatorname{adj} \left[(I - A_k) X_{k+1} \right])_{\nu,n} \\
&= \sum_{\nu=1}^{n} ((I - A_k) X_{k+1} - B_k U_k)_{n,\nu} \left(\left[(I - A_k) X_{k+1} \right]^{-1} \right)_{\nu,n} \det \left[(I - A_k) X_{k+1} \right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\nu=1}^{n} (X_k)_{n,\nu} \left(X_{k+1}^{-1} \tilde{A}_k \right)_{\nu,n} \det(I - A_k) \det X_{k+1} \\
&= \det X_{k+1} \left(X_k X_{k+1}^{-1} \tilde{A}_k \right)_{n,n} = r_k^{(n)} \det X_{k+1} \left(X_k X_{k+1}^{-1} \tilde{A}_k B_k \right)_{n,n} \\
&= r_k^{(n)} \det X_{k+1} \left(D_k \right)_{n,n},
\end{aligned}
$$

so the desired result follows. □

In the next statement and its proof we suppose that the matrices $X, \mathcal{L}, \mathcal{M}$ are the same as in the previous section and that the matrices A, B, C in (H) are given by (9).

THEOREM 4.2. *Suppose that* $X_n, \ldots, X_{N+1}$ *are nonsingular and denote*

$$
d_k := (D_k)_{n,n} = \frac{\det X_k}{r_k^{(n)} \det X_{k+1}}, \qquad k = n, \ldots, N. \tag{16}
$$

Then there exists an $nN \times (N - n + 1)$ *matrix* $\mathcal{N}$ *such that*

$$
\mathcal{N}^T \mathcal{L} \mathcal{N} = \operatorname{diag}\{d_n, \ldots, d_N\}.
$$

Proof. Observe that the assumption of nonsingularity of $X_n, \ldots, X_{N+1}$ corresponds to the assumption $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ in Theorem 3.1 because of Lemma 2.5 (ii). Denote by $\mathcal{M}^{[j]} \in \mathbb{R}^{nN}$, $j = 1, \ldots, nN$, the columns of the matrix $\mathcal{M}$ and let $\mathcal{N}$ be the $nN \times (N - n + 1)$ matrix which results from $\mathcal{M}$ after omitting all zero columns, i.e.,

$$
\mathcal{N} = \left[\mathcal{M}^{[n^2]} \; \mathcal{M}^{[n(n+1)]} \ldots \mathcal{M}^{[nN]} \right].
$$

Since $D_1 = \cdots = D_{n-1} = 0$ for LHdS corresponding to (SL) by Lemma 2.5 (ii), we have $\mathcal{M}^T \mathcal{L} \mathcal{M} = \operatorname{diag}\{0, \ldots, 0, D_n, \ldots, D_N\}$ by Theorem 3.1. By Lemma 4.1, $D_j = \operatorname{diag}\{0, \ldots, 0, d_j\}$, $j = n, \ldots, N$, and the statement follows from the relation between $\mathcal{M}$ and $\mathcal{N}$. □

Sturm–Liouville equations may be investigated also directly, i.e., not as a special case of LHdS. Let $t_1, \ldots, t_n$ be n-dimensional vectors with entries

$$
t_\nu^{(\mu)} = (-1)^{n-\nu} \binom{\mu - 1}{n - \nu}, \qquad \mu = 1, \ldots, n,
$$

with the usual convention that $\binom{n}{k} = 0$ if $k > n$ or $k < 0$, and denote by T the $n \times n$ matrix whose columns are the vectors t_ν, i.e., $T = [t_1 \ldots t_n]$. Furthermore, let $\mathcal{P} = (p_{i,j})$ be the $nN \times (N - n + 1)$ matrix with n-vector entries $p_{i,j} \in \mathbb{R}^n$ given by $p_{i,j} = t_{n+j-i}$, with the convention that $t_l = 0$ if $l < 1$ or $l > n$.

It follows from Lemma 2.4 that if (H) corresponds to a Sturm–Liouville equation (SL), then (x, u) satisfying $x_0 = 0 = x_{N+1}$ is admissible iff there exists $y = \{y_k\}_{k=n}^N$ such that

$$
x_k = T \begin{pmatrix} y_k \\ \vdots \\ y_{k+n-1} \end{pmatrix};
$$

hence $(x_1^T,\dots,x_N^T)^T\in\mathcal{A}$ iff

$$\begin{pmatrix} x_1\\ \vdots\\ x_N\end{pmatrix}=\mathcal{P}\begin{pmatrix} y_n\\ \vdots\\ y_N\end{pmatrix}.$$

Consequently, if A,B,C are given by (9), T_k,S_k by (8), and $\mathcal{K}:=\mathcal{P}^T\mathcal{L}\mathcal{P}$, we have

$$\begin{aligned}\mathcal{F}(y)&=\sum_{k=0}^{N}\sum_{\nu=0}^{n} r_k^{(\nu)}(\Delta^\nu y_{k+n-\nu})^2=\sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1}+u_k^T B_k u_k\}\\ &=\begin{pmatrix} y_n\\ \vdots\\ y_N\end{pmatrix}^T\mathcal{P}^T\mathcal{L}\mathcal{P}\begin{pmatrix} y_n\\ \vdots\\ y_N\end{pmatrix}=\begin{pmatrix} y_n\\ \vdots\\ y_N\end{pmatrix}^T\mathcal{K}\begin{pmatrix} y_n\\ \vdots\\ y_N\end{pmatrix}.\end{aligned}$$

Expanding the differences in (10), it is easy to see that $\mathcal{K}$ is a $2n+1$-diagonal matrix. Our next computations extend the results of the first section by relating the quantities d_k from (16) (in section 1 for $n=1$ denoted by δ_k) to the principal minors of the matrix $\mathcal{K}$.

First we look for a relation between the $nN\times(N-n+1)$ matrices $\mathcal{P}$ and $\mathcal{N}$ and the corresponding representation for $(x_1^T,\dots,x_N^T)^T\in\mathcal{A}$. By the previous considerations and Theorem 3.3, $(x_1^T,\dots,x_N^T)^T\in\mathcal{A}$ iff there exists $c=(c_n,\dots,c_N)^T$ such that

$$\begin{pmatrix} x_1\\ \vdots\\ x_N\end{pmatrix}=\mathcal{N}\begin{pmatrix} c_n\\ \vdots\\ c_N\end{pmatrix}=\mathcal{P}\begin{pmatrix} y_n\\ \vdots\\ y_N\end{pmatrix}.$$

From the last equality we will find a relation between vectors $c=(c_n,\dots,c_N)^T$ and $y=(y_n,\dots,y_N)^T$. Note that these vectors are determined uniquely since the matrices $\mathcal{P}$ and $\mathcal{N}$ have full rank. Denote by d_i^j the last column of the matrix $X_iX_j^\dagger D_j$. Taking into account the form of the matrices $\mathcal{P}$ and $\mathcal{N}$, we get the system of equations

$$\left\{\begin{array}{rcl} y_Nt_1&=&c_Nd_N^N,\\ y_{N-1}t_1+y_Nt_2&=&c_{N-1}d_{N-1}^{N-1}+c_nd_{N-1}^N,\\ &\vdots&\\ y_nt_1&=&c_nd_1^n+\ldots+c_Nd_1^N.\end{array}\right.\tag{17}$$

From this system of equations, we see that there exists an upper triangular matrix $\mathcal{B}=(\mathcal{B}_{i,j})\in\mathbb{R}^{(N-n+1)\times(N-n+1)}$ such that $y=\mathcal{B}c$ and that this is just the matrix which reduces the $2n+1$-diagonal matrix $\mathcal{K}=\mathcal{P}^T\mathcal{L}\mathcal{P}$ to the diagonal form. Indeed, we have

$$\begin{aligned}\mathcal{F}(y)&=\begin{pmatrix} y_n\\ \vdots\\ y_N\end{pmatrix}^T\mathcal{K}\begin{pmatrix} y_n\\ \vdots\\ y_N\end{pmatrix}=\begin{pmatrix} x_1\\ \vdots\\ x_N\end{pmatrix}^T\mathcal{L}\begin{pmatrix} x_1\\ \vdots\\ x_N\end{pmatrix}\\ &=\begin{pmatrix} c_n\\ \vdots\\ c_N\end{pmatrix}^T\mathcal{N}^T\mathcal{L}\mathcal{N}\begin{pmatrix} c_n\\ \vdots\\ c_N\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} y_n \\ \vdots \\ y_N \end{pmatrix}^T (\mathcal{B}^T)^{-1} \mathrm{diag}\{d_n, \dots, d_N\} \mathcal{B}^{-1} \begin{pmatrix} y_n \\ \vdots \\ y_N \end{pmatrix}$$

for any $y = (y_n, \dots, y_N)^T \in \mathbb{R}^{N-n+1}$; hence

$$\mathcal{B}^T \mathcal{K} \mathcal{B} = \mathrm{diag}\{d_n, \dots, d_N\}.$$

Observe also that the diagonal entries of the matrix $\mathcal{B}$ are $\mathcal{B}_{k,k} = (-1)^{n-1} d_{n+k-1}$, $k = 1, \dots, N-n+1$ (this follows directly from (17)), and that

$$(\mathcal{K}\mathcal{B})_{j,k} = (\mathcal{B}^T \mathcal{K})_{k,j} = \left(\mathrm{diag}\{d_n, \dots, d_N\} \mathcal{B}^{-1}\right)_{k,j} = \begin{cases} 0 & j = 1, \dots, k-1, \\ (-1)^{n-1} & j = k. \end{cases}$$

Here we used the information about diagonal entries of $\mathcal{B}$. The last expression may be regarded as a system of linear equations for $\mathcal{B}_{1,k}, \dots, \mathcal{B}_{k,k}$, and by Cramer's rule we have

$$\mathcal{B}_{k,k} = (-1)^{n-1} d_{n+k-1} = \frac{(-1)^{n-1} \Delta_{k-1}}{\Delta_k}.$$

Now we can summarize our previous computations and relate the quantities d_k from (16), $k = n, \dots, N$, to the principal minors Δ_k of $\mathcal{K}$. This statement may be viewed as a direct extension of (7) to higher order Sturm–Liouville difference equations. Here, similarly as in the previous theorem, (X, U) is the principal solution of (H) with A, B, C given by (9), and the assumption of nonsingularity of the matrices $X_n, \dots, X_{N+1}$ has the same meaning as in Theorem 4.2.

THEOREM 4.3. *Suppose $X_n, \dots, X_{N+1}$ are nonsingular and let Δ_k be the principal minors of the matrix $\mathcal{K}$. Then, for all $1 \le k \le N-n+1$,*

$$d_{n+k-1} = \frac{\Delta_{k-1}}{\Delta_k}, \quad \Delta_0 := 1.$$

REFERENCES

[1] C. D. AHLBRANDT AND A. C. PETERSON, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer Academic Publishers, Boston, MA, 1996.

[2] M. BOHNER, *Linear Hamiltonian difference systems: Disconjugacy and Jacobi-type conditions*, J. Math. Anal. Appl., 199 (1996), pp. 804–826.

[3] M. BOHNER, *On disconjugacy for Sturm-Liouville difference equations*, J. Difference Equations and Appl., 2 (1996), pp. 227–237.

[4] W. A. COPPEL, *Disconjugacy*, Lecture Notes in Math. 220, Springer-Verlag, Berlin, 1971.

[5] O. DOŠLÝ, *Factorization of disconjugate higher order Sturm-Liouville difference operators*, Comput. Math. Appl., to appear.

[6] W. T. REID, *Ordinary Differential Equations*, John Wiley, New York, 1971.

[7] W. T. REID, *Sturmian Theory for Ordinary Differential Equations*, Springer-Verlag, New York, 1980.