Introduction and Implementation for Finite Element Methods

Chapter 2: 2D/3D finite element spaces

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Outline

1. 2D uniform Mesh
2. Triangular elements
3. Rectangular elements
4. 3D elements
5. More discussion
Outline

1. 2D uniform Mesh
2. Triangular elements
3. Rectangular elements
4. 3D elements
5. More discussion
Consider \( \Omega = [\text{left}, \text{right}] \times [\text{bottom}, \text{top}] \).

First, we form a uniform rectangular partition of \( \Omega \) into \( N_1 \) elements in \( x \) – \( axis \) and \( N_2 \) elements in \( y \) – \( axis \) with mesh size

\[
h = [h_1, h_2] = \left[ \frac{\text{right} - \text{left}}{N_1}, \frac{\text{top} - \text{bottom}}{N_2} \right].
\]
For example, when $N_1 = N_2 = 8$, we have
Then we divide each rectangular element into two triangular elements by connecting the left-top corner and the right-lower corner of the rectangular element.

For example, when $N_1 = N_2 = 8$, we have
This would give an uniform triangular partition.

There are $N = 2N_1 N_2$ elements and $N_m = (N_1 + 1)(N_2 + 1)$ mesh nodes.
Define your global indices for all the mesh elements $E_n \ (n = 1, \ldots, N)$ and mesh nodes $Z_k \ (k = 1, \ldots, N_m)$.

For example, when $N_1 = N_2 = 2$, we have
Let $N_i$ denote the number of local mesh nodes in a mesh element. Define your index for the local mesh nodes in a mesh element.
Triangular mesh: information matrices

- Define matrix $P$ to be an information matrix consisting of the coordinates of all mesh nodes.
- Define matrix $T$ to be an information matrix consisting of the global node indices of the mesh nodes of all the mesh elements.
- We can use the $j^{th}$ column of the matrix $P$ to store the coordinates of the $j^{th}$ mesh node and the $n^{th}$ column of the matrix $T$ to store the global node indices of the mesh nodes of the $n^{th}$ mesh element. For example, when $N_1 = N_2 = 2$, we have

$$P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \end{pmatrix}.$$
Triangular mesh: boundary edge index

- Define your index for the boundary edges.
- For example, when $N_1 = N_2 = 2$, we have

```
  6  5
7   4

  8  3
1   2
```
Triangular mesh: boundary edge information matrix

- Matrix `boundaryedges`:
  - `boundaryedges(1, k)` is the type of the $k^{th}$ boundary edge $e_k$: Dirichlet (-1), Neumann (-2), Robin (-3)......
  - `boundaryedges(2, k)` is the index of the element which contains the $k^{th}$ boundary edge $e_k$.
  - Each boundary edge has two end nodes. We index them as the first and the second counterclockwise along the boundary.
  - `boundaryedges(3, k)` is the global node index of the first end node of the $k^{th}$ boundary boundary edge $e_k$.
  - `boundaryedges(4, k)` is the global node index of the second end node of the $k^{th}$ boundary boundary edge $e_k$.
  - Set $nbe = \text{size}(boundaryedges, 2)$ to be the number of boundary edges;
For the mesh with \( N_1 = N_2 = 2 \) and all Dirichlet boundary condition, we have:

\[
\text{boundaryedges} = \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 5 & 6 & 8 & 8 & 4 & 3 & 1 \\
1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \\
4 & 7 & 8 & 9 & 6 & 3 & 2 & 1
\end{pmatrix}.
\]
What are the information matrices $P$, $T$, \textit{boundaryedges} for the following mesh?
What are the information matrices $P$, $T$, $boundaryedges$ for a general uniform triangular mesh with the mesh size $h = [h_1, h_2] = \left[ \frac{right - left}{N_1}, \frac{top - bottom}{N_2} \right]$ in the domain $\Omega = [left, right] \times [bottom, top]$?
Rectangular mesh: uniform partition

- Consider $\Omega = [left, right] \times [bottom, top]$.

- Consider a uniform rectangular partition of $\Omega$ into $N_1$ elements in $x$ – axis and $N_2$ elements in $y$ – axis with mesh size

  $$h = [h_1, h_2] = \left[ \frac{right - left}{N_1}, \frac{top - bottom}{N_2} \right].$$

- There are $N = N_1 N_2$ elements and $N_m = (N_1 + 1)(N_2 + 1)$ mesh nodes.
For example, when $N_1 = N_2 = 8$, we have
Rectangular mesh: global indices

- Define your global indices for all the mesh elements $E_n \ (n = 1, \cdots, N)$ and mesh nodes $Z_k \ (k = 1, \cdots, N_m)$.

- For example, when $N_1 = N_2 = 2$, we have

```
1 2
2 3
4 5
7 8
9   
```
Let $N_l$ denote the number of local mesh nodes in a mesh element. Define your index for the local mesh nodes in a mesh element.
Rectangular mesh: information matrices

- Define matrix $P$ to be an information matrix consisting of the coordinates of all mesh nodes.

- Define matrix $T$ to be an information matrix consisting of the global node indices of the mesh nodes of all the mesh elements.

- For example, when $N_1 = N_2 = 2$, we have

$$P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 4 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \\ 2 & 3 & 5 & 6 \end{pmatrix}. $$
Rectangular mesh: boundary edge index

- Define your index for the boundary edges.

- For example, when $N_1 = N_2 = 2$, we have

```
  6  5
  7   
  8

  1  2
```

$\frac{21}{100}$
Rectangular mesh: boundary edge information matrix

- Matrix $\text{boundaryedges}$:
  - $\text{boundaryedges}(1, k)$ is the type of the $k^{th}$ boundary edge $e_k$: Dirichlet (-1), Neumann (-2), Robin (-3)....
  - $\text{boundaryedges}(2, k)$ is the index of the element which contains the $k^{th}$ boundary edge $e_k$.
- Each boundary edge has two end nodes. We index them as the first and the second counterclockwise along the boundary.
  - $\text{boundaryedges}(3, k)$ is the global node index of the first end node of the $k^{th}$ boundary boundary edge $e_k$.
  - $\text{boundaryedges}(4, k)$ is the global node index of the second end node of the $k^{th}$ boundary boundary edge $e_k$.
- Set $\text{nbe} = \text{size}(\text{boundaryedges}, 2)$ to be the number of boundary edges;
For example, when $N_1 = N_2 = 2$ and all the boundary are Dirichlet type, we have:

$$
\text{boundaryedges} = \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 3 & 3 & 4 & 4 & 2 & 2 & 1 \\
1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \\
4 & 7 & 8 & 9 & 6 & 3 & 2 & 1
\end{pmatrix}.
$$
What are the information matrices $P$, $T$, $boundaryedges$ for the following mesh?
What are the information matrices $P$, $T$, $boundaryedges$ for a general uniform rectangular mesh with the mesh size $h = [h_1, h_2] = \left[ \frac{right - left}{N_1}, \frac{top - bottom}{N_2} \right]$ in the domain $\Omega = [left, right] \times [bottom, top]$?
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2D linear finite element: reference basis functions

- The “reference → local → global” framework will be used to construct the finite element spaces.
- We only consider the nodal basis functions (Lagrange type) in this course.
- We first consider the reference 2D linear basis functions on the reference triangular element \( \hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3 \) where \( \hat{A}_1 = (0, 0) \), \( \hat{A}_2 = (1, 0) \), and \( \hat{A}_3 = (0, 1) \).
- Define three reference 2D linear basis functions
  \[
  \hat{\psi}_j(\hat{x}, \hat{y}) = a_j \hat{x} + b_j \hat{y} + c_j, \quad j = 1, 2, 3,
  \]
  such that
  \[
  \hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 
  0, & \text{if } j \neq i, \\
  1, & \text{if } j = i,
  \end{cases}
  \]
  for \( i, j = 1, 2, 3 \).
Then it's easy to obtain

\[ \hat{\psi}_1(\hat{A}_1) = 1 \Rightarrow c_1 = 1, \]
\[ \hat{\psi}_1(\hat{A}_2) = 0 \Rightarrow a_1 + c_1 = 0, \]
\[ \hat{\psi}_1(\hat{A}_3) = 0 \Rightarrow b_1 + c_1 = 0, \]
\[ \hat{\psi}_2(\hat{A}_1) = 0 \Rightarrow c_2 = 0, \]
\[ \hat{\psi}_2(\hat{A}_2) = 1 \Rightarrow a_2 + c_2 = 1, \]
\[ \hat{\psi}_2(\hat{A}_3) = 0 \Rightarrow b_2 + c_2 = 0, \]
\[ \hat{\psi}_3(\hat{A}_1) = 0 \Rightarrow c_3 = 0, \]
\[ \hat{\psi}_3(\hat{A}_2) = 0 \Rightarrow a_3 + c_3 = 0, \]
\[ \hat{\psi}_3(\hat{A}_3) = 1 \Rightarrow b_3 + c_3 = 1. \]
Hence

\[ a_1 = -1, \quad b_1 = -1, \quad c_1 = 1, \]
\[ a_2 = 1, \quad b_2 = 0, \quad c_2 = 0, \]
\[ a_3 = 0, \quad b_3 = 1, \quad c_3 = 0. \]

Then the three reference 2D linear basis functions are

\[ \hat{\psi}_1(\hat{x}, \hat{y}) = -\hat{x} - \hat{y} + 1, \]
\[ \hat{\psi}_2(\hat{x}, \hat{y}) = \hat{x}, \]
\[ \hat{\psi}_3(\hat{x}, \hat{y}) = \hat{y}. \]
2D linear finite element: reference basis functions

- Plots of the three linear basis functions on the reference triangle:
2D linear finite element: local node index

- Let $N_{lb}$ denote the number of local finite element nodes (local finite element basis functions) in a mesh element. Here $N_{lb} = 3$. Define your index for the local finite element nodes in a mesh element.
2D linear finite element: information matrices

- The mesh information matrices $P$ and $T$ are for the mesh nodes.

- We also need similar finite element information matrices $P_b$ and $T_b$ for the finite elements nodes, which are the nodes corresponding to the finite element basis functions.

- **Note:** For the nodal finite element basis functions, the correspondence between the finite elements nodes and the finite element basis functions is one-to-one in a straightforward way. But it could be more complicated for other types of finite element basis functions in the future.

- Let $N_b$ denote the total number of the finite element basis functions (= the number of unknowns = the total number of the finite element nodes). Here $N_b = N_m = (N_1 + 1)(N_2 + 1)$. 

2D linear finite element: information matrices

- Define your global indices for all the mesh elements $E_n \ (n = 1, \cdots, N)$ and finite element nodes $X_j \ (j = 1, \cdots, N_b)$ (or the finite element basis functions).

- For example, when $N_1 = N_2 = 2$, we have
Define matrix $P_b$ to be an information matrix consisting of the coordinates of all finite element nodes.

Define matrix $T_b$ to be an information matrix consisting of the global node indices of the finite element nodes of all the mesh elements.
For the 2D linear finite elements, $P_b$ and $T_b$ are the same as the $P$ and $T$ of the triangular mesh since the nodes of the 2D linear finite element basis functions are the same as those of the mesh. For example, when $N_1 = N_2 = 2$, we have

$$P_b = P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T_b = T = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \end{pmatrix}.$$
Define your index for the boundary finite element nodes.

For example, when $N_1 = N_2 = 2$, we have,
Matrix \textit{boundarynodes}:

\textit{boundarynodes}(1, k) \text{ is the type of the } k^{th} \text{ boundary finite element node: Dirichlet (-1), Neumann (-2), Robin (-3)....}

The intersection nodes of Dirichlet boundary condition and other boundary conditions usually need to be treated as Dirichlet boundary nodes.

\textit{boundarynodes}(2, k) \text{ is the global node index of the } k^{th} \text{ boundary boundary finite element node.}

Set \textit{nbn} = \text{size(}\textit{boundarynodes}, 2\text{)} \text{ to be the number of boundary finite element nodes;}

For the above example with all Dirichlet boundary condition, we have:

\[
\textit{boundarynodes} = \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 4 & 7 & 8 & 9 & 6 & 3 & 2
\end{pmatrix}.
\]
2D linear finite element: affine mapping

- Now we can use the affine mapping between an arbitrary triangle $E = \triangle A_1 A_2 A_3$ and the reference triangle $\hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ to construct the local basis functions from the reference ones.

- Assume

$$A_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad i = 1, 2, 3.$$ 

- Consider the affine mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_2 - A_1, A_3 - A_1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + A_1$$

$$= \begin{pmatrix} x_2 - x_1, x_3 - x_1 \\ y_2 - y_1, y_3 - y_1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$
2D linear finite element: affine mapping

- The affine mapping actually maps
  
  \[
  \hat{A}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A_1,
  \]
  
  \[
  \hat{A}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = A_2,
  \]
  
  \[
  \hat{A}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = A_3.
  \]

- Hence the affine mapping maps \( \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3 \) to \( \triangle A_1 A_2 A_3 \).

- Also,
  
  \[
  \hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)},
  \]
  
  \[
  \hat{y} = \frac{(y_2 - y_1)(x - x_1) - (x_2 - x_1)(y - y_1)}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}.
  \]
2D linear finite element: affine mapping

- Define the Jacobi matrix:

\[ J = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}. \]

- Then

\[ |J| = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1), \]

and

\[ \hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|}, \]

\[ \hat{y} = \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}. \]
2D linear finite element: affine mapping

For a given function $\hat{\psi}(\hat{x}, \hat{y})$ where $(\hat{x}, \hat{y}) \in \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$, we can define the corresponding function for $(x, y) \in \triangle A_1 A_2 A_3$ as follows:

$$\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),$$

where

$$\hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|},$$

$$\hat{y} = \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}.$$
Then by chain rule, we get

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{y_3 - y_1}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{y_1 - y_2}{|J|},
\]

\[
\frac{\partial \psi}{\partial y} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{x_1 - x_3}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{x_2 - x_1}{|J|}.
\]
2D linear finite element: local basis functions

- Consider the \( n^{th} \) element \( E_n = \triangle A_{n1}A_{n2}A_{n3} \) where

\[
A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix} \quad (i = 1, 2, 3).
\]

- The three local 2D linear basis functions are

\[
\psi_{ni}(x, y) = \hat{\psi}_i(\hat{x}, \hat{y}), \quad i = 1, 2, 3,
\]

where

\[
\begin{align*}
\hat{x} &= \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|}, \\
\hat{y} &= -\frac{(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|}, \\
|J_n| &= (x_{n2} - x_{n1})(y_{n3} - y_{n1}) - (x_{n3} - x_{n1})(y_{n2} - y_{n1}).
\end{align*}
\]
And for $i = 1, 2, 3$,

$$
\frac{\partial \psi_{ni}}{\partial x} = \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \left( \frac{y_n - y_{n1}}{|J_n|} \right) + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \left( \frac{y_{n1} - y_{n2}}{|J_n|} \right),
$$

$$
\frac{\partial \psi_{ni}}{\partial y} = \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \left( \frac{x_n - x_{n3}}{|J_n|} \right) + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \left( \frac{x_{n2} - x_{n1}}{|J_n|} \right).
$$

The reference and local basis functions defined in this section are what you need to input into the code in order to use the “reference → local” framework to define the local basis functions.
2D linear finite element: local basis functions

In more details, we have

\[ \psi_{n1}(x, y) = \hat{\psi}_1(\hat{x}, \hat{y}) = -\hat{x} - \hat{y} + 1 = \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|} \]

\[ -\frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|} + 1, \]

\[ \psi_{n2}(x, y) = \hat{\psi}_2(\hat{x}, \hat{y}) = \hat{x} = \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|}, \]

\[ \psi_{n3}(x, y) = \hat{\psi}_3(\hat{x}, \hat{y}) = \hat{y} = \frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|}. \]
2D linear finite element: local basis functions

And

\[
\frac{\partial \psi_{n1}}{\partial x} = -\frac{y_{n3} - y_{n1}}{|J_n|} + \frac{y_{n2} - y_{n1}}{|J_n|} = \frac{y_{n2} - y_{n3}}{|J_n|},
\]

\[
\frac{\partial \psi_{n2}}{\partial x} = \frac{y_{n3} - y_{n1}}{|J_n|},
\]

\[
\frac{\partial \psi_{n3}}{\partial x} = -\frac{y_{n2} - y_{n1}}{|J_n|},
\]

\[
\frac{\partial \psi_{n1}}{\partial y} = \frac{x_{n3} - x_{n1}}{|J_n|} - \frac{x_{n2} - x_{n1}}{|J_n|} = \frac{x_{n3} - x_{n2}}{|J_n|},
\]

\[
\frac{\partial \psi_{n2}}{\partial y} = -\frac{x_{n3} - x_{n1}}{|J_n|},
\]

\[
\frac{\partial \psi_{n3}}{\partial y} = \frac{x_{n2} - x_{n1}}{|J_n|}.
\]

You can also directly input these local basis functions and their derivatives into your code.
In another way, the local basis functions can be also directly formed on the $n^{th}$ element $E_n = \triangle A_{n1}A_{n2}A_{n3}$ as follows:

$$\psi_{nj}(x, y) = a_{nj}x + b_{nj}y + c_{nj}, \ j = 1, 2, 3,$$

such that

$$\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 
0, & \text{if } j \neq i, \\
1, & \text{if } j = i,
\end{cases}$$

for $i, j = 1, 2, 3$.

Obtain the local basis functions in the above way and compare them with the $\psi_{n1}, \psi_{n2}$, and $\psi_{n3}$ obtained before.

They are the same!
“local $\rightarrow$ global” framework:

- Define the local finite element space

$$S_h(E_n) = \text{span}\{\psi_{n1}, \psi_{n2}, \psi_{n3}\}.$$ 

- At each finite element node $X_j$ ($j = 1, \cdots, N_b$), define the corresponding global linear basis function $\phi_j$ such that $\phi_j|_{E_n} \in S_h(E_n)$ and

$$\phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \cdots, N_b$.

- Then define the global finite element space to be

$$U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}.$$
Hence

\[ \phi_j|_{E_n} = \begin{cases} 
\psi_{n1}, & \text{if } j = T_b(1, n), \\
\psi_{n2}, & \text{if } j = T_b(2, n), \\
\psi_{n3}, & \text{if } j = T_b(3, n), \\
0, & \text{otherwise.}
\end{cases} \]

for \( j = 1, \cdots, N_b \) and \( n = 1, \cdots, N \).
We first consider the reference 2D quadratic basis functions on the reference triangular element \( \hat{E} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3 \) where \( \hat{A}_1 = (0, 0), \hat{A}_2 = (1, 0), \) and \( \hat{A}_3 = (0, 1) \). Define \( \hat{A}_4 = (0.5, 0), \hat{A}_5 = (0.5, 0.5), \) and \( \hat{A}_6 = (0, 0.5) \).

Define six reference 2D linear basis functions

\[
\hat{\psi}_j(\hat{x}, \hat{y}) = a_j \hat{x}^2 + b_j \hat{y}^2 + c_j \hat{x} \hat{y} + d_j \hat{y} + e_j \hat{x} + f_j, \quad j = 1, \cdots, 6,
\]

such that

\[
\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 
0, & \text{if } j \neq i, \\
1, & \text{if } j = i,
\end{cases}
\]

for \( i, j = 1, \cdots, 6 \).
2D quadratic finite element: reference basis functions

- For \( \hat{\psi}_1 \), it’s easy to obtain
  
  \[
  \hat{\psi}_1(\hat{A}_1) = 1 \implies f_1 = 1, \\
  \hat{\psi}_1(\hat{A}_2) = 0 \implies a_1 + e_1 + f_1 = 0, \\
  \hat{\psi}_1(\hat{A}_3) = 0 \implies b_1 + d_1 + f_1 = 0, \\
  \hat{\psi}_1(\hat{A}_4) = 0 \implies 0.25a_1 + 0.5e_1 + f_1 = 0, \\
  \hat{\psi}_1(\hat{A}_5) = 0 \implies 0.25a_1 + 0.25b_1 + 0.25c_1 + 0.5d_1 + 0.5e_1 + f_1 = 0, \\
  \hat{\psi}_1(\hat{A}_6) = 0 \implies 0.25b_1 + 0.5d_1 + f_1 = 0.
  \]

- Hence
  
  \[
  a_1 = 2, \ b_1 = 2, \ c_1 = 4, \ d_1 = -3, \ e_1 = -3, \ f_1 = 1.
  \]

- Then
  
  \[
  \hat{\psi}_1(\hat{x}, \hat{y}) = 2\hat{x}^2 + 2\hat{y}^2 + 4\hat{x}\hat{y} - 3\hat{y} - 3\hat{x} + 1.
  \]
Similarly, we can obtain all the six reference 2D quadratic basis functions:

\[
\begin{align*}
\hat{\psi}_1(\hat{x}, \hat{y}) &= 2\hat{x}^2 + 2\hat{y}^2 + 4\hat{x}\hat{y} - 3\hat{y} - 3\hat{x} + 1, \\
\hat{\psi}_2(\hat{x}, \hat{y}) &= 2\hat{x}^2 - \hat{x}, \\
\hat{\psi}_3(\hat{x}, \hat{y}) &= 2\hat{y}^2 - \hat{y}, \\
\hat{\psi}_4(\hat{x}, \hat{y}) &= -4\hat{x}^2 - 4\hat{x}\hat{y} + 4\hat{x}, \\
\hat{\psi}_5(\hat{x}, \hat{y}) &= 4\hat{x}\hat{y}, \\
\hat{\psi}_6(\hat{x}, \hat{y}) &= -4\hat{y}^2 - 4\hat{x}\hat{y} + 4\hat{y}.
\end{align*}
\]
2D quadratic finite element: reference basis functions

- Plots of the six quadratic basis functions on the reference triangle:
2D quadratic finite element: local node index

- Define your index for the local finite element nodes in a mesh element with \( N_{lb} = 6 \).
Define your global indices for all the mesh elements $E_n \ (n = 1, \cdots, N)$ and finite element nodes $X_j \ (j = 1, \cdots, N_b)$ (or the finite element basis functions) with $N_b = (2N_1 + 1)(2N_2 + 1) \neq N_m$. 
2D quadratic finite element: information matrices

For example, when $N_1 = N_2 = 2$, we have
The $P_b$ and $T_b$ for 2D quadratic finite element are different from the $P$ and $T$ for the triangular mesh. For the above example we have

$$P_b = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & 1 & 1 & 1 & 1 & 1 \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 & \cdots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1
\end{pmatrix},$$

$$T_b = \begin{pmatrix}
1 & 3 & 3 & 5 & 11 & 13 & 13 & 15 \\
11 & 11 & 13 & 13 & 21 & 21 & 23 & 23 \\
3 & 13 & 5 & 15 & 13 & 23 & 15 & 25 \\
6 & 7 & 8 & 9 & 16 & 17 & 18 & 19 \\
7 & 12 & 9 & 14 & 17 & 22 & 19 & 24 \\
2 & 8 & 4 & 10 & 12 & 18 & 14 & 20
\end{pmatrix}. $$
2D quadratic finite element: boundary node index

- Define your index for the boundary finite element nodes.

- For example, when $N_1 = N_2 = 2$, we have,
2D quadratic finite element: boundary node information matrix

- Matrix $\text{boundarynodes}$:
- For example, when $N_1 = N_2 = 2$ and all the boundary is Dirichlet type, we have:

$$\text{boundarynodes} = 
\begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & \cdots & -1 & \cdots & -1 & \cdots & -1 \\
1 & 6 & 11 & 16 & 21 & \cdots & 25 & \cdots & 5 & \cdots & 2
\end{pmatrix}.$$
2D quadratic finite element: affine mapping

- The affine mapping we use here is exactly the same as the previous one!

- Recall: for a given function $\hat{\psi}(\hat{x}, \hat{y})$ where $(\hat{x}, \hat{y}) \in \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$, we can define the corresponding function for $(x, y) \in \triangle A_1 A_2 A_3$ as follows:

$$\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),$$

where

$$\hat{x} = \frac{(y_3 - y_1)(x - x_1) - (x_3 - x_1)(y - y_1)}{|J|},$$

$$\hat{y} = \frac{-(y_2 - y_1)(x - x_1) + (x_2 - x_1)(y - y_1)}{|J|}.$$
2D quadratic finite element: affine mapping

- Recall: by chain rule, we get

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{y_3 - y_1}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{y_1 - y_2}{|J|},
\]

\[
\frac{\partial \psi}{\partial y} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{x_1 - x_3}{|J|} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{x_2 - x_1}{|J|}.
\]
By chain rule again, we get

\[
\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial x} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{x}}{\partial x} \frac{y_1 - y_2}{|J|} \\
+ \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial x} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{\partial \hat{y}}{\partial x} \frac{y_1 - y_2}{|J|} \\
= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{(y_3 - y_1)^2}{|J|^2} + 2 \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(y_3 - y_1)(y_1 - y_2)}{|J|^2} \\
+ \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{(y_1 - y_2)^2}{|J|^2}.
\]
2D quadratic finite element: affine mapping

And

\[
\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial y} \frac{x_1 - x_3}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{x}}{\partial y} \frac{x_2 - x_1}{|J|} \\
+ \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \frac{x_1 - x_3}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{\partial \hat{y}}{\partial y} \frac{x_2 - x_1}{|J|} \\
= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{(x_1 - x_3)^2}{|J|^2} + 2 \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(x_2 - x_1)}{|J|^2} \\
+ \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{(x_2 - x_1)^2}{|J|^2}.
\]
And

\[
\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial y} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{x}}{\partial y} \frac{y_1 - y_2}{|J|} \\
+ \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial y} \frac{y_3 - y_1}{|J|} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{\partial \hat{y}}{\partial y} \frac{y_1 - y_2}{|J|} \\
= \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{(x_1 - x_3)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(y_1 - y_2)}{|J|^2} \\
+ \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{(x_2 - x_1)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} \frac{(x_2 - x_1)(y_1 - y_2)}{|J|^2}.
\]
2D quadratic finite element: local basis functions

- Consider the $n^{th}$ element $E_n = \triangle A_{n1}A_{n2}A_{n3}$ where
  
  $$A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix}, \quad i = 1, 2, 3.$$ 

- Define
  
  $$A_{n4} = \frac{A_{n1} + A_{n2}}{2}, \quad A_{n5} = \frac{A_{n2} + A_{n3}}{2}, \quad A_{n6} = \frac{A_{n3} + A_{n1}}{2}.$$
2D quadratic finite element: local basis functions

- The six local 2D linear basis functions are

\[ \psi_{ni}(x, y) = \hat{\psi}_i(\hat{x}, \hat{y}), \quad i = 1, \ldots, 6, \]

where

\[ \hat{x} = \frac{(y_{n3} - y_{n1})(x - x_{n1}) - (x_{n3} - x_{n1})(y - y_{n1})}{|J_n|}, \]

\[ \hat{y} = \frac{-(y_{n2} - y_{n1})(x - x_{n1}) + (x_{n2} - x_{n1})(y - y_{n1})}{|J_n|}, \]

\[ |J_n| = (x_{n2} - x_{n1})(y_{n3} - y_{n1}) - (x_{n3} - x_{n1})(y_{n2} - y_{n1}). \]
2D quadratic finite element: local basis functions

And for $i = 1, \cdots, 6,$

\[
\frac{\partial \psi_{ni}}{\partial x} = \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{y_{n3} - y_{n1}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{y_{n1} - y_{n2}}{|J_n|},
\]

\[
\frac{\partial \psi_{ni}}{\partial y} = \frac{\partial \hat{\psi}_i}{\partial \hat{x}} \frac{x_{n1} - x_{n3}}{|J_n|} + \frac{\partial \hat{\psi}_i}{\partial \hat{y}} \frac{x_{n2} - x_{n1}}{|J_n|},
\]

\[
\frac{\partial^2 \psi_{ni}}{\partial x^2} = \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(y_3 - y_1)^2}{|J|^2} + 2 \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(y_3 - y_1)(y_1 - y_2)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(y_1 - y_2)^2}{|J|^2},
\]

\[
\frac{\partial^2 \psi_{ni}}{\partial y^2} = \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(x_1 - x_3)^2}{|J|^2} + 2 \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(x_2 - x_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(x_2 - x_1)^2}{|J|^2},
\]

\[
\frac{\partial^2 \psi_{ni}}{\partial x \partial y} = \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x}^2} \frac{(x_1 - x_3)(y_3 - y_1)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}} \frac{(x_1 - x_3)(y_1 - y_2)}{|J|^2} + \frac{\partial^2 \hat{\psi}_i}{\partial \hat{y}^2} \frac{(x_2 - x_1)(y_1 - y_2)}{|J|^2}.
\]
In another way, the local basis functions can be also directly formed on the \( n^{th} \) element \( E_n = \triangle A_{n1}A_{n2}A_{n3} \) with edge middle points \( A_{n4}, A_{n5}, \) and \( A_{n6} \): Define

\[
\psi_{nj}(x, y) = a_{nj}x^2 + b_{nj}y^2 + c_{nj}xy + d_{nj}y + e_{nj}x + f_{nj},
\]

\( j = 1, \ldots, 6, \)

such that

\[
\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 
0, & \text{if } j \neq i, \\
1, & \text{if } j = i,
\end{cases}
\]

for \( i, j = 1, \ldots, 6. \)
2D quadratic finite element: global basis functions

“local → global” framework:

- Define the local finite element space

\[ S_h(E_n) = \text{span}\{\psi_{n1}, \cdots, \psi_{n6}\}. \]

- At each finite element node \( X_j \) \((j = 1, \cdots, N_b)\), define the corresponding global linear basis function \( \phi_j \) such that

\[ \phi_j|_{E_n} \in S_h(E_n) \text{ and } \phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases} \]

for \( i, j = 1, \cdots, N_b \).

- Then define the global finite element space to be

\[ U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}. \]
Hence

\[ \phi_j|_{E_n} = \begin{cases} 
\psi_{n1}, & \text{if } j = T_b(1, n), \\
\psi_{n2}, & \text{if } j = T_b(2, n), \\
\psi_{n3}, & \text{if } j = T_b(3, n), \\
\psi_{n4}, & \text{if } j = T_b(4, n), \\
\psi_{n5}, & \text{if } j = T_b(5, n), \\
\psi_{n6}, & \text{if } j = T_b(6, n), \\
0, & \text{otherwise.} 
\end{cases} \]

for \( j = 1, \cdots, N_b \) and \( n = 1, \cdots, N \).
Outline

1. 2D uniform Mesh
2. Triangular elements
3. Rectangular elements
4. 3D elements
5. More discussion
Bilinear finite element: reference basis functions

- If we consider the reference bilinear basis functions on the reference rectangular element \( \hat{E} = \Box \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4 \) where \( \hat{A}_1 = (0, 0), \hat{A}_2 = (1, 0), \hat{A}_3 = (1, 1), and \hat{A}_4 = (0, 1) \), then the formation of these basis functions is very similar that of the reference 2D linear basis functions.

- Also, the affine mapping between \( \hat{E} = \Box \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4 \) and \( e = \Box A_1 A_2 A_3 A_4 \) is very similar to the one we use for the triangular mesh. The only change is to use \( \hat{A}_4 \) and \( A_4 \) to replace \( \hat{A}_3 \) and \( A_3 \) respectively. Think about why!

- Hence the formation of the local and global bilinear basis functions is also very similar to that of the local and global 2D linear basis functions.

- Derive the reference, local and global bilinear basis functions in the above way by yourself.
Bilinear finite element: reference basis functions

- In this section, we consider the reference bilinear basis functions on another reference rectangular element $\hat{E} = \hat{\bar{A}}_1 \hat{\bar{A}}_2 \hat{\bar{A}}_3 \hat{\bar{A}}_4$ where $\hat{\bar{A}}_1 = (-1, -1)$, $\hat{\bar{A}}_2 = (1, -1)$, $\hat{\bar{A}}_3 = (1, 1)$, and $\hat{\bar{A}}_4 = (-1, 1)$. We will also take a look at a different affine mapping.

- Define four reference bilinear basis functions

$$\hat{\psi}_j(\hat{x}, \hat{y}) = a_j + b_j \hat{x} + c_j \hat{y} + d_j \hat{x} \hat{y}, \quad j = 1, 2, 3, 4$$

such that

$$\hat{\psi}_j(\hat{\bar{A}}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3, 4$. 
Then the four reference bilinear basis functions are

\[
\hat{\psi}_1(\hat{x}, \hat{y}) = \frac{1 - \hat{x} - \hat{y} + \hat{x}\hat{y}}{4},
\]

\[
\hat{\psi}_2(\hat{x}, \hat{y}) = \frac{1 + \hat{x} - \hat{y} - \hat{x}\hat{y}}{4},
\]

\[
\hat{\psi}_3(\hat{x}, \hat{y}) = \frac{1 + \hat{x} + \hat{y} + \hat{x}\hat{y}}{4},
\]

\[
\hat{\psi}_4(\hat{x}, \hat{y}) = \frac{1 - \hat{x} + \hat{y} - \hat{x}\hat{y}}{4}.
\]
Bilinear finite element: reference basis functions

- Plots of the four bilinear basis functions on the reference triangle:
Define your index for the local finite element nodes in a mesh element with $N_{lb} = 4$. 
Bilinear finite element: information matrices

- Define your global indices for all the mesh elements $E_n (n = 1, \cdots, N)$ and finite element nodes $X_j (j = 1, \cdots, N_b)$ (or the finite element basis functions) with $N_b = N_m = (N_1 + 1)(N_2 + 1)$.

- For example, when $N_1 = N_2 = 2$, we have
For the bilinear finite elements, $P_b$ and $T_b$ are the same as the $P$ and $T$ of the rectangular mesh since the nodes of the bilinear finite element basis functions are the same as those of the mesh. For example, when $N_1 = N_2 = 2$, we have

$$P_b = P = \begin{pmatrix} 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \end{pmatrix},$$

$$T_b = T = \begin{pmatrix} 1 & 2 & 4 & 5 \\ 4 & 5 & 7 & 8 \\ 5 & 6 & 8 & 9 \\ 2 & 3 & 5 & 6 \end{pmatrix}.$$
Define your index for the boundary finite element nodes.

For example, when $N_1 = N_2 = 2$, we have
Bilinear finite element: boundary node information matrix

- Matrix `boundarynodes`:

- For example, when $N_1 = N_2 = 2$ and all the boundary is Dirichlet type, we have:

  $$boundarynodes = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 4 & 7 & 8 & 9 & 6 & 3 & 2 \end{pmatrix}.$$
Bilinear finite element: affine mapping

- Now we can use the affine mapping between an arbitrary rectangle \( E = \square A_1 A_2 A_3 A_4 \) and the reference triangle \( \hat{E} = \square \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4 \) to construct the local basis functions from the reference ones.

- Assume \( A_1, A_2, A_3, \) and \( A_4 \) are the left-lower, right-lower, right-upper, and left-upper vertices respectively.

- Assume

  \[
  A_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad (i = 1, 2, 3, 4), \quad h_1 = x_2 - x_1, \quad h_2 = y_4 - y_1.
  \]

- Consider the affine mapping

  \[
  \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} h_1 & 0 \\ 0 & \frac{1}{2} h_2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 + \frac{1}{2} h_1 \\ y_1 + \frac{1}{2} h_2 \end{pmatrix}.
  \]
Bilinear finite element: affine mapping

- The affine mapping actually maps
  \[ \hat{A}_i \rightarrow A_i, \quad i = 1, 2, 3, 4. \]

- Hence the affine mapping maps \( \square \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4 \) to \( \square A_1 A_2 A_3 A_4 \).

- Also,
  \[ \hat{x} = \frac{2x - 2x_1 - h_1}{h_1}, \]
  \[ \hat{y} = \frac{2y - 2y_1 - h_2}{h_2}. \]
Bilinear finite element: affine mapping

- For a given function \( \hat{\psi}(\hat{x}, \hat{y}) \) where \((\hat{x}, \hat{y}) \in \hat{\square}A_1 \hat{A}_2 \hat{A}_3 \hat{A}_4\), we can define the corresponding function for \((x, y) \in \square A_1 A_2 A_3 A_4\) as follows:

\[
\psi(x, y) = \hat{\psi}(\hat{x}, \hat{y}),
\]

where

\[
\hat{x} = \frac{2x - 2x_1 - h_1}{h_1}, \\
\hat{y} = \frac{2y - 2y_1 - h_2}{h_2}.
\]
Bilinear finite element: affine mapping

Then by chain rule, we get

\[
\frac{\partial \psi}{\partial x} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{2}{h_1},
\]

\[
\frac{\partial \psi}{\partial y} = \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y} = \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{2}{h_2},
\]

\[
\frac{\partial^2 \psi}{\partial x \partial y} = \frac{2}{h_1} \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} \frac{\partial \hat{x}}{\partial y} \frac{\partial \hat{y}}{\partial y} + \frac{2}{h_1} \frac{\partial^2 \hat{\psi}}{\partial \hat{x} \partial \hat{y}} \frac{\partial \hat{y}}{\partial y} = \frac{4}{h_1 h_2} \frac{\partial^2 \hat{\phi}}{\partial \hat{x} \partial \hat{y}}.
\]
Bilinear finite element: local basis functions

- Consider the $n^{th}$ element $E_n = \triangle A_{n1}A_{n2}A_{n3}A_{n4}$ where
  
  $$A_{ni} = \begin{pmatrix} x_{ni} \\ y_{ni} \end{pmatrix}. $$

  Recall that the mesh size $h = (h_1, h_2)$.

- The four local bilinear basis functions are
  
  $$\psi_{ni}(x, y) = \hat{\psi}_i(\hat{x}, \hat{y}), \quad i = 1, 2, 3, 4$$

where

$$\hat{x} = \frac{2x - 2x_{n1} - h_1}{h_1},$$

$$\hat{y} = \frac{2y - 2y_{n1} - h_2}{h_2}.$$
Bilinear finite element: local basis functions

- And for \( i = 1, 2, 3, 4 \),

\[
\frac{\partial \psi_{ni}}{\partial x} = \frac{2}{h_1} \frac{\partial \hat{\psi}_i}{\partial \hat{x}},
\]

\[
\frac{\partial \psi_{ni}}{\partial y} = \frac{2}{h_2} \frac{\partial \hat{\psi}_i}{\partial \hat{y}},
\]

\[
\frac{\partial^2 \psi_{ni}}{\partial x \partial y} = \frac{4}{h_1 h_2} \frac{\partial^2 \hat{\psi}_i}{\partial \hat{x} \partial \hat{y}}.
\]

- The reference and local functions defined in this section are what you will need to input into the code!
In another way, the local basis functions can be also directly formed on the $n^{th}$ element $E_n = \triangle A_{n1}A_{n2}A_{n3}A_{n4}$ as follows:

$$\psi_{nj}(x, y) = a_{nj} + b_{nj}x + c_{nj}y + d_{nj}, \quad j = 1, 2, 3, 4,$$

such that

$$\psi_{nj}(A_{ni}) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, 2, 3, 4$. 
Bilinear finite element: global basis functions

“local → global” framework:

- Define the local finite element space

\[ S_h(E_n) = \text{span}\{\psi_{n1}, \psi_{n2}, \psi_{n3}, \psi_{n4}\}. \]

- At each finite element node \( X_j \) \((j = 1, \ldots, N_b)\), define the corresponding global linear basis function \( \phi_j \) such that \( \phi_j|_{E_n} \in S_h(E_n) \) and

\[ \phi_j(X_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases} \]

for \( i, j = 1, \ldots, N_b \).

- Then define the global finite element space to be

\[ U_h = \text{span}\{\phi_j\}_{j=1}^{N_b}. \]
Bilinear finite element: global basis functions

Hence

\[ \phi_j|_{E_n} = \begin{cases} 
\psi_{n1}, & \text{if } j = T_b(1, n), \\
\psi_{n2}, & \text{if } j = T_b(2, n), \\
\psi_{n3}, & \text{if } j = T_b(3, n), \\
\psi_{n4}, & \text{if } j = T_b(4, n), \\
0, & \text{otherwise.} 
\end{cases} \]

for \( j = 1, \cdots, N_b \) and \( n = 1, \cdots, N \).
Biquadratic finite element: reference basis functions

- We consider the reference biquadratic basis functions on the reference rectangular element $\hat{E} = \hat{\square} \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ where $\hat{A}_1 = (-1, -1)$, $\hat{A}_2 = (1, -1)$, $\hat{A}_3 = (1, 1)$, and $\hat{A}_4 = (-1, 1)$. Define $\hat{A}_5 = (0, -1)$, $\hat{A}_6 = (1, 0)$, $\hat{A}_7 = (0, 1)$, $\hat{A}_8 = (-1, 0)$, and $\hat{A}_9 = (0, 0)$.

- Define nine reference biquadratic basis functions $\hat{\psi}_j(\hat{x}, \hat{y}) = a_j + b_j\hat{x} + c_j\hat{y} + d_j\hat{x}\hat{y} + e_j\hat{x}^2 + f_j\hat{y}^2 + g_j\hat{x}^2\hat{y} + h_j\hat{x}\hat{y}^2 + k_j\hat{x}^2\hat{y}^2$, $j = 1, \ldots, 9$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \ldots, 9$. 
Biquadratic finite element: reference basis functions

- Plots of the nine biquadratic basis functions on the reference triangle:
2D uniform Mesh

Triangular elements

Rectangular elements

3D elements

More discussion
3D linear finite element: reference basis functions

- We consider the reference 3D linear basis functions on the reference tetrahedron element \( E = \Delta \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4 \) where \( \hat{A}_1 = (0, 0, 0), \hat{A}_2 = (1, 0, 0), \), \( \hat{A}_3 = (0, 1, 0), \) and \( \hat{A}_4 = (0, 0, 1). \)

- Define four reference 3D linear basis functions

\[
\hat{\psi}_j(\hat{x}, \hat{y}, \hat{z}) = a_j \hat{x} + b_j \hat{y} + c_j \hat{z} + d_j, \quad j = 1, 2, 3, 4
\]

such that

\[
\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 
0, & \text{if } j \neq i, \\
1, & \text{if } j = i,
\end{cases}
\]

for \( i, j = 1, 2, 3, 4. \)
3D linear finite element: reference basis functions

- Then it's easy to obtain

\[
\hat{\psi}_1(\hat{A}_1) = 1 \quad \Rightarrow \quad d_1 = 1,
\hat{\psi}_1(\hat{A}_2) = 0 \quad \Rightarrow \quad a_1 + d_1 = 0,
\hat{\psi}_1(\hat{A}_3) = 0 \quad \Rightarrow \quad b_1 + d_1 = 0,
\hat{\psi}_1(\hat{A}_4) = 0 \quad \Rightarrow \quad c_1 + d_1 = 0,
\hat{\psi}_2(\hat{A}_1) = 0 \quad \Rightarrow \quad d_2 = 0,
\hat{\psi}_2(\hat{A}_2) = 1 \quad \Rightarrow \quad a_2 + d_2 = 1,
\hat{\psi}_2(\hat{A}_3) = 0 \quad \Rightarrow \quad b_2 + d_2 = 0,
\hat{\psi}_2(\hat{A}_4) = 0 \quad \Rightarrow \quad c_2 + d_2 = 0,
\]
3D linear finite element: reference basis functions

and

\[ \hat{\psi}_3(\hat{A}_1) = 0 \Rightarrow d_3 = 0, \]
\[ \hat{\psi}_3(\hat{A}_2) = 0 \Rightarrow a_3 + d_3 = 0, \]
\[ \hat{\psi}_3(\hat{A}_3) = 0 \Rightarrow b_3 + d_3 = 1, \]
\[ \hat{\psi}_3(\hat{A}_4) = 1 \Rightarrow c_3 + d_3 = 0, \]
\[ \hat{\psi}_4(\hat{A}_1) = 0 \Rightarrow d_4 = 0, \]
\[ \hat{\psi}_4(\hat{A}_2) = 0 \Rightarrow a_4 + d_4 = 0, \]
\[ \hat{\psi}_4(\hat{A}_3) = 0 \Rightarrow b_4 + d_4 = 0, \]
\[ \hat{\psi}_4(\hat{A}_4) = 1 \Rightarrow c_4 + d_4 = 1. \]
Hence

\[ a_1 = -1, \ b_1 = -1, \ c_1 = -1, \ d_1 = 1, \]
\[ a_2 = 1, \ b_2 = 0, \ c_2 = 0, \ d_2 = 0, \]
\[ a_3 = 0, \ b_3 = 1, \ c_3 = 0, \ d_3 = 0, \]
\[ a_4 = 0, \ b_4 = 0, \ c_4 = 1, \ d_4 = 0. \]

Then the four reference 3D linear basis functions are

\[ \hat{\psi}_1(\hat{x}, \hat{y}, \hat{z}) = -\hat{x} - \hat{y} - \hat{z} + 1, \]
\[ \hat{\psi}_2(\hat{x}, \hat{y}, \hat{z}) = \hat{x}, \]
\[ \hat{\psi}_3(\hat{x}, \hat{y}, \hat{z}) = \hat{y}, \]
\[ \hat{\psi}_4(\hat{x}, \hat{y}, \hat{z}) = \hat{z}. \]
Trilinar finite element: reference basis functions

- We consider the reference trilinear basis functions on the reference cube element $E = \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4 \hat{A}_5 \hat{A}_6 \hat{A}_7 \hat{A}_8$ where
  $\hat{A}_1 = (0,0,0)$, $\hat{A}_2 = (1,0,0)$, $\hat{A}_3 = (1,1,0)$, $\hat{A}_4 = (0,1,0)$,
  $\hat{A}_5 = (0,0,1)$, $\hat{A}_6 = (1,0,1)$, $\hat{A}_7 = (1,1,1)$, and
  $\hat{A}_8 = (0,1,1)$.

- Define eight reference 3D trilinear basis functions

$$\hat{\psi}_j(\hat{x}, \hat{y}, \hat{z}) = a_j + b_j \hat{x} + c_j \hat{y} + d_j \hat{z} + e_{j} \hat{x}\hat{y} + f_{j} \hat{x}\hat{z} + g_{j} \hat{y}\hat{z} + h_{j} \hat{x}\hat{y}\hat{z}, \ j = 1, \cdots, 8$$

such that

$$\hat{\psi}_j(\hat{A}_i) = \delta_{ij} = \begin{cases} 0, & \text{if } j \neq i, \\ 1, & \text{if } j = i, \end{cases}$$

for $i, j = 1, \cdots, 8$. 
Outline

1. 2D uniform Mesh
2. Triangular elements
3. Rectangular elements
4. 3D elements
5. More discussion
Approximation capability of the finite element spaces

- Question: Given (1) a function \( u \) in a Sobolev space \( H \) with norm \( \| \cdot \| \); (2) a finite element space \( U_h = \text{span}\{\phi_j\}_{j=1}^{N_b} \) with finite element nodes \( X_j \) \((j = 1, \cdots, N_b)\) and basis functions \( \phi_j \) \((j = 1, \cdots, N_b)\), how small is \( \inf_{w \in U_h} \| u - w \| \)?

- Finite element interpolation:

\[
i_h u = \sum_{j=1}^{N_b} u(X_j) \phi_j \in U_h.
\]

- Since \( u_I \in U_h \), then

\[
\inf_{w \in U_h} \| u - w \| \leq \| u - i_h u \|.
\]

- Hence the finite element interpolation error \( \| u - i_h u \| \) is a traditional tool to evaluate the approximation capability of a finite element space. (see Chapter 3 and the literature for more details)
More topics for finite elements

- Higher degree finite elements
- Mixed finite elements: Raviart-Thomas elements, Taylor-Hood elements, Mini elements
- Hermitian types of finite elements
- Nonconforming finite elements
- Another way to construct the basis functions: use the product of 1D basis functions to form the corresponding basis functions on rectangle or cube elements.