Form Errors in Precision Metrology:

A Survey of Measurement Techniques

Abhijit Gosavi

Email:gosavia@mst.edu

219 Engineering Management Building

Missouri S & T, Rolla, MO 65409

Elizabeth Cudney

Email: cudney@mst.edu

217 Engineering Management Building

Missouri S & T, Rolla, MO 65409

Abstract

This paper presents an overview of foundational concepts and techniques used in metrology of form errors such as straightness, flatness, circularity, sphericity, and cylindricity. While there exists a significant body of literature on form-error metrology, to the best of our knowledge, no review paper has been written on this topic. Our aim here is to (i) present a unified view of the mathematical foundations of form-error metrology and (ii) to uncover the relative strengths and weaknesses of a wide spectrum of techniques from the literature. Our analysis concludes with a discussion of opportunities for future research.

Keywords: form errors, metrology, least-squares, survey, minimum zone.

INTRODUCTION

Our goal in this paper is to present a survey of the measurement techniques employed for surface metrology features belonging in particular to the class known as *form errors*. Examples of such features include straightness, flatness, circularity, sphericity, and cylindricity. Coordinate Measuring Machines (CMMs) are widely used in the industry for automated form-error measurement. The *least-squares* technique, which was one of the first techniques to have been developed for form-error measurement, is still widely used in most industrial CMMs. Although, a number of more sophisticated and powerful techniques have now appeared in the literature, to the best of our knowledge, no survey paper has been written till date that analyzes the plethora of techniques now available for form-error measurement.

We note at the very outset that in order to keep the scope of this survey manageable, we limit our attention to form errors (other types of surface errors include waviness and roughness) and to techniques that can be used with CMMs. Alternatives to CMMs, e.g., vernier callipers, which are hand-held, or laser trackers, which generate voluminous amounts of data, are not covered. We also do not cover in this survey issues related to optimal sample sizes, distributions, or other statistical issues related to form-error measurement.

Our overall objective here is to present an overview of techniques presented in the literature for analyzing form errors. This survey covers a vast spectrum of techniques and identifies two broad groups (classes), namely *algebraic* and *computational-geometry-based*, into which the techniques surveyed can be classified. The algebraic methods can be further divided into two sub-groups: white-box (model-based) and blackbox (model-free). A key goal is to highlight differences and similarities in the various form-error-computation methods developed in the literature. White-box and black-box methods rely on *algebraic properties* of mathematical functions constructed from abstractions of the measurement processes. White-box methods make use of the *closed forms* of the underlying functions and *analytical-optimization* techniques, while black-box methods use *numeric values* of the functions and *computational-optimization* techniques. In contrast to white-box and black-box methods, computational-geometry methods use *geometric properties* of the underlying functions to compute form errors.

Another goal here is to present a *unified* view of the mathematical foundations of this important topic. With this in mind, we characterize form errors as functions of vector functions, e.g., the Euclidean norm (used in least-square methods), the max norm (used in minimax methods), and the span semi-norm (used in minimum zone methods). In addition, we highlight the relationship between the minimum zone and the least-squares approaches (Gota and Lizuka, 1977) and the same between normal and vertical deviations (Murthy and Abdin, 1980; Shunmugam, 1987a).

It is hoped that new research ideas will be stimulated from reading this survey. The rest of this article is organized as follows. The following section provides a brief discussion of the mathematical foundations of this subject. The sections entitled "White-Box Methods," "Black-Box Methods," and "Computational Geometry Techniques" discuss white-box, black-box, and computational geometric techniques respectively. The final section concludes this survey with a discussion of possible topics for future research.

MATHEMATICAL FOUNDATIONS

In this section, our goal is to provide an overview of the mathematical foundations of the science underlying form-error measurement. In the first subsection, we define reference forms (profiles) and form deviations, while in the second, we define norms and form errors.

Reference forms and form deviations

To define the form (or profile) error, one first needs to define the *reference form.* For straightness error, the reference form is an imaginary *perfect* straight line. The same for flatness, circularity (roundness), sphericity, and cylindricity error is a perfect plane, circle, sphere, and cylinder, respectively. The reference form is constructed using the data collected from the surface of the part being inspected. Throughout this paper, (x_i, y_i, z_i) will denote the ith data point's coordinates in the x, y, and z axis. The form error is generally a function of the *deviations* from the reference form.

Straightness deviation: Let $y = mx + c$ denote the straight line in two dimensions, where m and c are the parameters of the equation, i.e., the *defining parameters.* The vertical deviation of a point, (x_i, y_i), on the edge from the reference form will be given by: $e_i^v = y_i - (mx_i + c)$, where the superscript v stands for vertical. This is the distance (deviation) along the Y-axis. The normal deviation of the same point from the reference edge is:

$$e_i^n = \frac{y_i - (mx_i + c)}{\sqrt{1 + m^2}}, \tag{1}$$

where the superscript n stands for normal. It is easier to minimize vertical deviations rather than normal deviations, via the well-known *least-squares* method. This is the reason for their popularity in commercial software programs. But the normal deviations measure the deviations correctly (ISO, 1983; ANSI: Dimensioning, 1995; ANSI: Definitions, 1995). Unfortunately, normality introduces non-linearity in the definition of error thereby complicating its analysis.

Flatness deviation: The equation of a plane in three dimensions is defined as: $z = mx + ly + c$. Then the normal deviation of any sampled point, (x_i, y_i, z_i), from the reference plane is given by

$$e_i^n = \frac{z_i - (mx_i + ly_i + c)}{\sqrt{(1 + m^2 + l^2)}}, \tag{2}$$

and the vertical deviation (distance) is given by: $e_i^v = z_i - (mx_i + ly_i + c)$, which is measured along the Z-axis.

Circularity deviation: A perfect circle is typically defined by: $(x - a)^2 + (y - b)^2 = R^2$ if R denotes the radius of circle and the coordinates of the circle's center are: (a, b). The defining parameters of the equation of a circle are thus: a, b, and R. For circularity, the deviation of interest is called the *radial deviation*, which is measured from a given point *along the radius* to the circle's center. Hence the radial deviation for a point, (x_i, y_i), is: $e_i = \sqrt{(x_i - a)^2 + (y_i - b)^2} - R$, in which the square-rooted term represents the distance from the circle's center to (x_i, y_i).

Sphericity deviation: The equation of a sphere is: $(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$, where R is the radius of the sphere and the sphere's center is: (a, b, c). The radial deviation for sphericity, which is an extension of the same for circularity to three dimensions, is defined as: $e_i = \sqrt{(x_i - a)^2 + (y_i - b)^2 + (z_i - c)^2} - R$. The defining parameters of the equation of a sphere are thus: a, b, c, and R. In the deviation, the square-rooted term is the distance from (x_i, y_i, z_i) to the center of the sphere.

Cylindricity deviation: The equation of a cylinder (Gosavi and Phatakwala, 2006), whose radius is R and whose axis is defined by: $(x-u)/m = (y-v)/l = (z-w)/k$, is $(x-mt-u)^2+(y-lt-v)^2+(z-kt-w)^2 = R^2$ where $t = ((x - u)m + (y - v)l + (z - w)k) / (m^2 + l^2 + k^2)$. The defining parameters of the equation are: l, m, k, u, v, w, and R. The deviation of the ith point on a real cylinder from a perfect cylinder can then be defined as: $e_i = \sqrt{(x_i - mt_i - u)^2 + (y_i - lt_i - v)^2 + (z_i - kt_i - w)^2} - R$, where the square-rooted term is the normal distance from a point, (x_i, y_i, z_i), on the surface of the cylinder, to the axis of the cylinder.

Norms and form errors

The deviation vector, $\vec{e}$, is defined as the vector of all the deviations, i.e., $\vec{e} = \{e_1, e_2, \ldots, e_n\}$, if n measurements are made. There are three commonly-used measurement criteria, which use norms or semi-norms of the deviation vector (see any standard text on vector algebra for definitions of norms and semi-norms). They are: the Euclidean norm, the max (or maximum) norm, and the span semi-norm.

Euclidean norm: The Euclidean norm of the deviation vector $\vec{e}$ is defined as: $||\vec{e}||_2 = \sqrt{\sum_{i=1}^{n} e_i^2}$. When the Euclidean norm of the deviation vector is minimized to obtain values for the defining parameters of the reference surface, the surface obtained is the so-called "least-squares" surface, which is used in most CMMs for many features.

Max norm: The max (or maximum) norm or the Chebyshev norm of the deviation vector $\vec{e}$ is defined as: $||\vec{e}||_\infty = \max_i |e_i|$, where $|x|$ denotes the absolute value of x. When the max norm is minimized, one obtains the so-called "minimax" surface.

Span semi-norm: The span semi-norm, or *zone*, is the difference between the maximum and the minimum of the deviations. It is defined as:

$$sp(\vec{e}) \equiv (\max_i e_i - \min_i e_i) = |\max_i e_i| + |\min_i e_i|. \tag{3}$$

In (3), for the second equality to hold, it is assumed that the set of deviations contains *both* positive and negative signs. When the zone is minimized to obtain values for the defining parameters of the reference surface, the surface obtained is the so-called "minimum-zone" surface. The associated error is called the zone error, and according to (ISO, 1983) and (ANSI: Dimensioning, 1995; ANSI: Definitions, 1995) is the true error. Hence, this criterion is of critical importance in form-error metrology.

Form error: The form error is a function of the deviation vector. It is important to note that the defining parameters of the reference surface are not always computed by minimizing the zone of the deviation vector. However, after the defining parameters of the reference surface are determined, i.e., the reference surface is determined, the form error is declared to equal the zone of the deviation vector. The question that now arises is: why would one use some criterion *other than* the zone to determine the defining parameters, but subsequently use the zone to compute the form error? The answer is: Although, one uses the zone of the deviation vector to compute the form error, it is usually *easier* to use other criteria, e.g., the Euclidean norm, to compute the defining parameters of the reference surface. That is, one performs an optimization with a *surrogate* but an imprecise objective function, but after the optimization is performed, the objective function value is computed with respect to the true objective function, i.e., the zone. Obviously, this is done only if there is a strong motivation to use an imprecise objective function during the optimization. It is indeed the case that in form-error metrology, the correct objective function (the zone) is algebraically complex and not easily optimizable. The Euclidean norm on the other hand can be optimized more easily, and is hence popular. In general, the procedure of form-error estimation is a three-step procedure described next.

Step 1: Let (x_i, y_i) or (x_i, y_i, z_i) for $i = 1, 2, \ldots, n$ denote the data for the n samples gathered from the surface to be measured. Determine the values of the defining parameters of the reference surface that minimize a pre-defined function of the deviations (the Euclidean norm, the max norm, or the zone) using an algorithm.

Step 2: Use the defining parameters to calculate the deviations.

Step 3: Compute the zone of the vector of deviations and declare the zone to be the form error.

In this three-step procedure, there are a number of factors, and each of them introduces an error that will be called a *bias*. The first factor arises from the fact that although there is an infinite number of points on the actual surface, we use only a finite sample. This is called the *sampling* bias. This issue is beyond the scope of this paper, but we refer the interested reader to papers that study it in detail: (Badar et al., 2003; Dowling et al., 1997; Namboothiri and Shunmugam, 1999; Traband et al., 2004; Kim and Raman, 2000; Kurfess and Banks, 1995; Obeidat, 2008; Gilbert et al., 2009). The second factor, discussed above, arises when the function used in Step 1, for finding the defining parameters, is *not* the zone. Furthermore, in the case of straightness and flatness, using vertical distances in Step 1 but normal deviations in Step 2 can lead to an additional bias. Finally, a third source of bias can be traced to machine-measurement error. We now review the three classes of methods alluded to earlier in the first section.

WHITE-BOX METHODS

For the most part, white-box methods develop a surrogate objective function that can be optimized using well-known optimization techniques which guarantee convergence. Of course, the bias in these methods arises out of the use of an imprecise objective function. A major strength of these methods is their robustness and a reduced computational burden. Robustness arises out of the use of provably convergent optimization techniques, many of which also have a computational burden lower than that of black-box methods that we will discuss later. In this section, we describe a wide spectrum of white-box methods described in the literature. We begin with the most popular white-box method.

The least-squares method

The method used to minimize the Euclidean norm of the deviation vector is often called the least-squares method or ordinary least squares or regression. Because it exploits the closed form, the least-squares method is considered to be white-box.

Consider straightness using vertical distances. Step 1 of the generalized procedure minimizes the Euclidean norm, i.e., determines m and c to minimize $\sum_{i=1}^{n}(y_i - mx_i - c)^2$. The defining parameters can be determined by solving simultaneously the following two linear equations in which the unknowns are m and c:

$$\sum_{i=1}^{n} y_i = m\sum_{i=1}^{n} x_i + nc. \tag{4}$$

$$\sum_{i=1}^{n} x_i y_i = m\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i. \tag{5}$$

From the values of m and c, one computes the deviations via Equation (1), and then the form error via Equation (3).

The computational elegance of Equations (4) and (5) makes this method attractive for commercial software; the method is robust and all it needs is an algorithm for solving linear equations. The weakness is that the form error is usually *over-estimated*, which can be attributed to two sources of bias, which are: (i) the use of the Euclidean norm — and not the zone — as the objective function, and (ii) the use of vertical distances — and not normal distances — in the objective function.

Step 1 in flatness measurement involves finding m, l, and c to minimize $\sum_{i=1}^{n}(z_i - mx_i - ly_i - c)^2$. Then the following equations are solved to obtain m, l, and c:

$$\sum_{i=1}^{n} z_i = m\sum_{i=1}^{n} x_i + l\sum_{i=1}^{n} y_i + nc,$$

$$\sum_{i=1}^{n} x_i z_i = m \sum_{i=1}^{n} x_i^2 + l \sum_{i=1}^{n} x_i y_i + c \sum_{i=1}^{n} x_i, \text{ and}$$

$$\sum_{i=1}^{n} y_i z_i = m \sum_{i=1}^{n} x_i y_i + l \sum_{i=1}^{n} y_i^2 + c \sum_{i=1}^{n} y_i.$$

One then computes the deviations at each point using Equation (2), and then the form error via Equation (3).

The least-squares approach for flatness and straightness using vertical distances can be modeled as *linear* least-squares, which have a straightforward solution. This does not generalize to circularity and sphericity the functions of which are non-linear. A widely-cited formula for circularity is (Whitehouse, 2002): $a = 2\frac{\sum_{i=1}^{n} x_i}{n}, b = 2\frac{\sum_{i=1}^{n} y_i}{n}$, and $R = \frac{\sum_{i=1}^{n} \sqrt{x_i^2+y_i^2}}{n}$. This requires data from *equally-spaced* points, i.e., points which have an equal angular spacing. A large number of researchers have worked in the area of developing least-squares-based methods for solving the circularity problem (Chaudhuri and Kundu, 1993; Cooper, 1993; Takiyama and Ono, 1989; Guu and Tsai, 1999).

For sphericity, a formula analogous to the one above which also needs equally-spaced data is: $a = 2\frac{\sum_{i=1}^{n} x_i}{n}, b = 2\frac{\sum_{i=1}^{n} y_i}{n}, c = 2\frac{\sum_{i=1}^{n} z_i}{n}$, and $R = \frac{\sum_{i=1}^{n} \sqrt{x_i^2+y_i^2+z_i^2}}{n}$. Least-squares solutions and other approximations for cylindricity have been covered in the literature (Marshall et al., 2001; Shunmugam, 1986; Murthy, 1982). For least-squares problems, gradient methods of the Gauss-Newton type (Forbes, 1989) and Levenberg-Marquardt (Shakarji, 1998) type have been recommended by NPL (UK) and NIST (USA), respectively.

Mathematical programming methods

To the best of our knowledge, Murthy and Abdin (1980) were the first to advocate the minimization of the zone in form-error metrology. Also, they used normal distances in measurement of straightness and flatness. Chetwynd (1985) first proposed the use of linear programming methods to engineering metrology. His approach was aimed at minimizing the zone using vertical distances in straightness and flatness. For straightness, the linear program (LP) of Chetwynd (1985) is:

$$\text{Minimize } h \text{ subject to } h \geq 0,$$

$$mx_i + c + h \geq y_i \quad \forall i, \text{ and } mx_i + c - h \leq y_i \quad \forall i.$$

For flatness, the corresponding LP is: Minimize h subject to $h \geq 0$,

$$mx_i + ly_i + c + h \geq z_i \quad \forall i, \text{ and } mx_i + ly_i + c - h \leq z_i \quad \forall i.$$

Wang (1992) sought to use the correct distance, i.e., the normal distance, but minimized the max norm. His remarkable formulation resulted in the following non-linear program, in fact, a constrained quadratic program (QP) which was solved exactly via sequential quadratic programming (Bazaraa et al., 1993):

$$\text{Minimize } h \text{ subject to } h \geq 0,$$

$$d_i \leq h \qquad \forall i, \text{ where}$$

$$\text{and } d_i = \frac{y_i - (mx_i + c)}{\sqrt{1 + m^2}} \text{ for straightness and } d_i = \frac{z_i - (mx_i + ly_i + c)}{\sqrt{1 + m^2 + l^2}} \text{ for flatness.}$$

Although the objective function in the formulation is linear, the constraints are quadratic which makes it a QP. The step size in the optimization algorithm was determined using an augmented Lagrangian. He also minimized circularity and cylindricity using the same strategy and provided computational evidence of outperforming the least-squares method. Lin et al. (1995) also used sequential quadratic programming in form-error estimation.

Gass et al. (1998) presented a somewhat distinctive approach for handling the non-linearity in circularity and sphericity. They used a surrogate objective function which can be optimized via linear programming. They also provided encouraging computational experience which shows that their approach approximates the true error. We now describe their approach for circularity. Essentially, the mathematical program they employ is:

$$\text{Minimize } \max_i |M_i| - \min_i |M_i| \text{ where } M_i = (x_i - a)^2 + (y_i - b)^2 - R^2.$$

Note that the ith component of the true deviation vector in circularity is $\sqrt{(x_i - a)^2 + (y_i - b)^2} - R$. Clearly, $M_i = x_i^2 + y_i^2 - 2x_i a - 2y_i b - \rho$ in which ρ is defined as: $\rho = R^2 - a^2 - b^2$. However, the surrogate, M_i, used above, leads to an elegant LP. Then, the mathematical program given above becomes: Minimize h subject to $h \geq 0$

$$h \geq x_i^2 + y_i^2 - 2x_i a - 2y_i b - \rho \quad \forall i,$$

$$-h \leq x_i^2 + y_i^2 - 2x_i a - 2y_i b - \rho \quad \forall i.$$

In the above, the decision variables are h, a, b, and ρ. When the LP is solved, the values of a and b are used in the definition of ρ to determine the value of R. A simple extension exists to sphericity using: $M_i = (x_i - a)^2 + (y_i - b)^2 + (z_i - c)^2 - R^2$.

Several other researchers have used mathematical programming in the computation of form error. Cheraghi et al. (1996) and Weber et al. (2002) developed a linear approximation of the objective function using Taylor's series. Their approach required a least-squares solution in the first phase, and in the second phase, the solution of an LP. A non-linear programming approach in which the objective function was exploited to derive a feasible search direction (Wang et al., 2001, 1999) along with convergence guarantees to local optima. Ventura and Yerelan (1989) also developed a non-linear optimization procedure to solve the circularity problem. Prakasvudhisarn et al. (2003) use *support-vector regression* that employs mathematical programming for minimizing the zone.

Linearization

A large number of linear approximations (also called limacon approximations) to non-linear features have been devised in the literature (Chetwynd, 1979; Murthy, 1986; Shunmugam, 1986; Landau, 1987; Yerelan and Ventura, 1988; Lai and Chen, 1996). These approximations develop a linear surrogate for the non-linear feature. We now describe them for circularity and sphericity.

Circularity: For circularity, the polar coordinates (r_i, θ_i) and the Cartesian coordinates, (x_i, y_i), share the following relation: $x_i = r_i \cos\theta_i$, $y_i = r_i \sin\theta_i$, $r_i = \sqrt{x_i^2 + y_i^2}$, and $\theta_i = \tan^{-1}(y_i/x_i)$. The radial deviation at any point can be expressed approximately using the following linearized equation $e_i \approx r_i - (R + a\cos\theta_i + b\sin\theta_i)$, where R denotes the circle radius and (a, b) is the center.

The power of this linearization can be exploited by using it in a least-squares method in which one seeks to minimize $\sum_i e_i^2$. There will be three unknowns, R, a, and b, in the following set of linear equations, which have to be solved simultaneously:

$$\sum_{i=1}^{n} r_i = nR + a\sum_{i=1}^{n}\cos\theta_i + b\sum_{i=1}^{n}\sin\theta_i.$$

$$\sum_{i=1}^{n} r_i\cos\theta_i = R\sum_{i=1}^{n}\cos\theta_i + a\sum_{i=1}^{n}\cos^2\theta_i + b\sum_{i=1}^{n}\sin\theta_i\cos\theta_i.$$

$$\sum_{i=1}^{n} r_i\sin\theta_i = R\sum_{i=1}^{n}\sin\theta_i + a\sum_{i=1}^{n}\sin\theta_i\cos\theta_i + b\sum_{i=1}^{n}\sin^2\theta_i.$$

Sphericity: For sphericity, one can use spherical cooridinates: (r_i, θ_i, ϕ_i), whose relationship with Cartesian coordinates, (x_i, y_i, z_i), is defined below:

$$r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}, \theta_i = \tan^{-1}\left(\frac{y_i}{x_i}\right), \text{ and } \phi_i = \cos^{-1}\left(\frac{z_i}{r_i}\right).$$

The radial deviation is: $e_i \approx r_i - (R + a\cos\phi_i\cos\theta_i + b\cos\phi_i\sin\theta_i + c\sin\phi_i)$, where the sphere's center is (a, b, c) and the radius is R. For cylindricity, see Shunmugam (1986) for an analogous approximation.

Chebyshev approximations

A method that minimizes the max norm, i.e., any "minimax" method, in practice outperforms the least-squares method. As a result, the minimax criterion has attracted considerable attention. Chebyshev approximations of a function (see any standard text on numerical analysis, e.g., Ralston and Rabinowitz (2001)) can be used to determine a minimax approximation of the function. Loebritz (1993) is one of the first works to have used this approximation. Dhanish and Shunmugam (1991) used a Chebyshev approximation in an iterative algorithm for straightness, flatness, circularity, sphericity, and cylindricity. They develop an approach in which one begins with a subset of the data and modifies the data set progressively. Chatterjee and Roth (1997) present a minimax algorithm for straightness and flatness. Fukuda and Shimokohbe (1984) used a minimax approximation to minimize the zone. Grinde and Ventura (1994) use the minimax criterion for center estimation in sphericity. Gosavi (1995) used a rational function Chebyshev approximation (Press et al., 1992) using the second algorithm of Remes (Ralston and Rabinowitz, 2001) for straightness, flatness, and circularity.

BLACK-BOX METHODS

In contrast to white-box methods which employ analytical properties of the objective function, black-box methods treat the objective function as a black box of which only numeric values are of interest. Consequently, black-box methods do not rely on any specific property of the objective function, and, in principle, can work on any objective function. Hence the bias that arises from approximating the objective function, which is often present in white-box methods, is non-existent with black-box methods. Unfortunately, black-box methods are (i) not guaranteed to converge to global optima and (ii) may require a large number of function evaluations (and hence a long time). It is to be noted, however, that with the increasing power of computers, the latter point does not pose a serious challenge nowadays.

A large number of computational methods are available in the literature, which can be used in form-error metrology. Some of the well-known ones are: the Nelder-Mead simplex procedure (Nelder and Mead, 1965), the Hooke-Jeeves procedure (Hooke and Jeeves, 1961), derivative-based methods, and meta-heuristics, e.g., genetic algorithms, simulated annealing, and tabu search. The Nelder-Mead approach, also called flexible polygon search, has been exploited extensively (Murthy and Abdin, 1980; Kanada and Suzuki, 1993; Kanada, 1995; Damodarasamy and Anand, 1999). Elmaraghy et al. (1990) employed a Hooke-Jeeves search and Lai *et al.* (Lai et al., 2000) make use of a genetic algorithm. Derivative descent has been used in white-box methods (Cardou et al., 1972; Zhu et al., 2003) and black-box methods (Gosavi and Phatakwala, 2006).

In the Nelder-Mead algorithm, one starts with a simplex, which is essentially a convex hull of $(D+1)$ points selected randomly from the solution space, where D denotes the number of defining parameters. The best point in the simplex, i.e., the point with the lowest function value, and the worst (with the highest function value) point are identified. Then the *centroid* of the points in the simplex other than the worst point is computed. A so-called reflection is performed, and a new vertex is generated. If the new vertex is better (has a lower function value) than the worst vertex in the simplex, the latter is replaced by the new vertex. Although it is of a heuristic nature, the algorithm works well for a small value for D

In steepest gradient descent, the iterative algorithm begins at an arbitrary solution. Let the solution for the ith defining parameter in the pth iteration of the algorithm be denoted by q_i^p, and the vector $\vec{q^p} = \{q_1^p, q_2^p, \ldots, q_D^p\}$. The heart of the algorithm is as follows:

$$q_i^{p+1} \leftarrow q_i^p - \mu \left. \frac{\partial f(.)}{\partial q_i} \right|_{\vec{q}=\vec{q^p}}, \text{ for all } i = 1, 2, \ldots, D,$$

where $f(.)$ is usually the zone. The algorithm is terminated by checking whether the gradient norm is less than a stopping tolerance, i.e., if

$$\sqrt{\sum_{i=0}^{D} \left(\left. \frac{\partial f(.)}{\partial q_i} \right|_{\vec{q}=\vec{q^p}} \right)^2} < \epsilon,$$

where ϵ denotes the stopping tolerance, which is generally set to a small number.

Black-box methods can be used on complex features, but tend to be slower than white-box methods. Also, some researchers are uncomfortable with using them, because they do not exploit analytical properties, relying on numeric values instead. Computational-geometry methods carry the promise of a *considerably-reduced* computational time, and hence form the most exciting methods on the frontier of form-error metrology. We now present a review of these methods.

COMPUTATIONAL GEOMETRY TECHNIQUES

A number of researchers have used computational geometry (see e.g., Preparata and Shamos (1985) for a textbook) for form-error computation. A central idea in such methods is to construct at least two surfaces which (i) enclose within them all the measured points and (ii) have the minimum separating distance; in case of straightness and flatness, the two surfaces are parallel straight lines and planes, respectively, in case of circularity and sphericity, the two surfaces are concentric circles and spheres, respectively, and finally for cylindricity, the surfaces are co-axial cylinders with different radii. Most of the algorithms in this class are iterative, and start with a good solution, which is progressively improved until the optimal or near-optimal solution is obtained. These algorithms generally use the so-called Voronoi diagrams and employ the theory of convex sets. A positive aspect of these algorithms is that most of them attempt to minimize the zone.

Elzinga and Hearn (1971) is an early work that presents a fundamental geometry method for a minimax problem in location theory. Traband et al. (1989) used a convex-hull method that can be applied for measuring flatness. Huang et al. (1993) developed a so-called Control Plane Rotation Scheme for minimum zone evaluation. Carr and Ferreira (1995) developed a procedure that searches, by rotation, the direction vector at which two parallel planes (or straight lines for straightness error) have the minimum distance. Damodarasamy and Anand (1999) embed a computational-geometry mechanism within a flexible polygon search (Nelder and Mead, 1965), and produce some remarkable computational results. Shunmugam (1987b) has developed a "median search technique" that constructs in an iterative fashion a so-called median plane as the reference plane and has shown that when the defining parameters of this plane are used in calculating the form error, the latter is smaller than that obtained with the least-squares plane. Huang (1999) developed an approach to determine straightness by using points on the vertices of the convex hull of the measured points. Rajagopal and Anand (1999) developed a "selective data-partitioning" technique for determining circularity error. Lai and Wang (1988), Le and Lee (1991) and Kim et al. (2000) used a Voronoi-diagram technique, which employs the concepts of convex hull, to compute circularity. Roy and Zhang (1992) formulated a technique that considers all possible pairs of concentric circles for measuring the circularity error. Etesami and Qiao (1990) provide a Voronoi diagram to determine the centers of the maximum inscribed and the minimum circumscribed circles. Roy and Xu (1995) used a method that develops co-axial cylinders coupled with Voronoi diagrams for measuring cylindricity. Cheraghi et al. (2003) developed a procedure that rotates the cylinder in a fashion such that the circularity error in a projected circle, which can be computed relatively easily, can be exploited to determine the cylindricity error.

We note that research in computational geometry for form-error measurement is relatively novel and new, and it is also likely to have a significant influence on this field. As of now, however, most industrial strength CMMs tend to use algebraic methods. This may change in the future.

CONCLUSIONS

Form-error computation is a critical function of the quality-control department in any precision-manufacturing industry. Some textbooks on surface metrology have appeared (Miller, 1962; Whitehouse, 2002) in the open literature. However, there is no book that deals elaborately with form-error computation, which is one reason why we felt the need for writing a review. Our review focused on optimization techniques used in form-error metrology.

Some of the highlights of our findings are as follows. The literature on the measurement techniques is broadly classifiable into two streams, namely algebraic and geometric, and within the algebraic class, one finds two different sub-classes of techniques, namely white-box and black-box. The complexity of the correct objective function, i.e., the zone, which is difficult to optimize as a result, appears to be major source of the approximations that are difficult to avoid. The approximations appear in developing a surrogate (white-box) or in the technique itself (black-box). The appeal of a linearization approach became evident because of its ability to make the objective function more easily optimizable and amendable to linear regression/linear programming. The black-box techniques, which use this function as it is in the optimization, appear to be of an iterative nature may be time-consuming on the computer. It is important to note that reliance on linear regression (least squares) in commercial CMMs can be traced to the complexity of analyzing the zone.

An aspect beyond the scope of this paper is that of statistical issues related to sampling. However, it must be pointed out that statistical issues can play a role in deciding whether it is worthwhile using normal distances instead of vertical distances (Badar et al., 2003; Shunmugam, 1987a) because of the increased computational time. Also, according to Dowling et al. (1995), the least-squares method may be safer in practice than the minimum zone error. There is also a growing body of literature that seeks to use laser trackers for function fitting (see e.g., Yang and Qian (2008)). Laser trackers gather data for millions of points (*point cloud*) on the surface of the part. Sampling becomes very critical as a result, since most metrology algorithms are likely to become very slow in processing millions of points. Least squares, which requires only additions and is not of an iterative nature, is likely to fare better in this scenario; however, the approximations it introduces are only likely to be amplified.

A number of other issues related to form-error metrology were not covered in this review. We would like to provide the reader with some pointers in these directions however. Roughness and waviness form two other types of surface errors, distinct from form errors, and have been discussed widely in the literature, e.g., Whitehouse (2002) and Raja et al. (2002). An interesting paper on form-error measurement with traditional instruments such as gauges is Griffith (2002). Sample-size determination for form-error measurement, i.e., determining the number of data points to be selected, is a critical issue that has been examined in many papers, e.g., Dowling et al. (1997). For an understanding of tolerance evaluation in the form-error context, please see Traband et al. (2004). An interesting paper that considers the distribution of flatness measurements with the aim of improving fits in assemblies is Berrado et al. (2006). Finally, form errors of complicated surfaces, e.g., cones, have been covered in Chung and Raman (1999).

We conclude this paper with some directions for future research in form-error metrology that we believe our review uncovered.

- The *combined* use of normal distances (as in Wang (1992)) and the zone (as in Chetwynd (1985)) has not been attempted in the literature and should form an interesting avenue for future research for all features. This is because the sequential quadratic procedure is guaranteed to converge to the global optimum (Bazaraa et al., 1993).

- The surrogate-function approach of Gass et al. (1998) could perhaps be exploited in cylindricity. The power of their approach is that the objective function can be optimized by linear programming for which efficient software programs are available. Linearization (Chetwynd, 1979) is yet another powerful approximation that could be combined with the linear programming approach of Chetwynd (1985) for circularity and sphericity.

- There is a need for developing CMM software based on computational-geometry-based techniques. While software based on linear programming and statistical techniques are readily available, software for computational geometry have to be tailor-made for the technique at hand. However, the development of such software programs is likely to increase the usage of these techniques in industrial strength CMMs.

- There is an urgent need to study six sigma methodologies that can benefit from improved form-error metrology. The literature on the interface of surface metrology and six sigma appears to be scant.

ABOUT THE AUTHORS

Abhijit Gosavi has a Ph.D. in Industrial Engineering from the University of South Florida, an M. Tech in Mechanical Engineering from the Indian Institute of Technology, Madras, and a B.E. in Mechanical Engineering from Jadavpur University, Calcutta. His research interests are in simulation modeling, Markov decision theory, and quality control. He has published in these areas, and some of his work has appeared in *Automatica*, *INFORMS Journal on Computing*, and *Management Science*. He is the Division Chair of the Industrial Engineering Division of the American Society of Engineering Education (2010-2011). He joined as an Assistant Professor in the Engineering Management and Systems Engineering Department of Missouri University of Science and Technology in 2008. He is a member of IIE, INFORMS, and ASEE.

Elizabeth Cudney is an Assistant Professor in the Engineering Management and Systems Engineering Department at Missouri University of Science and Technology. She received her B.S. in Industrial Engineering from North Carolina State University, Master of Engineering in Mechanical Engineering and Master of Business Administration from the University of Hartford, and her doctorate in Engineering Management from the University of Missouri - Rolla. In 2010, Dr. Cudney was inducted into the ASQ International Academy for Quality. She received the 2008 ASQ A.V. Feigenbaum Medal and the 2006 SME Outstanding Young Manufacturing Engineering Award. She is an ASQ Certified Quality Engineer, Manager of Quality/Operational Excellence, and Certified Six Sigma Black Belt. She is a member of the ASEE, ASEM, ASME, ASQ, IIE, SAE, and the Japan Quality Engineering Society (JQES).

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