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THE GAMMA FUNCTION ON TIME SCALES

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Abstract. We introduce the generalized Gamma function on time scales and prove some of its properties, which coincide with the ones well known in the continuous case. We also define an appropriate factorial function for computing the values of the generalized Gamma function in some special cases.

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1 Introduction

The so-called *Gamma function*, which is defined by the convergent improper integral

$$\Gamma(x) := \int_0^\infty \eta^{x-1}\mathrm{e}^{-\eta}\mathrm{d}\eta \quad \text{for } x \in \mathbb{R}^+, \tag{1}$$

has been a focus of interest almost in every branches of mathematics. It is shown easily by performing a partial integration to (1) that the Gamma function satisfies the functional relationship

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for } x \in \mathbb{R}^+. \tag{2}$$

Therefore, the values of the Gamma function have been tabulated for the interval $(0, 1]$, and using these values one can evaluate Γ on $\mathbb{R}^+$. Suppose for instance that one needs to compute $\Gamma(5/2)$. Using the recursion formula in (2), we see that $\Gamma(5/2) = (3/2)\Gamma(3/2) = (3/2)(1/2)\Gamma(1/2)$. From the table it is known that $\Gamma(1/2) = \sqrt{\pi}$, and thus $\Gamma(5/2) = 3\sqrt{\pi}/4$. Let $x \in \mathbb{N}$, then $\Gamma(x+1) = x!$ (here $\cdot!$ stands for the usual factorial function) since one can show (or look up in the table) that $\Gamma(1) = 1$ is valid. On the other hand, it is not hard to see that the improper integral in (1) diverges for $x \in \mathbb{R}_0^-$, and thus the definition of the Gamma function makes no sense in this case. However, for negative values, the definition of the Gamma function is extended by the functional recursion formula (2), i.e., $\Gamma(x) = \Gamma(x+1)/x$ for $x \in \mathbb{R}^-\backslash\mathbb{Z}_0^-$.

The relation between the Gamma function Γ and the Laplace transform $\mathcal{L}$ is given by

$$\mathcal{L}\left\{\left(\frac{\mathrm{I}}{s}\right)^{x-1}\right\}(z) = \frac{1}{s^{x-1}}\frac{\Gamma(x)}{z^x} \quad \text{for } s, x, z \in \mathbb{R}^+,$$

or particularly

$$\mathcal{L}\big\{\mathrm{I}^{x-1}\big\}(1) = \Gamma(x) \quad \text{for } x \in \mathbb{R}^+$$

(see [6, § 2.1]). Above, we have denoted by I, the identity function on $\mathbb{R}$ and for $s, t \in \mathbb{R}^+$, $\big(\mathrm{I}(t)/s\big)^{x-1}$ means the $(x-1)$-st power of (t/s).

In this paper, motivated by the relation between the Gamma function and the Laplace transform in the continuous case we give the definition of the generalized Gamma function for arbitrary time scales, and prove some of its typical properties on arbitrary time scales which match with the well-known ones from its continuous counterpart. Our observation shows that the results are not very "nice" for arbitrary time scales (of course, unbounded above and including the origin), but for the particular choices of the time scales, for instance, the set of reals and/or the set of quantum numbers, we get "nice" results. We also would like to mention here that our results exactly coincide with the ones obtained in [5, § 21].

To be able to talk more about the main results, we find useful to introduce the following basic definitions and facilities for a reader not familiar with the time scale calculus. A *time scale*, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here, and later throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf(t, \infty)_{\mathbb{T}}$ while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined to be $\mu := \sigma - \mathrm{I}$, where I is the identity function on $\mathbb{T}$. A point $t \in \mathbb{T}$ is called *right-dense* if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds; otherwise, it is called *right-scattered*, and similarly *left-dense* and *left-scattered* points are defined with respect to the backward jump operator.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the Δ-derivative $f^\Delta(t)$ of f at the point t is defined to be the number, provided it exists, with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f^\sigma(t) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U,$$

where $\mathbb{T}^\kappa := \mathbb{T}\backslash\{t \in \mathbb{T} : \ t = \max \mathbb{T} \text{ and } \rho(t) < t\}$ and $f^\sigma := f \circ \sigma$ on $\mathbb{T}$. We shall mean the Hilger derivative of a function when we only say derivative unless otherwise specified. A function f is called *rd-continuous* provided that it is continuous at right-dense points in $\mathbb{T}$, and has finite limit at left-dense points, and the *set of rd-continuous functions* are denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. The set of functions $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is consist of the functions whose derivative is in $\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ too. For $s,t\in\mathbb{T}$ and a function $f\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, the Δ-integral of f is defined by

$$\int_s^t f(\eta)\Delta\eta = F(t)-F(s) \quad \text{for } s,t\in\mathbb{T},$$

where $F\in\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is an anti-derivative of f, i.e., $F^\Delta=f$ on $\mathbb{T}^\kappa$. Table 1 gives the explicit forms of the forward jump, delta derivative and the integral on the well-known time scales reals, integers and quantum set, respectively.

Table 1: The explicit forms of the forward jump, graininess, delta derivative and the integral on some time scales.

$\mathbb{T}$	$\mathbb{R}$	$h\mathbb{Z}$, $(h>0)$	$\overline{q^{\mathbb{Z}}}$, $(q>1)$
$\sigma(t)$	t	$t+h$	qt
$f^\Delta(t)$	$f'(t)$	$\dfrac{f(t+h)-f(t)}{h}$	$\dfrac{f(qt)-f(t)}{(q-1)t}$
$\displaystyle\int_s^t f(\eta)\Delta\eta$	$\displaystyle\int_s^t f(\eta)\mathrm{d}\eta$	$\displaystyle h\sum_{\eta=s/h}^{t/h-1} f(h\eta)$	$\displaystyle (q-1)\sum_{\eta=\log_q(s)}^{\log_q(t)-1} f(q^\eta)q^\eta$

$\log_q$ located in the last row and the last column of Table 1 stands for the common logarithm function with the base of q.

A function $f\in\mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is called *regressive* if $1+f\mu\neq 0$ on $\mathbb{T}^\kappa$, and *positively regressive* if $1+f\mu>0$ on $\mathbb{T}^\kappa$. The *set of regressive functions* and the *set of positively regressive functions* are denoted by $\mathcal{R}(\mathbb{T},\mathbb{R})$ and $\mathcal{R}^+(\mathbb{T},\mathbb{R})$, respectively, and $\mathcal{R}^-(\mathbb{T},\mathbb{R})$ is defined similarly.

Let $f\in\mathcal{R}(\mathbb{T},\mathbb{R})$ and $s\in\mathbb{T}$, then the *generalized exponential function* $\mathrm{e}_f(\cdot,s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the initial value problem

$$\begin{cases} y^\Delta = fy & \text{on } \mathbb{T}^\kappa \\ y(s)=1. \end{cases}$$

The exponential function can also be written in the form

$$\mathrm{e}_f(t,s) := \exp\left\{\int_s^t \xi_{\mu(\eta)}\big(f(\eta)\big)\Delta\eta\right\} \quad \text{for } s,t\in\mathbb{T},$$

where the *cylinder transformation* ξ_h for $h\in\mathbb{R}_0^+$ is defined by

$$\xi_h(z) := \lim_{r\to h}\frac{1}{r}\operatorname{Log}(1+zr) \quad \text{for } z\in\mathbb{C} \text{ with } 1+zh\neq 0.$$

Table 2 illustrates the exponential function on some well-known time scales.

Table 2: The explicit form of the exponential function on some time scales.

$\mathbb{T}$	$\mathbb{R}$	$h\mathbb{Z}$, $(h>0)$	$\overline{q^{\mathbb{Z}}}$, $(q>1)$
$\mathrm{e}_f(t,s)$	$\exp\left\{\int_s^t f(\eta)\mathrm{d}\eta\right\}$	$\prod_{\eta=s/h}^{t/h-1}\big(1+f(h\eta)h\big)$	$\prod_{\eta=\log_q(s)}^{\log_q(t)-1}\big(1+f(q^\eta)(q-1)q^\eta\big)$

It is known that the exponential function $\mathrm{e}_f(\cdot,s)$ is strictly positive on $[s,\infty)_{\mathbb{T}}$ provided that $f\in\mathcal{R}^+([s,\infty)_{\mathbb{T}},\mathbb{R})$, while $\mathrm{e}_f(\cdot,s)$ alternates in sign at right-scattered points in $[s,\infty)_{\mathbb{T}}$ provided that $f\in\mathcal{R}^-([s,\infty)_{\mathbb{T}},\mathbb{R})$. For $h\in\mathbb{R}_0^+$, let $z,w\in\mathbb{C}_h$, the *circle plus* and the *circle minus* are respectively defined by

$$z\oplus_h w:=z+w+zwh\quad\text{and}\quad z\ominus_h w:=\frac{z-w}{1+wh}.$$

It is also known that $\big(\mathcal{R}^+(\mathbb{T},\mathbb{R}),\oplus_\mu\big)$ is a subgroup of $\big(\mathcal{R}(\mathbb{T},\mathbb{R}),\oplus_\mu\big)$, i.e., $0\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$, $f,g\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$ implies $f\oplus_\mu g\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$ and $\ominus_\mu f\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$, where $\ominus_\mu f:=0\ominus_\mu f$ on $\mathbb{T}$. If, for $h\in\mathbb{R}_0^+$ and $z,w\in\mathbb{C}$, we define the *circle dot* by

$$z\odot_h w:=\lim_{r\to h}\frac{1}{r}\big((1+wr)^z-1\big)$$

then $\big(\mathcal{R}^+(\mathbb{T},\mathbb{R}),\oplus_\mu,\odot_\mu\big)$ becomes a real vector space. The readers are referred to [1, 2] for further details in the time scale theory.

2 Definitions and preliminaries

In this section, we construct the necessary information for the definition of the generalized Gamma function on time scales.

To this end, we introduce new binary operations as follows. First, we define the set of functions $\mathcal{P}(\mathbb{T},\mathbb{R}):=\{f\in\mathrm{C_{rd}}(\mathbb{T},\mathbb{R}):f/\mathrm{I}\in\mathcal{R}(\mathbb{T}\backslash\{0\},\mathbb{R})\}$, and similarly, we define $\mathcal{P}^+(\mathbb{T},\mathbb{R})$ and $\mathcal{P}^-(\mathbb{T},\mathbb{R})$. For $f,g\in\mathcal{P}(\mathbb{T},\mathbb{R})$, we define the "*boxplus*" addition and the "*boxminus*" subtraction operations by

$$f\boxplus_\mu g:=f+g+\frac{1}{\mathrm{I}}fg\mu\quad\text{and}\quad f\boxminus_\mu g:=\frac{(f-g)\mathrm{I}}{\mathrm{I}+g\mu}\quad\text{on }\mathbb{T}\backslash\{0\}$$

or implicitly

$$f\boxplus_\mu g=f\oplus_{\frac{\mu}{\mathrm{I}}}g\quad\text{and}\quad f\boxminus_\mu g=f\ominus_{\frac{\mu}{\mathrm{I}}}g\quad\text{on }\mathbb{T}\backslash\{0\}.$$

The proof of the lemma below therefore is straight forward (see [1, Theorem 2.7]).

Lemma 1. $\big(\mathcal{P}(\mathbb{T},\mathbb{R}),\boxplus_\mu\big)$ *is an Abelian group.*

The following properties directly follow from the properties of $\oplus_\mu$ and $\ominus_\mu$ (see [1, Exercise 2.28]).

Lemma 2. *If $f, g \in \mathcal{P}(\mathbb{T}^+, \mathbb{R})$, where $\mathbb{T}^+ := (0, \infty)_{\mathbb{T}}$, then*

(i) $f \boxplus_\mu g, f \boxminus_\mu g \in \mathcal{P}(\mathbb{T}, \mathbb{R})$,

(ii) $f \boxminus_\mu f = 0$ *on* $\mathbb{T}^+$,

(iii) $\boxminus_\mu(\boxminus_\mu f) = f$ *on* $\mathbb{T}^+$,

(iv) $\boxminus_\mu(f \boxminus_\mu g) = g \boxminus_\mu f$ *on* $\mathbb{T}^+$,

(v) $\boxminus_\mu(f \boxplus_\mu g) = (\boxminus_\mu f) \boxplus_\mu (\boxminus_\mu g)$ *on* $\mathbb{T}^+$.

Now, for $\alpha \in \mathbb{R}$ and $f \in \mathcal{P}(\mathbb{T}, \mathbb{R})$, we define the "*boxdot*" multiplication by

$$\alpha \boxdot_\mu f := \alpha \odot_{\frac{\mu}{\mathrm{I}}} f \quad \text{on } \mathbb{T} \backslash \{0\}.$$

Lemma 3. $\big(\mathcal{P}^+(\mathbb{T}\backslash\{0\}, \mathbb{R}), \boxplus_\mu, \boxdot_\mu\big)$ *is a real vector space.*

Lemma 4. *If $f, g \in \mathcal{P}^+(\mathbb{T}^+, \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then*

(i) $\alpha \boxdot_\mu f \in \mathcal{P}^+(\mathbb{T}^+, \mathbb{R})$,

(ii) $\alpha \boxdot_\mu (\beta \boxdot_\mu f) = (\alpha\beta) \boxdot_\mu f$ *on* $\mathbb{T}^+$,

(iii) $1 \boxdot_\mu f = f$ *on* $\mathbb{T}^+$,

(iv) $\alpha \boxdot_\mu (f \boxplus_\mu g) = (\alpha \boxdot_\mu f) \boxplus_\mu (\alpha \boxdot_\mu g)$ *on* $\mathbb{T}^+$,

(v) $(\alpha + \beta) \boxdot_\mu f = (\alpha \boxdot_\mu f) \boxplus (\beta \boxdot_\mu f)$ *on* $\mathbb{T}^+$.

Below, we define the function p, which plays the major role in this paper.

Definition 1. *For $f \in \mathcal{P}(\mathbb{T}, \mathbb{R})$, we define*

$$\mathrm{p}_f(t, s) := \mathrm{e}_{f/\mathrm{I}}(t, s) \quad \textit{for } s, t \in \mathbb{T}^+,$$

and $\mathrm{p}_f(0, s) := 0$ provided that $0 \in \mathbb{T}$ with $\mu(0) > 0$.

Table 3 illustrates the function p on some well-known time scales.

The following properties also follow from the properties of the exponential function (see [1, Theorem 2.36]), and thus we omit most of the proofs. But we first would like to present the following lemma.

Lemma 5. *If $f \in \mathcal{P}^+(\mathbb{T}^+, \mathbb{R})$ and $\alpha \in \mathbb{R}$, then*

$$\alpha \odot_\mu \frac{f}{\mathrm{I}} = \frac{\alpha \boxdot_\mu f}{\mathrm{I}} \quad \textit{on } \mathbb{T}\backslash\{0\}.$$

Table 3: The function p on some time scales. Here, it is assumed that $t \geq s$ and $x \in \mathbb{R}_0^+$ (note that if $x \in \mathbb{R}_0^+$, then $\mathrm{p}_x(t,s)$ is well-defined) and Γ is the usual Gamma function.

$\mathbb{T}_0^+ := [0,\infty)_{\mathbb{T}}$	$\mathrm{p}_x(t,s)$, $(x \in \mathbb{R}^+)$
$\mathbb{R}_0^+$	$\left(\frac{t}{s}\right)^x$
$h\mathbb{N}_0$, $(h>0)$	$\frac{\Gamma(t/h+x)\Gamma(s/h)}{\Gamma(t/h)\Gamma(s/h+x)}$
$\overline{q^{\mathbb{Z}}}$, $(q>1)$	$\left(\frac{t}{s}\right)^{\log_q(1+x(q-1))}$

Proof. Let $t \in \mathbb{T}^+$. Then we find by using the definition that

$$\begin{aligned}\alpha \odot_{\mu(t)} \frac{f(t)}{t} &= \lim_{r\to\mu(t)} \frac{1}{r}\left(\left(1+\frac{f(t)}{t}r\right)^{\alpha}-1\right) = \frac{1}{t}\lim_{r\to\mu(t)} \frac{t}{r}\left(\left(1+f(t)\frac{r}{t}\right)^{\alpha}-1\right)\\ &= \frac{1}{t}\lim_{r\to\frac{\mu(t)}{t}} \frac{1}{r}\left(\left(1+f(t)r\right)^{\alpha}-1\right) = \frac{1}{t}\left(\alpha \boxdot_{\mu(t)} f(t)\right),\end{aligned}$$

which completes the proof. $\square$

Lemma 6. *If $f,g \in \mathcal{P}^+(\mathbb{T}^+,\mathbb{R})$, $\alpha \in \mathbb{R}$ and $r,s,t \in \mathbb{T}^+$, then*

(i) $\mathrm{p}_0(t,s) \equiv 1$ *and* $\mathrm{p}_f(t,t) \equiv 1$,

(ii) $\mathrm{p}_f^{\sigma}(t,s) = (1+f\mu/\mathrm{I})\mathrm{p}_f(t,s)$,

(iii) $\mathrm{p}_f(s,t) = 1/\mathrm{p}_f(t,s) = \mathrm{p}_{\boxminus_\mu f}(t,s)$,

(iv) $\mathrm{p}_f(t,s)\mathrm{p}_f(s,r) = \mathrm{p}_f(t,r)$,

(v) $\mathrm{p}_f(t,s)\mathrm{p}_g(t,s) = \mathrm{p}_{f\boxplus_\mu g}(t,s)$,

(vi) $\mathrm{p}_f(t,s)/\mathrm{p}_g(t,s) = \mathrm{p}_{f\boxminus_\mu g}(t,s)$,

(vii) $\left(\mathrm{p}_f(t,s)\right)^{\alpha} = \mathrm{p}_{\alpha\boxdot_\mu f}(t,s)$

Proof. We shall only prove the part (vii) since the rest of the items follow by using similar arguments. Let $f \in \mathcal{P}^+(\mathbb{T}^+,\mathbb{R})$, $\alpha \in \mathbb{R}$ and $s,t \in \mathbb{T}^+$. Then by [2, Theorem 2.44] and Lemma 5, we have

$$\left(\mathrm{p}_f(t,s)\right)^{\alpha} = \left(\mathrm{e}_{f/\mathrm{I}}(t,s)\right)^{\alpha} = \mathrm{e}_{\alpha\odot_\mu(f/\mathrm{I})}(t,s) = \mathrm{e}_{(\alpha\boxdot_\mu f)/\mathrm{I}}(t,s) = \mathrm{p}_{\alpha\boxdot_\mu f}(t,s).$$

This completes the proof. $\square$

Remark 1. In particular, if $f \in \mathcal{P}(\mathbb{T},\mathbb{R})$ and $s,t \in \mathbb{T}^+$, then $\mathrm{p}_{f\boxminus_\mu 1}(t,s) = \mathrm{e}_{(f-1)/\sigma}(t,s)$.

Remark 2. If $f \in \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}_0^+)$, then $\mathrm{p}_f(t,s) > 0$ for $s, t \in \mathbb{T}^+$ since $f/\mathrm{I} \in \mathcal{R}^+(\mathbb{T}^+, \mathbb{R})$ (see [1, Theorem 2.44(i)]).

We need the following properties of the function p in the sequel.

Theorem 1. *Let $x \in \mathbb{R}^+$, $0 \in \mathbb{T}$ and $s \in \mathbb{T}^+$. Then the following properties hold.*

(i) $\lim_{t\to 0^+} \mathrm{p}_x(t,s) = 0$ *provided that* $\mu(0) = 0$.

(ii) $\mathrm{p}_x(\cdot, s)$ *is of exponential order for any positive number on* $\mathbb{T}_0^+ := [0,\infty)_\mathbb{T}$ *provided that* $\sup \mathbb{T} = \infty$, *i.e.*

$$\lim_{t\to\infty}\big(\mathrm{p}_x(t,s)\mathrm{e}_{\ominus_\mu \alpha}(t,0)\big) = 0 \quad \textit{for any } \alpha \in \mathbb{R}^+. \tag{3}$$

Proof. (i) We shall first show that the limit exists. Clearly, $\mathrm{p}_x^\Delta(t,s) = (x/t)\mathrm{p}_x(t,s) > 0$ for all $t \in \mathbb{T}^+$, which implies that $\mathrm{p}_x(\cdot,s)$ is increasing on $\mathbb{T}^+$. Thus, $\ell_s := \lim_{t\to 0^+} \mathrm{p}_x(t,s)$ exists and satisfies $\ell_s \in [0,1)_\mathbb{R}$ by Lemma 6 (i). To prove $\ell_s = 0$, assume the contrary that $\ell_s \in (0,1)_\mathbb{R}$. Then, $\mathrm{p}_x^\Delta(t,s) \geq x\ell_s/t$ for all $t \in (0,s]_\mathbb{T}$. We estimate by [3, Theorem 5.1] for all $t \in (0,s]_\mathbb{T}$ that

$$\begin{aligned} 1 - \mathrm{p}_x(t,s) = &\int_t^s \mathrm{p}_x^\Delta(\eta,s)\Delta\eta \geq x\ell_s \int_t^s \frac{1}{\eta}\Delta\eta \\ \geq &x\ell_s \int_t^s \frac{1}{\eta}\mathrm{d}\eta = x\ell_s \ln\Big(\frac{s}{t}\Big), \end{aligned}$$

which yields a contradiction by letting $t \to 0^+$ since the right-hand diverges but the right-hand side is $(1-\ell_s)$, which is finite. This contradiction proves that $\ell_s = 0$.

(ii) As $\mathbb{T}$ is unbounded above, for every fixed $\alpha \in \mathbb{R}^+$, we may find $r_\alpha \in [s,\infty)_\mathbb{T}$ such that $x/t \leq \alpha$ for all $t \in [r_\alpha, \infty)_\mathbb{T}$. Therefore, we have

$$0 < \mathrm{p}_x(t,s) = \mathrm{p}_x(r_\alpha,s)\mathrm{p}_x(t,r_\alpha) \leq \mathrm{p}_x(r_\alpha,s)\mathrm{e}_\alpha(t,r_\alpha) \tag{4}$$

for all $t \in [r_\alpha,\infty)_\mathbb{T}$. Since $\mathrm{p}_x(\cdot,s) \in \mathrm{C_{rd}}(\mathbb{T}^+, \mathbb{R}_0^+)$ and $\lim_{t\to 0^+} \mathrm{p}_x(t,s) = 0$ provided that $\mu(0) = 0$ by the part (i), we may find $M_\alpha \in \mathbb{R}^+$ such that $\mathrm{p}_x(t,s) \leq M_\alpha$ for all $t \in [0,r_\alpha]_\mathbb{T}$ (see [1, Theorem 1.65]). Therefore, from (4), we have

$$0 \leq \mathrm{p}_x(t,s) \leq K_\alpha \mathrm{e}_\alpha(t,0) \quad \text{for all } t \in \mathbb{T}^+,$$

where $K_\alpha := \max\{M_\alpha, \mathrm{p}_x(r_\alpha,s)\}\mathrm{e}_{\ominus_\mu \alpha}(r_\alpha,0)$, and (3) is a consequence of [4, Lemma 4.4].

Thus, the proof is completed. ☐

The following property shows that the function p_1 is a first-order polynomial in the usual sense.

Lemma 7. *If $s \in \mathbb{T}$, then*

$$\mathrm{e}_{f^\Delta/f}(t,s) = \frac{f(t)}{f(s)} \quad \text{for all } t \in \mathbb{T}$$

for any $f \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ with $f^\Delta/f \in \mathcal{R}(\mathbb{T},\mathbb{R})$ provided that f does not vanish on $[s,t)_\mathbb{T}$.

Proof. The proof follows directly by showing that the function $y = f/f(s)$ solves the initial value problem

$$\begin{cases} y^\Delta = \dfrac{f^\Delta}{f} y & \text{on } [s,t]^\kappa_\mathbb{T} \\ y(s) = 1. \end{cases}$$

□

Corollary 1. If $s \in \mathbb{T}^+$, then

$$\mathrm{p}_1(t,s) = \frac{t}{s} \quad \text{for all } t \in \mathbb{T}^+.$$

Proof. The proof follows by letting $f = \mathrm{I}$ in Lemma 7. □

Below, we show that the derivative and the integral of the function p satisfy properties which we are familiar from the continuous case.

Theorem 2. *If $x \in \mathbb{R}^+$, then*

(i) $\mathrm{p}^\Delta_x(t,s) = (x/s)\mathrm{p}_{x\boxminus_\mu 1}(t,s)$ *for all $t \in \mathbb{T}^+_0$, where $s \in \mathbb{T}^+$,*

(ii) $\mathrm{p}^\Delta_{\boxminus_\mu x}(t,s) = -(x/s)\mathrm{p}^\sigma_1(t,t)\mathrm{p}^\sigma_{\boxminus_\mu(x\boxplus_\mu 1)}(t,s)$ *for all $t \in \mathbb{T}^+$, where $s \in \mathbb{T}^+$,*

(iii) $\int_s^t \mathrm{p}_{x\boxminus_\mu 1}(\eta,r)\Delta\eta = (r/x)\big(\mathrm{p}_x(t,r) - \mathrm{p}_x(s,r)\big)$ *for all $s,t \in \mathbb{T}^+_0$, where $r \in \mathbb{T}^+$,*

(iv) $\int_s^t \mathrm{p}^\sigma_1(\eta,\eta)\mathrm{p}^\sigma_{\boxminus_\mu(x\boxplus_\mu 1)}(\eta,r)\Delta\eta = (s/x)\big(\mathrm{p}_{\boxminus_\mu x}(s,r) - \mathrm{p}_{\boxminus_\mu x}(t,r)\big)$ *for all $s,t \in \mathbb{T}^+_0$, where $r \in \mathbb{T}^+$.*

Proof. (i) Using Definition 1, Lemma 6 (iv) and Theorem 1, for all $t \in \mathbb{T}^+$, we have

$$\mathrm{p}^\Delta_x(t,s) = \mathrm{e}^\Delta_{x/\mathrm{I}}(t,s) = \frac{x}{t}\mathrm{p}_x(t,s) = \frac{x}{s}\frac{1}{\mathrm{p}_1(t,s)}\mathrm{p}_x(t,s) = \frac{x}{s}\mathrm{p}_{x\boxminus_\mu 1}(t,s).$$

(ii) We can easily compute by using the part (i) of the proof that

$$\begin{aligned}\mathrm{p}^\Delta_{\boxminus_\mu x}(t,s) &= \left(\frac{1}{\mathrm{p}_x(t,s)}\right)^\Delta = -\frac{\mathrm{p}^\Delta_x(t,s)}{\mathrm{p}_x(t,s)\mathrm{p}^\sigma_x(t,s)} = -\frac{x}{s}\frac{\mathrm{p}_{x\boxminus_\mu 1}(t,s)}{\mathrm{p}_x(t,s)\mathrm{p}^\sigma_x(t,s)} \\ &= -\frac{x}{s}\frac{1}{\mathrm{p}_1(t,s)\mathrm{p}^\sigma_x(t,s)}\end{aligned}$$

for all $t \in \mathbb{T}^+$. Applying now Lemma 6 (iii)–(v) and making some arrangements, for all $t \in \mathbb{T}^+$, we get

$$\begin{aligned}\mathrm{p}^{\Delta}_{\boxminus_\mu x}(t,s) &= -\frac{x}{s}\frac{\mathrm{p}^{\sigma}_1(t,t)}{\mathrm{p}^{\sigma}_1(t,s)\mathrm{p}^{\sigma}_x(t,s)} = -\frac{x}{s}\frac{\mathrm{p}^{\sigma}_1(t,t)}{\mathrm{p}^{\sigma}_{x\boxplus_\mu 1}(t,s)} = -\frac{x}{s}\mathrm{p}^{\sigma}_1(t,t)\frac{1}{\mathrm{p}^{\sigma}_{x\boxplus_\mu 1}(t,s)}\\ &= -\frac{x}{s}\mathrm{p}^{\sigma}_1(t,t)\mathrm{p}^{\sigma}_{\boxminus_\mu(x\boxplus_\mu 1)}(t,s).\end{aligned}$$

(iii) We will use the conclusion of the part (i). For all $s,t \in \mathbb{T}^+$, we have

$$\begin{aligned}\int_s^t \mathrm{p}_{x\boxminus_\mu 1}(\eta,r)\Delta\eta =& \frac{r}{x}\int_s^t \mathrm{p}^{\Delta}_x(\eta,r)\Delta\eta = \frac{r}{x}\mathrm{p}_x(\eta,r)\Big|_{\eta=s}^{\eta=t}\\ =& \frac{r}{x}\big(\mathrm{p}_x(t,r)-\mathrm{p}_x(s,r)\big).\end{aligned}$$

(iv) The proof of the part (iv) can be given similar to that of the part (iii), thus we omit it here.

The proof is therefore completed. □

The following lemma plays an important role in proving asymptotic properties of the Gamma function.

Lemma 8. *If $s,t \in \mathbb{T}^+$ with $t > s$, then*

$$\lim_{x\to\infty}\frac{\mathrm{p}_x(t,s)}{x} = \infty.$$

Proof. As $x \to \infty$, we may suppose that $x \in \mathbb{R}^+$. For $k \in \mathbb{N}$, we define

$$\mathrm{m}^k_x(t,s) := \int_s^t \frac{\mathrm{m}^{k-1}_x(\eta,s)}{1+x\mu(\eta)}\Delta\eta \quad \text{for } s,t \in \mathbb{T} \text{ and } x \in \mathbb{R}^+.$$

where $\mathrm{m}^0_x(t,s) := 1$. Then, as in the proof [4, Theorem 7.1], for $k \in \mathbb{N}$, we have

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\mathrm{e}_x(t,s) = \mathrm{m}^k_x(t,s)\mathrm{e}_x(t,s) > 0 \quad \text{for all } x \in \mathbb{R}^+,$$

which implies that the functions $\mathrm{e}_\cdot(t,s)$ and $\mathrm{e}'_\cdot(t,s)$ are positive increasing and convex on $\mathbb{R}^+$, and thus, $\lim_{x\to\infty}\mathrm{e}_x(t,s) = \infty$ and $\lim_{x\to\infty}\mathrm{e}'_x(t,s) = \infty$. Therefore, we see that

$$\lim_{x\to\infty}\frac{\mathrm{e}_x(t,s)}{x} = \lim_{x\to\infty}\mathrm{e}'_x(t,s) = \infty,$$

where we have applied the usual L'Hôpital's rule in the last step. By Definition 1, we have the inequality $\mathrm{p}_x(t,s) \geq \mathrm{e}_{x/t}(t,s)$ for all $x \in \mathbb{R}^+$, which completes the proof. □

3 The Gamma operator

In this section, we give our definitions and lemmas to prove the main results stated in the sequel. Throughout the paper, we shall assume that $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$.

As we have mentioned previously, we shall define the generalized Gamma function by means of the generalized Laplace transform on time scales. The definition of the Laplace transform of a function $f \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}_0^+, \mathbb{R})$ is given by

$$\mathcal{L}_{\mathbb{T}}\{f\}(x) = \int_0^\infty f(\eta) \mathrm{e}^\sigma_{\ominus_\mu x}(\eta, 0) \Delta\eta \quad \text{for } x \in (\alpha, \infty)_{\mathbb{R}},$$

where $\alpha \in \mathbb{R}$ is the exponential order of the function f (see [4, Definition 4.1]).

Now, we are ready to introduce the definition of the generalized Gamma function.

Definition 2 (The Gamma operator)**.** *For $f \in \mathcal{P}^+(\mathbb{T}_0^+, \mathbb{R}^+)$ and $s \in \mathbb{T}^+$, we define the generalized Gamma function centered at s by*

$$\Gamma_{\mathbb{T}}(f; s) := \mathcal{L}_{\mathbb{T}}\{\mathrm{p}_{f \boxminus_\mu 1}(\cdot, s)\}(1),$$

or explicitly

$$\Gamma_{\mathbb{T}}(f; s) = \int_0^\infty \mathrm{p}_{f \boxminus_\mu 1}(\eta, s) \mathrm{e}^\sigma_{\ominus_\mu 1}(\eta, 0) \Delta\eta$$

provided that the improper integral exists.

Remark 3. In view of Remark 1, the generalized Gamma function can be rewritten as

$$\Gamma_{\mathbb{T}}(f; s) = \int_0^\infty \mathrm{e}_{(f-1)/\sigma}(\eta, s) \mathrm{e}^\sigma_{\ominus_\mu 1}(\eta, 0) \Delta\eta \quad \text{for } f \in \mathcal{P}(\mathbb{T}_0^+, \mathbb{R}^+),$$

where $s \in \mathbb{T}^+$ is fixed.

Table 4 illustrates the generalized Gamma function on some well-known time scales.

Figure 1 shows the plot of graphic of $\Gamma_{\mathbb{T}}(\cdot; 1)$ on particular cases of the time scales given in Table 4.

3.1 Convergence of the Gamma function

Out first result is on the convergence of the Gamma function.

Theorem 3. *If $s \in \mathbb{T}^+$, then $\Gamma_{\mathbb{T}}(x; s)$ converges for any $x \in \mathbb{R}^+$.*

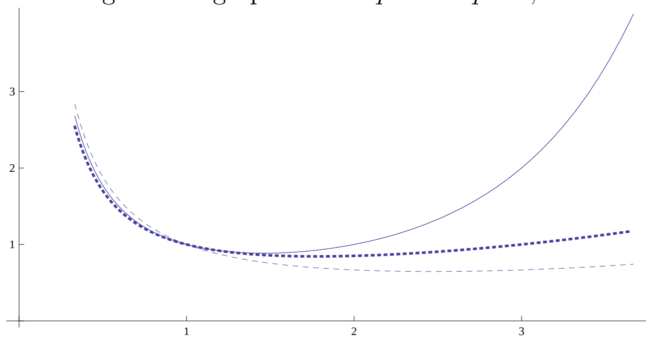
Proof. Using Theorem 1 (i), Theorem 2 (iii) and [1, Theorem 3.87], we get

$$\begin{aligned}\Gamma_{\mathbb{T}}(x; s) =& \mathcal{L}_{\mathbb{T}}\{\mathrm{p}_{x \boxminus_\mu 1}(\cdot, s)\}(1) = \frac{s}{x} \mathcal{L}_{\mathbb{T}}\{\mathrm{p}_x^\Delta(\cdot, s)\}(1) \\ =& \frac{s}{x}\Big(\mathcal{L}_{\mathbb{T}}\{\mathrm{p}_x(\cdot, s)\}(1) - \lim_{t \to 0^+} \mathrm{p}_x(t, s)\Big) \\ =& \frac{s}{x} \mathcal{L}_{\mathbb{T}}\{\mathrm{p}_x(\cdot, s)\}(1). \end{aligned} \tag{5}$$

Table 4: The generalized Gamma function on some particular time scales.

$\mathbb{T}$	$\Gamma_{\mathbb{T}}(x;s)$, $(x\in\mathbb{R}^{+})$
$\mathbb{R}_0^+$	$\displaystyle\int_0^\infty \left(\frac{\eta}{s}\right)^{x-1}\mathrm{e}^{-\eta}\mathrm{d}\eta$
$h\mathbb{N}_0$, $(h>0)$	$\displaystyle h\sum_{\eta=0}^{\infty}\left(\prod_{\zeta=s/h}^{\eta-1}\frac{\zeta+x}{\zeta+1}\right)\frac{1}{(h+1)^{\eta+1}}$
$\overline{q^{\mathbb{Z}}}$, $(q>1)$	$\displaystyle\frac{(q-1)s}{\big(1+(q-1)x\big)^{\log_q(s)}}\sum_{\eta=-\infty}^{\infty}\frac{\big(1+(q-1)x\big)^{\eta}}{\prod_{\zeta=-\infty}^{\eta}\big(1+(q-1)q^{\zeta}\big)}$

Figure 1: The solid curve belongs to graph of $\mathbb{T}=\mathbb{R}_0^+$ with $s=1$, and the dashed curve belongs to the graph of $\mathbb{T}=h\mathbb{N}_0$ with $h=1, s=1$, while the dotted curve belongs to the graph of $\mathbb{T}=\overline{q^{\mathbb{Z}}}$ with $q=2, s=1$.



As $\mathrm{p}_x(\cdot,s)$ is of exponential for order any positive number, we deduce from [4, Theorem 5.2] that $\mathcal{L}_{\mathbb{T}}\{\mathrm{p}_x(\cdot,s)\}(1)$ is finite (because of $1>0$). This completes the proof. $\square$

3.2 Asymptotic properties of the Gamma function

Theorem 4. *If* $s\in\mathbb{T}^{+}$, *then* $\lim_{x\to 0^+}\Gamma_{\mathbb{T}}(x;s)=\infty$.

Proof. Clearly, $\mathrm{e}_{\ominus_\mu 1}(\cdot,0)$ is positive and strictly decreasing on $\mathbb{T}_0^+$. On the other hand, by Lemma 6 (vi) and Remark 2, we have

$$\mathrm{p}_{x\boxminus_\mu 1}(t,s)=\frac{\mathrm{p}_x(t,s)}{\mathrm{p}_1(t,s)}>0\quad\text{for all } t\in\mathbb{T}_0^+.$$

By using the definition of the Gamma function, Theorem 1 (i) and Theorem 2 (iii), we get

$$\Gamma_{\mathbb{T}}(x;s) \geq \int_0^s \mathrm{p}_{x\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta \geq \mathrm{e}_{\ominus_\mu 1}(s,0)\int_0^s \mathrm{p}_{x\boxminus_\mu 1}(\eta,s)\Delta\eta$$
$$= \frac{s}{x}\mathrm{e}_{\ominus_\mu 1}(s,0)\big(\mathrm{p}_x(s,s)-\mathrm{p}_x(0,s)\big) = \frac{s}{x}\mathrm{e}_{\ominus_\mu 1}(s,0)$$

for all $x \in \mathbb{R}^+$, which yields the desired result by letting $x \to 0^+$. □

Theorem 5. *If* $s \in \mathbb{T}^+$, *then* $\lim_{x\to\infty}\Gamma_{\mathbb{T}}(x;s) = \infty$.

Proof. Let $t \in \mathbb{T}^+$ with $t > s$. For all $x \in \mathbb{R}^+$, we have

$$\Gamma_{\mathbb{T}}(x;s) \geq \int_s^t \mathrm{p}_{x\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta \geq \mathrm{e}_{\ominus_\mu 1}(s,0)\int_s^t \mathrm{p}_{x\boxminus_\mu 1}(\eta,s)\Delta\eta.$$

Using Theorem 2 (iii), we get

$$\Gamma_{\mathbb{T}}(x;s) \geq \frac{s}{x}\mathrm{e}_{\ominus_\mu 1}(s,0)\big(\mathrm{p}_x(t,s)-1\big)$$

for all $x \in \mathbb{R}^+$. Now, letting $x \to \infty$ and using Lemma 8, we see that the claim is true. □

3.3 Functional properties of the Gamma function

Theorem 6. *If* $s \in \mathbb{T}^+$, *then* $\Gamma_{\mathbb{T}}(1;s) = 1$.

Proof. We get

$$\Gamma_{\mathbb{T}}(1;s) = \int_0^\infty \mathrm{p}_{1\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta = \int_0^\infty \mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta$$
$$= -\,\mathrm{e}_{\ominus_\mu 1}(\eta,0)\Big|_{\eta=0}^{\eta\to\infty} = 1,$$

where we have applied [4, Theorem 3.4(iii)] in the last step to complete the proof. □

Next, we give another result emphasizing an important property of the generalized Gamma function which we are familiar from the continuous case.

Theorem 7. *If* $s \in \mathbb{T}^+$, *then*

$$\Gamma_{\mathbb{T}}(x \boxplus_\mu 1;s) = \frac{x}{s}\Gamma_{\mathbb{T}}(x;s) \quad \textit{for all } x \in \mathbb{R}^+.$$

Proof. Using (5) and Lemma 6 (v),(vi), we get

$$\Gamma_{\mathbb{T}}(x;s) = \frac{s}{x}\mathcal{L}_{\mathbb{T}}\{\mathrm{p}_x(\cdot,s)\}(1) = \frac{s}{x}\mathcal{L}_{\mathbb{T}}\{\mathrm{p}_{(x\boxplus_\mu 1)\boxminus_\mu 1}(\cdot,s)\}(1)$$
$$= \frac{s}{x}\Gamma_{\mathbb{T}}(x \boxplus_\mu 1;s).$$

The proof is completed. □

Remark 4. The conclusion of Theorem 7 can be rearranged to obtain

$$\Gamma_{\mathbb{T}}\Big(\frac{x\sigma}{\mathrm{I}}+1;s\Big)=\frac{x}{s}\Gamma_{\mathbb{T}}(x;s)\quad\text{for all } x\in\mathbb{R}^+,$$

where $s\in\mathbb{T}^+$ is fixed. Note that for $\mathbb{T}=\mathbb{R}$, we have $\sigma=\mathrm{I}$, for $\mathbb{T}=\mathbb{Z}$, we have $\sigma=\mathrm{I}+1$, and for $\mathbb{T}=\overline{q^{\mathbb{Z}}}$ with $q>1$, we have $\sigma=q\mathrm{I}$. For $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\overline{q^{\mathbb{Z}}}$, the quantity σ/I is a constant, we therefore get a "nice" recursion formula.

3.4 An inequality related to the Gamma function

The following result can be regarded as the generalization of the well-known logarithmic convexity of the usual Gamma function.

Theorem 8 (Logarithmic convexity)**.** *If $s\in\mathbb{T}^+$, then*

$$\log\left(\Gamma_{\mathbb{T}}\left(\Big(\frac{1}{\alpha}\boxdot_\mu x\Big)\boxplus_\mu\Big(\frac{1}{\beta}\boxdot_\mu y\Big);s\right)\right)\le\frac{1}{\alpha}\log\big(\Gamma_{\mathbb{T}}(x;s)\big)+\frac{1}{\beta}\log\big(\Gamma_{\mathbb{T}}(y;s)\big)$$

for all $x,y\in\mathbb{R}^+$ and all $\alpha,\beta\in(1,\infty)_{\mathbb{R}}$ with $1/\alpha+1/\beta=1$.

Proof. We can compute that

$$\begin{aligned}
&\Gamma_{\mathbb{T}}\left(\Big(\frac{1}{\alpha}\boxdot_\mu x\Big)\boxplus_\mu\Big(\frac{1}{\beta}\boxdot_\mu y\Big);s\right)\\
&\quad=\int_0^\infty \mathrm{p}_{((\alpha^{-1}\boxdot_\mu x)\boxplus_\mu(\beta^{-1}\boxdot_\mu y))\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta\\
&\quad=\int_0^\infty \mathrm{p}_{\alpha^{-1}\boxdot_\mu x}(\eta,s)\mathrm{p}_{\beta^{-1}\boxdot_\mu y}(\eta,s)\mathrm{p}_{\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta\\
&\quad=\int_0^\infty \big(\mathrm{p}_x(\eta,s)\big)^{1/\alpha}\big(\mathrm{p}_y(\eta,s)\big)^{1/\beta}\mathrm{p}_{\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta\\
&\quad=\int_0^\infty \big(\mathrm{p}_{x\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\big)^{1/\alpha}\big(\mathrm{p}_{y\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\big)^{1/\beta}\Delta\eta,
\end{aligned}$$

which yields by an application of the Hölder's inequality [1, Theorem 6.13] that

$$\begin{aligned}
\Gamma_{\mathbb{T}}\left(\Big(\frac{1}{\alpha}\boxdot_\mu x\Big)\boxplus_\mu\Big(\frac{1}{\beta}\boxdot_\mu y\Big);s\right)\le&\left(\int_0^\infty \mathrm{p}_{x\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta\right)^{1/\alpha}\\
&\times\left(\int_0^\infty \mathrm{p}_{y\boxminus_\mu 1}(\eta,s)\mathrm{e}^\sigma_{\ominus_\mu 1}(\eta,0)\Delta\eta\right)^{1/\beta}\\
=&\Big(\Gamma_{\mathbb{T}}(x;s)\Big)^{1/\alpha}\Big(\Gamma_{\mathbb{T}}(y;s)\Big)^{1/\beta}.
\end{aligned}$$

Finally, taking the logarithm of both sides we get the desired inequality. $\square$

3.5 Extension of the Gamma function

Motivated by the definition of the usual Gamma function and the functional relation proved in Theorem 7, we conclude the section with an extended definition of the generalized Gamma function.

Definition 3 (Extension of the Gamma function)**.** *For $x \in \mathbb{R}$ and $s \in \mathbb{T}^+$, we extend the definition of the generalized Gamma function by setting*

$$\Gamma_{\mathbb{T}}(x;s) := \frac{s}{x}\Gamma_{\mathbb{T}}(x \boxplus_\mu 1; s)$$

provided that $\Gamma_{\mathbb{T}}(x \boxplus_\mu 1; s)$ is computable.

4 The Bracket numbers and the Factorial operator

In this section, we introduce the factorial operator on time scales.

Definition 4. *We define the bracket number operator $[\cdot]_{\mathbb{T}} : \mathbb{N}_0 \to \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^+, \mathbb{R})$ by*

$$[n]_{\mathbb{T}} := \begin{cases} 0, & n = 0 \\ [n-1]_{\mathbb{T}} \boxplus_\mu 1, & n \in \mathbb{N}, \end{cases}$$

and the bracket factorial operator $[\cdot]_{\mathbb{T}}! : \mathbb{N}_0 \to \mathrm{C}_{\mathrm{rd}}(\mathbb{T}^+, \mathbb{R})$ by

$$[n]_{\mathbb{T}}! := \begin{cases} 1, & n = 0 \\ \displaystyle\prod_{k=1}^{n} [k]_{\mathbb{T}}, & n \in \mathbb{N}. \end{cases}$$

Table 5 illustrates the bracket number operators on some well-known time scales.

Table 5: The bracket numbers on some particular time scales. For the continuous and the quantum cases, the bracket numbers are constant functions.

$\mathbb{T}$	$[n]_{\mathbb{T}}$, $(n \in \mathbb{N})$	$[n]_{\mathbb{T}}!$, $(n \in \mathbb{N})$
$\mathbb{R}_0^+$	n	$n!$
$h\mathbb{N}_0$, $(h > 0)$	$\frac{\mathrm{I}}{h}\left(\left(\frac{h}{\mathrm{I}}+1\right)^n - 1\right)$	$\left(\frac{\mathrm{I}}{h}\right)^n \prod_{k=1}^{n}\left(\left(\frac{h}{\mathrm{I}}+1\right)^k - 1\right)$
$\overline{q^{\mathbb{Z}}}$, $(q > 1)$	$\frac{q^n - 1}{q - 1}$	$\frac{1}{(q-1)^n}\prod_{k=1}^{n}\left(q^k - 1\right)$

Remark 5. It is not hard to see that

$$[n]_{\mathbb{T}} = [n-1]_{\mathbb{T}}\frac{\sigma}{\mathrm{I}} + 1 \quad \text{for all } n \in \mathbb{N} \text{ on } \mathbb{T}^{+}.$$

Table 5 suggests us the conclusion of the following lemma.

Lemma 9. *If $n \in \mathbb{N}$, then*

$$[n]_{\mathbb{T}} = \sum_{k=1}^{n} \binom{n}{k} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1} \quad \textit{and} \quad [n]_{\mathbb{T}} = \sum_{k=1}^{n} \Big(\frac{\sigma}{\mathrm{I}}\Big)^{k-1}$$

are alternative forms of the bracket operator $[\cdot]_{\mathbb{T}}$ on $\mathbb{T}^{+}$.

Proof. We proceed by mathematical induction to prove the first equality. It is obvious that the claim is true for $n = 1$. Suppose now that the claim is true for some $n \in \mathbb{N}$, then we have

$$\begin{aligned}
[n+1]_{\mathbb{T}} =& [n]_{\mathbb{T}} \boxplus_{\mu} 1 = \left(\sum_{k=1}^{n} \binom{n}{k} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1} \right) \boxplus_{\mu} 1 \\
=& \sum_{k=1}^{n} \binom{n}{k} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1} + 1 + \sum_{k=1}^{n} \binom{n}{k} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k} \\
=& \sum_{k=1}^{n} \binom{n}{k} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1} + 1 + \sum_{k=2}^{n+1} \binom{n}{k-1} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1} \\
=& \sum_{k=1}^{n+1} \binom{n}{k} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1} + \sum_{k=1}^{n+1} \binom{n}{k-1} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1} \\
=& \sum_{k=1}^{n+1} \left[\binom{n}{k} + \binom{n}{k-1} \right] \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1} \\
=& \sum_{k=1}^{n+1} \binom{n+1}{k} \Big(\frac{\mu}{\mathrm{I}}\Big)^{k-1}
\end{aligned}$$

on $\mathbb{T}^{+}$, hence the claim is also true when n is replaced by $(n+1)$. This completes the proof of the first part. The proof of the latter equality makes use of mathematical induction and the conclusion of Remark 5, and we omit it here. □

Theorem 9. *Let $n \in \mathbb{N}$, and assume that $[k]_{\mathbb{T}}$ is a constant function on $\mathbb{T}^{+}$ for all $k \in [1, n]_{\mathbb{N}}$. Then*

$$\Gamma_{\mathbb{T}}([n]_{\mathbb{T}}; s) = \frac{[n-1]_{\mathbb{T}}!}{s^{n-1}},$$

where $s \in \mathbb{T}^{+}$.

Proof. We will give the proof by using mathematical induction again. The claim is true for $n = 1$ since $[1]_{\mathbb{T}} = 1$ and $[0]_{\mathbb{T}}!$ is assumed to be 1. We suppose that the claim is true for some $n \in \mathbb{N}$, and then by using Theorem 7 and Definition 4, we get

$$\begin{aligned}\Gamma_{\mathbb{T}}([n+1]_{\mathbb{T}}; s) =& \Gamma_{\mathbb{T}}([n]_{\mathbb{T}} \boxplus_{\mu} 1; s) = \frac{[n]_{\mathbb{T}}}{s}\Gamma_{\mathbb{T}}([n]_{\mathbb{T}}; s)\\ =& \frac{[n]_{\mathbb{T}}}{s}\frac{[n-1]_{\mathbb{T}}!}{s^{n-1}} = \frac{[n]_{\mathbb{T}}!}{s^{n}},\end{aligned}$$

which shows that the claim is true for $(n + 1)$ too. The proof is therefore completed. □

Remark 6. As an immediate consequence of Lemma 9, we see that the assumptions of Theorem 9 hold for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ with $q > 1$.

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References

[1] M. Bohner and A. Peterson, Dynamic Equations on Time Scales. An Introduction with Applications, Birkhäuser Boston, Inc., Boston, 2001.

[2] M. Bohner and A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser Boston, Inc., Boston, 2003.

[3] M. Bohner and G. Sh. Guseinov, Improper integrals on time scales, *Dynam. Systems Appl.* **12** (2003), no. 1-2, 45–65.

[4] M. Bohner, G. Sh. Guseinov, and B. Karpuz, Properties of the Laplace transform on time scales with arbitrary graininess, *Integral Transforms Spec. Funct.* **22** (2011), no. 11, 785–800.

[5] V. Kac and P. Cheung, Quantum Calculus, Universitext, Springer, 2001.

[6] J. L. Schiff. *The Laplace Transform. Theory and Applications*, Springer-Verlag, New York, 1999.

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