

On the fast convergence of random perturbations of the gradient flow

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Abstract. We consider in this work small random perturbations (of multiplicative noise type) of the gradient flow. We prove that under mild conditions, when the potential function is a Morse function with additional strong saddle condition, the perturbed gradient flow converges to the neighborhood of local minimizers in $O(\ln(\varepsilon^{-1}))$ time on the average, where ε is the scale of the random perturbation. Under a change of time scale, this indicates that for the diffusion process that approximates the stochastic gradient method, it takes (up to logarithmic factor) only a linear time of inverse stepsize to evade from all saddle points. This can be regarded as a manifestation of fast convergence of the discrete-time stochastic gradient method, the latter being used heavily in modern statistical machine learning.

Keywords: Random perturbations of dynamical systems, saddle point, exit problem, stochastic gradient descent, diffusion approximation

1. Introduction

1.1. Setup and the main results

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function in class $\mathbf{C}^{(3)}$ with bounded first and second derivatives and let ∇F be its gradient vector field on \mathbb{R}^n . In this paper we consider small random perturbations (of multiplicative noise type) of the gradient flow associated with the function F . Let Y_t^ε be defined as the solution to the stochastic differential equation (SDE) with small parameter $\varepsilon > 0$:

$$dY_t^\varepsilon = -\nabla F(Y_t^\varepsilon) dt + \varepsilon \sigma(Y_t^\varepsilon) dW_t, \quad Y_0^\varepsilon = x. \quad (1)$$

Here $\sigma(\bullet)$ is an $n \times n$ matrix-valued function with bounded coefficients in class $\mathbf{C}^{(2)}$ and bounded first derivatives of these coefficients, such that the diffusion matrix $a(x) = \sigma(x)\sigma^T(x)$ is uniformly positive definite (which indicates that the noise is sufficiently omnidirectional).

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In the case where $\varepsilon = 0$, the SDE (1) reduces to the gradient flow ODE:

$$\frac{d}{dt}S^t x = -\nabla F(S^t x), \quad S^0 x = x. \quad (2)$$

It is well-known, see e.g., Section 10.2 of [7], that the equilibria of the gradient flow differential equation (2) are the critical points of $F(\bullet)$ (i.e., the gradient vector evaluated at the critical point is 0). Classical results state that under mild regularity conditions for $F(\bullet)$, every solution $S^t x$ of the gradient flow (2) on \mathbb{R}^n exists for all $t \geq 0$, and $S^t x$ converges to a connected component of the set of critical points of F as $t \rightarrow \infty$. Additionally, following the trajectories of gradient flow (2) the function value F is nonincreasing, but they may be trapped in the neighborhood of a saddle point for a substantial amount of time. To see the first conclusion, note that for any solution of ODE, $F(\bullet)$ itself serves as a Lyapunov function:

$$\frac{dF(S^t x)}{dt} = \nabla F(S^t x) \cdot \frac{dS^t x}{dt} = -|\nabla F(S^t x)|^2 \leq 0,$$

and hence $F(S^t x)$ is a non-increasing function.

Our goal in this paper is to use tools from stochastic analysis to prove the following result: adding a small amount of random perturbation (multiplicative noise type) enables fast evasion from saddle points and lands the process in a neighborhood of the set of local minimum points. Let x^* be a local minimum point of $F(\bullet)$ in the sense that for some open neighborhood $U(x^*)$ of x^* :

$$x^* = \arg \min_{x \in U(x^*)} F(x).$$

Assume $F(x) > F(x^*)$, then given any positive constant $0 < e < F(x) - F(x^*)$, we aim to characterize the distribution of the stopping time T_x^ε defined as

$$T_x^\varepsilon \equiv \inf\{t \geq 0 : Y_0^\varepsilon = x, F(Y_t^\varepsilon) \leq F(x^*) + e\} \quad (3)$$

as $\varepsilon \rightarrow 0^+$. Equivalently, when starting from $Y_0^\varepsilon = x$, T_x^ε is the first time the perturbed gradient flow dynamics Y_t^ε leads to at least the amount of $F(x) - F(x^*) - e$ decay in function value of F . In a hand-waiving manner, our main result (Theorem 3.4) can be formulated as the following

Main Result. *Let the function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse function satisfying the “strong saddle condition” that will be specified in Section 1.2. When $\varepsilon \rightarrow 0^+$, $\mathbf{E}T_x^\varepsilon$ is asymptotically bounded by $C \ln(\varepsilon^{-1})$, where $C > 0$ is some constant that is determined by the landscape of the function F .*

This result is significant in the sense that, when there exist many saddle points on the landscape of the function $F(\bullet)$ the ODE (2) risks being trapped at around saddle points. Nevertheless as long as the function satisfy some additional landscape property, by adding small random perturbation $\varepsilon \sigma(Y_t^\varepsilon) dW_t$ the random dynamics (1) pays a merely $\ln(\varepsilon^{-1})$ factor of time (multiplied by a constant) to enter into a neighborhood of the set of local minimum points.

1.2. Strong saddle condition

To detail the landscape condition of $F(\bullet)$, we first remind the readers of the definition of Morse function, as the following

Definition 1.1. A function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a *Morse function* if it is infinitely differentiable and has all its critical points being *non-degenerate*, i.e., for each critical point x , all eigenvalues of the Hessian matrix $\nabla^2 F(x)$ are nonzero.

Morse functions admit a local quadratic re-parametrization at each critical point, which is the content of the so-called *Morse Lemma* [22, Lemma 2.2]. To ensure that the perturbed gradient flow escapes from saddle points, we introduce the following “strict saddle property” (compare with [12,29]) as follows.

Definition 1.2 (Strict saddle property). Given fixed $\gamma_1 > 0$ and $\gamma_2 > 0$, we say a Morse function F defined on \mathbb{R}^n satisfies the “strict saddle property” if each point $x \in \mathbb{R}^n$ belongs to one of the following: (i) $|\nabla F(x)| \geq \gamma_2 > 0$; (ii) $|\nabla F(x)| < \gamma_2$ and $\lambda_{\min}(\nabla^2 F(x)) \leq -\gamma_1 < 0$; (iii) $|\nabla F(x)| < \gamma_2$ and $\lambda_{\min}(\nabla^2 F(x)) \geq \gamma_1 > 0$.

Here $\lambda_{\min}(\nabla^2 F(x))$ stands for the minimum eigenvalue of the Hessian matrix $\nabla^2 F(x)$. Notice that since F is defined on the whole space \mathbb{R}^n , to satisfy the strict saddle property, we usually have $F(x) \rightarrow \infty$ or $F(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$. We will call a saddle point $x \in \mathbb{R}^n$ of the function F a “strict saddle” if Definition 1.2(ii) holds at x . Thus a Morse function F that satisfies the strict saddle property has all its saddle points being strict saddle points.

For the sake of proof, it is natural to assume that all eigenvalues of the Hessian $\nabla^2 F$ at critical points are uniformly bounded away from 0. This leads to our new notion of “strong saddle property” as follows.

Definition 1.3 (Strong saddle property). Let the Morse function $F(\bullet)$ satisfy the strict saddle property with parameters $\gamma_1 > 0$ and $\gamma_2 > 0$. We say the Morse function $F(\bullet)$ satisfy the “strong saddle property” if for some $\gamma_3 > 0$ and any $x \in \mathbb{R}^n$ such that $\nabla F(x) = 0$, all eigenvalues $\lambda_i, i = 1, 2, \dots, n$ of the Hessian $\nabla^2 F(x)$ at x satisfying (ii) in Definition 1.2 are bounded away from zero by some $\gamma_3 > 0$ in absolute value, i.e., $|\lambda_i| \geq \gamma_3 > 0$ for any $1 \leq i \leq n$.

We will call a saddle point $x \in \mathbb{R}^n$ of the function F a “strong saddle” if Definition 1.3 holds at x . Thus a Morse function F that satisfies the strong saddle property has all its saddle points that are strong saddle points. Throughout this paper we will work under Definition 1.3 for the function F .

1.3. Linerization of the gradient flow near a strong saddle point

By the classical Hartman–Grobman Theorem (see [1, §13]), for any strong saddle point O that we consider, there exists an open neighborhood U of O , and a $C^{(0)}$ homeomorphism mapping $f : U \rightarrow \mathbb{R}^n$, such that the gradient flow (2) is mapped by f into a linear flow. The homeomorphism f is called a (linear) conjugacy mapping. It turns out, that this mapping can be taken to be h -Hölder continuous for some $0 < h \leq 1$ that depends only on the Hessian $\nabla^2 F(O)$ at O (see [5]).

We put here an additional assumption regarding the conjugacy mapping.

Linerization Assumption. The homeomorphism f provided by the Hartman–Grobman Theorem can be taken to be $C^{(2)}$.

This assumption is needed in the proof of Lemma 2.3, which is important for Theorem 2.1. It is known that a sufficient condition for the validity of the $\mathbf{C}^{(2)}$ Linerization Assumption is the so called non-resonance condition (see, for example, the Sternberg linerization Theorem [15, Theorem 6.6.6]). We refer the reader to [15, Theorem 6.6.6] for more discussions about our Linerization Assumption.

1.4. Relation with statistical machine learning

In this subsection we draw a connection between the perturbed gradient flow defined in (1) and the stochastic gradient method in modern statistical machine learning. Stochastic Gradient Descent (SGD, [2,8,10]) with constant learning rate is a stochastic analogue of the gradient descent algorithm, aiming at finding the local or global minimizers or maximizers of the function expectation parameterized by some random variable. One can schematically formulate the optimization problem as follows: under some initial distribution, we target at finding a local minimum point x^* of the expectation of function $F(x) \equiv \mathbf{E}[F(x; \zeta)]$, where the indexed random variable ζ follows some prescribed distribution. If the regularity conditions on the exchangeability of gradient operator and expectation operator hold, e.g., when ζ is supported on a finite set, the gradient method updates via the iteration

$$x_t = x_{t-1} - \beta \mathbf{E}[\nabla F(x_{t-1}; \zeta)].$$

However when the scale of the problem is extremely large, the access of expected gradients $\mathbf{E}[\nabla F(x_{t-1}; \zeta)]$ are often expensive, and SGD prevails due to its one-query of noisy gradient at each iteration. In particular, SGD iteration often takes the following form

$$x_t = x_{t-1} - \beta \nabla F(x_{t-1}; \zeta_t), \quad (4)$$

where β is the fixed step-size (called the learning rate¹) and $\{\zeta_t\}$ are i.i.d. random variables that have the same distribution as ζ . In particular, in the case of training a deep neural network, the random variable ζ samples mini-batches \mathcal{B} with size m ($m \leq n$) uniformly from an index set $\{1, 2, \dots, n\}$: $\mathcal{B} \subset \{1, 2, \dots, n\}$ and $|\mathcal{B}| = m$. In this case, given loss functions $f_1(x) = \ell_{(a_1, b_1)}(x), \dots, f_n(x) = \ell_{(a_n, b_n)}(x)$ on training data $(a_1, b_1), \dots, (a_n, b_n)$, we have $\nabla F(x; \zeta) = \frac{1}{m} \sum_{i \in \mathcal{B}} \nabla f_i(x)$ and $\mathbf{E}[\nabla F(x; \zeta)] = \frac{1}{n} \sum_{i=1}^n \nabla f_i(x)$. Due to its advantage in scalable data, SGD gains tremendous popularity in solving many large-scale statistical machine learning problems in the age of *Big Data*.

To analyze them, we associate the SGD with a diffusion process, i.e., the solution to a stochastic differential equation. Conversely, the SGD iteration can be viewed as a discrete-time, numerical scheme of such diffusion process. This point of view is closely related to stochastic approximation theory and the weak convergence theory of Markov processes, and has been thoroughly investigated in many previous works, notably the classical monographs [6,7,18]. In particular, [18, Chapters 7–9] describes the weak convergences of discrete iterations to SDE. Here we follow the setting of the recent work by [14] and analyze a constant small step-size, continuous-time analogue of the stochastic gradient method that takes the form

$$dX_t = -\beta \nabla F(X_t) dt + \beta \sigma(X_t) dW_t, \quad X_0 = x, \quad (5)$$

¹In the classical Robbins–Monro algorithm (see [26]), the coefficient β may be taken as a decreasing sequence. However, in many modern machine learning practices, the coefficient β is chosen to be constant.

where $\sigma(z) = [\text{Var}(\nabla F(z; \zeta))]^{1/2}$ is a positive semidefinite matrix. Equivalently, one can accelerate a factor of β^{-1} and obtain for $Y_t^{\sqrt{\beta}} \equiv X_{t/\beta}$,

$$dY_t^{\sqrt{\beta}} = -\nabla F(Y_t^{\sqrt{\beta}}) dt + \sqrt{\beta} \sigma(Y_t^{\sqrt{\beta}}) dW_t, \quad Y_0^{\sqrt{\beta}} = x.$$

In other words, (1) holds for $\varepsilon = \sqrt{\beta}$. Therefore under the small step-size regime $\beta \downarrow 0$, the stochastic process $Y_t^{\sqrt{\beta}}$ defined in (1) can be viewed as a continuous-time analogue of the SGD iteration, which addresses the stochasticity emerging from SGD. We further refer to [14,20,21] for the proof of the convergence of discrete SGD (4) to its continuous version (5). We also notice [28] that deals with a problem also related to (5). The equation (5) has been used as a continuous model for the discrete SGD iteration (4) in many earlier works (see for example [31]). Moreover, recent empirical studies further convinced that (5) is indeed a good model for the limiting process of SGD at least for large-batch training. Actually, in a most recent study [27], authors argued that the SGD noise is heavy-tailed and non-Gaussian, thus it might not be the best tool to approximate SGD using SDE driven by Brownian motion. This point of view has been empirically examined in [24], where the authors found that the conclusions of [27] are debateable, and indeed for batch sizes 256 and above, the distribution of the noise is best described as Gaussian at least in the early phases of training. These evidences further consolidate our use of (5) as a continuous model for SGD.

Thus from statistical machine learning point of view we are interested in the asymptotic as $\beta \downarrow 0$ of the hitting time

$$\tau^\beta = \inf\{t : F(X_t) \leq F(x^*) + e\}. \tag{6}$$

In plain words, how much time (upper bound) does it require for the function value to decay by at least $F(X_0) - F(x^*) + e$ using stochastic gradient descent?

By using our main result stated in Section 1.1, we observe the phenomenon that when the noise term $\beta\sigma(X_t) dW_t$ in (5) is sufficiently omnidirectional (i.e. the diffusion matrix $a(x) = \sigma(x)\sigma^T(x)$ is uniformly positive definite) it enables fast evasion from saddle points. Our Corollary 3.5 in this paper suggests a time complexity $\mathbf{E}\tau^\beta \lesssim O(\beta^{-1} \ln(\beta^{-1}))$ as $\beta \downarrow 0$ in discrete-time SGD (compare with [12], which requires $O(\beta^{-2})$ iteration). In other words, the hitting time to a point, in terms of function value, close to a local minimizer takes time that only introduces an additional factor of $O(\ln(\beta^{-1}))$ (compare with [12]).

It is worth mentioning that the previous work by Pemantle [25] proves from a gradient flow viewpoint that SGD can avoid all nondegenerate saddle points, whenever the diffusion matrix $a(x)$ is uniformly positive definite or “omnidirectional” (see also a recent work by [19] for the pure gradient descent case). However, neither of these works provides an analysis on the convergence rates. Within the scope of this work, we concentrate on the diffusion process associated with the discrete-time SGD, and we do not aim to quantify the approximation errors when bridging the discrete-time and continuous-time versions. We believe that using tools from numerical stochastic differential equations, such approximation error is small compared to the noise of dynamics and can be estimated rigorously, which potentially inspires another thread of interesting works.

It is also interesting to notice a recent work [30] related to the problem we are considering here. There the authors discuss algorithms that construct pathway maps on energy landscapes which are essentially connecting the saddle points over the landscape.

The rest of this paper is organized as follows. In Section 2 we quantify the decay of function value as well as the exit time in the neighborhood of one specific strong saddle point. We turn to analyze the general scenario in Section 3 where there are finitely many saddles that the process encounters along its trajectory. We provide the precise statement of theorem in the final part of this paper.

2. Analysis of the function decay and exit time in a neighborhood of one specific “strong saddle”

In this section we analyze the decay in function value of F and the exit time asymptotic in a neighborhood of one specific “strong saddle” point $x_s \in \mathbb{R}^n$. These problems have been discussed thoroughly in the literature on random perturbations of dynamical systems (see for example [3,4,16,23], among others), however, these results do not directly carry to our setting, as we may have a chain of saddle points that may occur on the landscape of the function F . It is worth mentioning here that the setting closest to our work is [4], where the author is also working with a chain of saddle points. The difference is that there these saddle points are connected by a hetero-clinic network, while in our setting, the saddles can be connected by gradient dynamics of (2).

We consider the trajectory of (1) in a neighborhood of one specific strong saddle point x_s . Without loss of generality we can assume that the saddle point $x_s = 0$ is the origin O inside $U \subset \mathbb{R}^n$: a bounded connected open set with smooth boundary. Let $G \subset U$ be a domain with smooth boundary, so that O is the only isolated saddle inside G .

Let O belong to the interior of G and there are no other equilibriums in G . Let $A = \nabla^2 F(O)$. We assume that $-\lambda_1 = -\lambda_2 = \dots = -\lambda_q < -\lambda_{q+1} \leq \dots \leq -\lambda_p < 0 < \lambda_{p+1} \leq \dots \leq \lambda_n$ are the (real) eigenvalues of A , and $\lambda_1 \geq \gamma_1 > 0$.

We shall work with the time-rescaled and perturbed process Y_t^ε as in (1). Let

$$\tau_x^\varepsilon = \inf\{t > 0 : Y_0 = x, Y_t^\varepsilon \in \partial G\}. \quad (7)$$

Let us also consider the corresponding deterministic gradient flow (2). We introduce a decomposition

$$G \cup \partial G = O \cup A_1 \cup A_2 \cup A_3,$$

where A_1 is a set of points $x \in G \cup \partial G$ such that if $x \in A_1$ then $S^u x \in G$ for $u > s$ and $S^u x \notin G \cup \partial G$ if $u \leq s$ for some $s = s(x) \leq 0$ and $S^t x \rightarrow O$ as $t \rightarrow \infty$; A_2 is a set of points $x \in G \cup \partial G$ such that if $x \in A_2$ then $S^u x \in G$ for $u < s$ and $S^u x \notin G \cup \partial G$ if $u > s$ for some $s = s(x) \geq 0$ and $S^t x \rightarrow O$ as $t \rightarrow -\infty$; A_3 is a set of points $x \in G \cup \partial G$ such that if $x \in A_3$ then $S^u x \in G$ provided $s_1 < u < s_2$ and $S^u x \notin G \cup \partial G$ if either $u > s_2$ or $u < s_1$ for some $s_1 = s_1(x) \leq 0$ and $s_2 = s_2(x) \geq 0$. Note that A_1 and A_3 may be simultaneously empty.

If $x \in A_2 \cup A_3$, then $S^t x$ leaves G after some time, so that there is a finite

$$t(x) = \inf\{t > 0 : S^t x \in \partial G\}. \quad (8)$$

Let us denote by Γ_{\max} the eigenspace of A which corresponds to the eigenvalues $-\lambda_1, \dots, -\lambda_q$. It is known that (see [13, Chapter 9]), there exists a q -dimensional sub-manifold W_{\max} tangent to Γ_{\max} at O and is invariant with respect to S^t . We see that $Q_{\max} = W_{\max} \cap \partial G$ is not empty. If $q > 1$ then Q_{\max} is a sub-manifold of $(q - 1)$ -dimensions on the boundary ∂G . If $q = 1$ then Q_{\max} consists of two points.

Let $U \subset G$ be an open neighborhood of the saddle point O . Let U be chosen so small that $\text{dist}(U \cup \partial U, \partial G) > 0$. Let $t(x)$ be defined as in (8). Set

$$\partial G_{U \cup \partial U \rightarrow \text{out}} = \{S^{t(x)}x \text{ for some } x \in U \cup \partial U \text{ with finite } t(x)\} \cup Q_{\max}.$$

For small $\mu > 0$ we let

$$Q^\mu = \{x \in \partial G, \text{dist}(x, \partial G_{U \cup \partial U \rightarrow \text{out}}) < \mu\}.$$

Then we have

Theorem 2.1. For any $r > 0$, there exist some $\varepsilon_0 > 0$ so that for all $x \in U \cup \partial U$ and all $0 < \varepsilon < \varepsilon_0$ we have

$$\frac{\mathbf{E}_x \tau_x^\varepsilon}{\ln(\varepsilon^{-1})} \leq \frac{1}{\lambda_1} + r. \tag{9}$$

For any small $\mu > 0$ and any $\rho > 0$, there exist some $\varepsilon_0 > 0$ so that for all $x \in U \cup \partial U$ and all $0 < \varepsilon < \varepsilon_0$ we have

$$\mathbf{P}_x(Y_{\tau_x^\varepsilon}^\varepsilon \in Q^\mu) \geq 1 - \rho. \tag{10}$$

which indicates that with high probability, the exit position $Y_{\tau_x^\varepsilon}^\varepsilon$ is close to $\partial G_{U \cup \partial U \rightarrow \text{out}}$.

Proof. Our strategy is to decompose $U \cup \partial U$ into disjoint regions according to their distances to $A_1 \cup O$. We prove the result when the initial point x is close to $A_1 \cup O$ within distance $\varepsilon > 0$ in Lemma 2.2, where we choose $0 < \varepsilon < \varepsilon_0$ and $\varepsilon_0 = \min(\varepsilon_0(r), \varepsilon_0(\rho, \mu))$. Then we prove the result when the initial point x is gradually farther away from $A_1 \cup O$, but still is “close” to $A_1 \cup O$ when $\varepsilon \rightarrow 0$, in the sense that its distance to $A_1 \cup O$ is between $\varepsilon^{1/2^k}$ and $\varepsilon^{1/2^{k-1}}$ for $k = 1, 2, \dots$. This is done in Lemma 2.3, where for each k , the statement is true for $0 < \varepsilon < \varepsilon_0$ such that $\varepsilon_0 = \min(\varepsilon_0(r, k), \varepsilon_0(\rho, \mu, k))$. Finally, we show that when k is larger than some finite $k_0 = k_0(r, \rho, \mu)$, if we start the process Y_t^ε from an initial point x that is of a distance at least $\varepsilon^{1/2^{k_0}}$ from $A_1 \cup O$, the statement is still true, provided that $0 < \varepsilon < \varepsilon_0$ and $\varepsilon_0 = \min(\varepsilon_0(r, k_0), \varepsilon_0(\rho, \mu, k_0))$. This is proved in Lemma 2.4. Thus we choose

$$0 < \varepsilon_0 < \min(\varepsilon_0(r), \varepsilon_0(\rho, \mu), \varepsilon_0(r, 1), \varepsilon_0(\rho, \mu, 1), \dots, \varepsilon_0(r, k_0), \varepsilon_0(\rho, \mu, k_0)),$$

where $k_0 = k_0(r, \rho, \mu)$, so that we can conclude the result of this Theorem. \square

Let us denote

$$(A_1 \cup O)_{(0, \varepsilon)} = \{x : \text{dist}(x, A_1 \cup O) < \varepsilon\}, \tag{11}$$

and for $k = 1, 2, \dots$ we denote

$$(A_1 \cup O)_{[\varepsilon^{1/2^{k-1}}, \varepsilon^{1/2^k})} = \{x : \varepsilon^{1/2^{k-1}} \leq \text{dist}(x, A_1 \cup O) < \varepsilon^{1/2^k}\}. \tag{12}$$

We provide the following three Lemmas, which settle the proof of Theorem 2.1.

Lemma 2.2.

- (i) For any $r > 0$, there exist some $\varepsilon_0 = \varepsilon_0(r)$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $x \in U \cup \partial U$ with $x \in (A_1 \cup O)_{[0, \varepsilon]}$ we have

$$\mathbf{E}\tau_x^\varepsilon \leq \left(\frac{1}{\lambda_1} + r \right) \ln(\varepsilon^{-1}).$$

- (ii) For any $\rho > 0$ and $\mu > 0$, there exist some $\varepsilon_0 = \varepsilon_0(\rho, \mu)$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $x \in U \cup \partial U$ with $x \in (A_1 \cup O)_{[0, \varepsilon]}$ we have

$$\mathbf{P}(Y_{\tau_x^\varepsilon}^\varepsilon \in Q^\mu) \geq 1 - \rho.$$

Proof. If $x \in A_1 \cup O$, then the proof of this lemma is more or less the same as in [16]. If $0 < \text{dist}(x, A_1 \cup O) < \varepsilon$, one can modify the proof in [4], and the argument is more or less the same as in the proof of the next Lemma, so that we refer to the proof of the next Lemma for more details. \square

Lemma 2.3.

- (i) For any $k \in \mathbb{N}$ and any $r > 0$, there exist some $\varepsilon_0 = \varepsilon_0(r, k)$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $x \in U \cup \partial U$ with $x \in (A_1 \cup O)_{[\varepsilon^{1/2^{k-1}}, \varepsilon^{1/2^k}]}$ we have

$$\mathbf{E}\tau_x^\varepsilon \leq \left(\frac{1}{2^{k-1}\lambda_1} + r \right) \ln(\varepsilon^{-1}).$$

- (ii) For any $k \in \mathbb{N}$ and any $\rho > 0$, any $\mu > 0$, there exist some $\varepsilon_0 = \varepsilon_0(\rho, \mu, k)$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $x \in U \cup \partial U$ with $x \in (A_1 \cup O)_{[\varepsilon^{1/2^{k-1}}, \varepsilon^{1/2^k}]}$ we have

$$\mathbf{P}(Y_{\tau_x^\varepsilon}^\varepsilon \in Q^\mu) \geq 1 - \rho.$$

Proof. Let $V \supset U$ be an open neighborhood of O . Let $f : V \rightarrow \mathbb{R}^n$ be the $\mathbf{C}^{(2)}$ conjugacy that maps the flow $S^t x$ in (2) within V into linear dynamics $\dot{y} = \Lambda y$, $f(O) = 0 \in \mathbb{R}^n$. Explicitly, the solution of the linear flow $\dot{y} = \Lambda y$, $y(0) = y_0 = (y_0^1, \dots, y_0^n)$ is given by $y(t) = (y^1(t), \dots, y^n(t))$ such that $y^l(t) = y_0^l e^{\lambda^l t}$ for $l = 1, 2, \dots, n$.² We will denote the linear flow corresponding to S^t to be S_Λ^t , so that $f(S^t x) = S_\Lambda^t(f(x))$.

We set $\mathcal{Y}_\varepsilon(t) = f(Y_t^\varepsilon)$. According to the argument that leads to (8.1) at the beginning of Section 8 of [4], we have

$$d\mathcal{Y}_\varepsilon(t) = \Lambda \mathcal{Y}_\varepsilon(t) + \varepsilon B(\mathcal{Y}_\varepsilon(t)) dW(t) + \varepsilon^2 C(\mathcal{Y}_\varepsilon(t)) dt, \quad (13)$$

where B and C are continuous and bounded in a neighborhood $f(V) = \{f(Y); Y \in V\}$ of the origin 0 , and B is non-degenerate.

²In Section 2 we use the eigenvalues $\lambda_1, \dots, \lambda_p, -\lambda_{p+1}, \dots, -\lambda_n$. But here for simplicity we denote the eigenvalues by $\lambda_1, \dots, \lambda_n$ and we understand that $\lambda_1, \dots, \lambda_p$ are positive eigenvalues and $\lambda_{p+1}, \dots, \lambda_n$ are negative eigenvalues of Λ .

From the assumption that $x \in (A_1 \cup O)_{[\varepsilon^{1/2^{k-1}}, \varepsilon^{1/2^k}]}$ we know that the initial condition

$$\mathcal{Y}_\varepsilon(0) = y_0 + \varepsilon^{1/2^k} \xi_\varepsilon, \tag{14}$$

where $y_0 \in f(A_1 \cup O)$ and ξ_ε is a point on $f(V)$ such that $\varepsilon^{1/2^k} \leq \text{dist}(\xi_\varepsilon, f(A_1 \cup O)) < 1$.

Let $\tilde{V} \subset V$ and $W = f(V)$, $\tilde{W} = f(\tilde{V})$, so that

$$\tilde{W} = \{y \in \mathbb{R}^n : |y^j| \leq R, j = 1, 2, \dots, n\} \subset W$$

for some small $R > 0$. Thus $y_0^l = 0$ for all $l \leq p$.

We would like to study the system (13) with initial data given by (14). Let us define the stopping times

$$t_{W,\varepsilon} = \inf\{t \geq 0 : \mathcal{Y}_\varepsilon(t) \in \partial W\},$$

$$t_\varepsilon = \inf\{t \geq 0 : \max(|\mathcal{Y}_\varepsilon^1(t)|, \dots, |\mathcal{Y}_\varepsilon^q(t)|) = R\} \wedge t_{W,\varepsilon}.$$

It turns out, and will be clear from the proof that follows, that $\mathbf{P}(t_\varepsilon < t_{W,\varepsilon}) \rightarrow 1$ as $\varepsilon \rightarrow 0$.

Similarly as in deriving equation (8.6) of [4], we have the solution $\mathcal{Y}_\varepsilon(t)$ in mild form

$$\mathcal{Y}_\varepsilon(t) = e^{\Lambda t} \mathcal{Y}_\varepsilon(0) + \varepsilon e^{\Lambda t} \int_0^t e^{-\Lambda s} B(\mathcal{Y}_\varepsilon(s)) dW_s + \varepsilon^2 e^{\Lambda t} \int_0^t e^{-\Lambda s} C(\mathcal{Y}_\varepsilon(s)) ds.$$

Fix some number $a \in (0, \frac{1}{2^k})$. For every $l = 1, 2, \dots, n$ let us define

$$\tau_\varepsilon^l = \inf\{t : |\mathcal{Y}_\varepsilon^l(t) - (S_\Lambda^t y_0)^l| = \varepsilon^a\},$$

and we define

$$\tau_\varepsilon = \min\{\tau_\varepsilon^l : l = 1, 2, \dots, n\}.$$

In very much the same way as the derivation of Lemmas 8.2–8.4 (see Section 11 of [4]) we see that we have

$$\sup_{t \geq 0} |\mathcal{Y}_\varepsilon(\tau_\varepsilon \wedge t) - S_\Lambda^{t \wedge \tau_\varepsilon} y_0| \xrightarrow{\mathbf{P}} 0 \quad \text{as } \varepsilon \rightarrow 0, \tag{15}$$

$$\tau_\varepsilon \xrightarrow{\mathbf{P}} \infty \quad \text{as } \varepsilon \rightarrow 0, \tag{16}$$

$$\mathbf{P}(\tau_\varepsilon = \min(\tau_\varepsilon^1, \dots, \tau_\varepsilon^q)) \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0. \tag{17}$$

By using the same reasoning as in [4], which derives equation (8.9) in [4], we see that we have, for $l \leq p$,

$$\mathcal{Y}_\varepsilon^l(\tau_\varepsilon) = e^{\lambda_l \tau_\varepsilon} (\varepsilon^{1/2^k} \xi_\varepsilon^k + \varepsilon N_\varepsilon^k(\tau_\varepsilon) + o_{\mathbf{P}}(\varepsilon)). \tag{18}$$

Here $N_\varepsilon(\tau_\varepsilon) = (N_\varepsilon^1(\tau_\varepsilon), \dots, N_\varepsilon^n(\tau_\varepsilon))$ and $N_\varepsilon(\tau_\varepsilon) \xrightarrow{\text{Law}} N_0$ as $\varepsilon \rightarrow 0$, where N_0 is a Gaussian random vector defined in (8.3) of [4]. We have used the notation that $\phi(\varepsilon) = o_{\mathbf{P}}(\psi(\varepsilon))$ for any families of random variables $\phi(\varepsilon), \psi(\varepsilon), \varepsilon > 0$ such that $\frac{\phi(\varepsilon)}{\psi(\varepsilon)} \xrightarrow{\mathbf{P}} 0$ as $\varepsilon \rightarrow 0$.

Put $\mathcal{K}_\varepsilon^l = \mathcal{Y}_\varepsilon^l(\tau_\varepsilon) e^{-\lambda_1 \tau_\varepsilon} \varepsilon^{-1/2^k} = \xi_\varepsilon^l + \varepsilon^{1-1/2^k} N_\varepsilon^l(\tau_\varepsilon) + o_{\mathbf{P}}(\varepsilon^{1-1/2^k})$, we rewrite (18) as

$$\mathcal{Y}_\varepsilon^l(\tau_\varepsilon) = e^{\lambda_1 \tau_\varepsilon} \varepsilon^{1/2^k} \mathcal{K}_\varepsilon^l. \quad (19)$$

From (17) we know that

$$\mathbf{P}(\max(|\mathcal{Y}_\varepsilon^1(\tau_\varepsilon)|, \dots, |\mathcal{Y}_\varepsilon^q(\tau_\varepsilon)|) = \varepsilon^a) \rightarrow 1. \quad (20)$$

So, we have $e^{\lambda_1 \tau_\varepsilon} \varepsilon^{1/2^k} \max(|\mathcal{K}_\varepsilon^1|, \dots, |\mathcal{K}_\varepsilon^q|) = \varepsilon^a$, which gives, with probability approaching 1, as $\varepsilon \rightarrow 0$,

$$\tau_\varepsilon - \frac{\frac{1}{2^k} - a}{\lambda_1} \ln(\varepsilon^{-1}) = -\frac{\ln(\max(|\mathcal{K}_\varepsilon^1|, \dots, |\mathcal{K}_\varepsilon^q|))}{\lambda_1}. \quad (21)$$

Since we know $\varepsilon^{1/2^k} \leq \text{dist}(\xi_\varepsilon, f(A_1 \cup O)) < 1$, we see that for all $x \in (A_1 \cup O)_{[\varepsilon^{1/2^{k-1}}, \varepsilon^{1/2^k}]}$, the corresponding $\xi_\varepsilon = (\xi_\varepsilon^1, \dots, \xi_\varepsilon^n)$ satisfies $\varepsilon^{1/2^k} \leq |\xi_\varepsilon^l| < 1$ for $l = 1, 2, \dots, n$.

For $k \geq 2$ we have, as $\varepsilon > 0$ is small,

$$\begin{aligned} |\mathcal{K}_\varepsilon^l| &= |\xi_\varepsilon^l + \varepsilon^{1-1/2^k} N_\varepsilon^l(\tau_\varepsilon) + o_{\mathbf{P}}(\varepsilon^{1-1/2^k})| \\ &= \varepsilon^{1/2^k} \left| \varepsilon^{-1/2^k} \xi_\varepsilon^l + \varepsilon^{1-1/2^k-1/2^k} N_\varepsilon^l(\tau_\varepsilon) + o_{\mathbf{P}}(\varepsilon^{1-1/2^k-1/2^k}) \right| \\ &\stackrel{(a)}{\geq} \varepsilon^{1/2^k} \left| 1 - \varepsilon^{1-1/2^{k-1}} |N_\varepsilon^l(\tau_\varepsilon)| - o_{\mathbf{P}}(\varepsilon^{1-1/2^{k-1}}) \right|, \end{aligned}$$

so that

$$-\ln|\mathcal{K}_\varepsilon^l| \leq \frac{1}{2^k} \ln(\varepsilon^{-1}) + o_{\mathbf{P}}(1). \quad (22)$$

Here in (a) we used the fact that $\varepsilon^{-1/2^k} |\xi_\varepsilon^l| \geq 1$, $1 - 1/2^k - 1/2^k = 1 - 1/2^{k-1}$, $|a+b| \geq ||a| - |b||$, and with high probability $\varepsilon^{1-1/2^{k-1}} |N_\varepsilon^l(\tau_\varepsilon)|$ is smaller than 1 as $\varepsilon \rightarrow 0$, since $N_\varepsilon(\tau_\varepsilon) \xrightarrow{\text{Law}} N_0$ for a Gaussian random vector N_0 defined in (8.3) of [4] (with fixed covariances).

For $k = 1$ we have, as $\varepsilon > 0$ is small,

$$\begin{aligned} |\mathcal{K}_\varepsilon^l| &= |\xi_\varepsilon^l + \varepsilon^{1/2} N_\varepsilon^l(\tau_\varepsilon) + o_{\mathbf{P}}(\varepsilon^{1/2})| \\ &\geq \varepsilon^{1/2} \left| |N_\varepsilon^l(\tau_\varepsilon)| - 1 - o_{\mathbf{P}}(\varepsilon^{1/2}) \right|, \end{aligned}$$

so that

$$-\ln|\mathcal{K}_\varepsilon^l| \leq \frac{1}{2} \ln(\varepsilon^{-1}) + \left| \ln(|N_\varepsilon^l(\tau_\varepsilon)| - 1 - o_{\mathbf{P}}(\varepsilon^{1/2})) \right|. \quad (23)$$

Equations (23) and (22) give that for any $r > 0$, there exist some $\varepsilon_0 > 0$ uniformly for all ξ_ε with $\varepsilon^{1/2^k} \leq \text{dist}(\xi_\varepsilon, f(A_1 \cup O)) < 1$, as $0 < \varepsilon < \varepsilon_0$, with probability approaching 1,

$$\frac{-\ln(\max(|\mathcal{K}_\varepsilon^1|, \dots, |\mathcal{K}_\varepsilon^q|))}{\ln(\varepsilon^{-1})} \leq \frac{1}{2^k} + r. \quad (24)$$

Combining (21) and (24) we can conclude that, for any $r > 0$, there exist some $\varepsilon_0 > 0$ uniformly for all ξ_ε with $\varepsilon^{1/2^k} \leq \text{dist}(\xi_\varepsilon, f(A_1 \cup O)) < 1$, as $0 < \varepsilon < \varepsilon_0$,

$$\frac{\mathbf{E}\tau_\varepsilon}{\ln(\varepsilon^{-1})} \leq \frac{\frac{1}{2^{k-1}} - a}{\lambda_1} + r. \quad (25)$$

Moreover, in very much the same way as the derivation of Lemmas 8.5 and 8.7 of [4], we know that for $q + 1 \leq l \leq p$ we have

$$\mathcal{Y}_\varepsilon^l(\tau_\varepsilon) = \varepsilon^{-\frac{\lambda_l}{\lambda_1}(\frac{1}{2^k} - a) + \frac{1}{2^k}} [\max(|\mathcal{K}_\varepsilon^1, \dots, \mathcal{K}_\varepsilon^q|)]^{-\frac{\lambda_l}{\lambda_1}} \mathcal{K}_\varepsilon^l, \quad (26)$$

and for $p + 1 \leq l \leq n$ and any $\beta > 0$ we have

$$\mathcal{Y}_\varepsilon^l(\tau_\varepsilon) = \varepsilon^{-\frac{\lambda_l}{\lambda_1}(\frac{1}{2^k} - a)} [\max(|\mathcal{K}_\varepsilon^1, \dots, \mathcal{K}_\varepsilon^q|)]^{-\frac{\lambda_l}{\lambda_1}} (y_0^l + \varepsilon^{\frac{1}{2^k}} \xi_\varepsilon^l) + o_{\mathbf{P}}(\varepsilon^{1-\beta}). \quad (27)$$

Let us now consider the process $\mathcal{Y}_\varepsilon(t)$ after τ_ε . Let $\bar{\mathcal{Y}}_\varepsilon(t) = \mathcal{Y}_\varepsilon(\tau_\varepsilon + t)$. Similarly as in (8.15) of [4], we have

$$\bar{\mathcal{Y}}_\varepsilon(t) = e^{\Lambda t} \left(\mathcal{Y}_\varepsilon(\tau_\varepsilon) + \varepsilon \int_0^t e^{-\Lambda s} B(\bar{\mathcal{Y}}_\varepsilon(s)) dW_s + \varepsilon^2 \int_0^t e^{-\Lambda s} C(\bar{\mathcal{Y}}_\varepsilon(s)) ds \right). \quad (28)$$

Set $\bar{\tau}_\varepsilon = \inf\{t : \max(|\bar{\mathcal{Y}}_\varepsilon^1(t)|, \dots, |\bar{\mathcal{Y}}_\varepsilon^q(t)|) = R\}$, then we have, in parallel to (8.16) of [4], that

$$\max(|\bar{\mathcal{Y}}_\varepsilon^1(\bar{\tau}_\varepsilon)|, \dots, |\bar{\mathcal{Y}}_\varepsilon^q(\bar{\tau}_\varepsilon)|) = e^{\lambda_1 \bar{\tau}_\varepsilon} \varepsilon^a (1 + \eta_\varepsilon), \quad (29)$$

with $\eta_\varepsilon \xrightarrow{\mathbf{P}} 0$ as $\varepsilon \rightarrow 0$. Thus (29) implies that

$$\bar{\tau}_\varepsilon = \frac{a}{\lambda_1} \ln(\varepsilon^{-1}) + \frac{1}{\lambda_1} \ln\left(\frac{R}{1 + \eta_\varepsilon}\right). \quad (30)$$

Back to part (i) of this Lemma that we are proving, we see that $t_\varepsilon = \tau_\varepsilon + \bar{\tau}_\varepsilon$ and $0 \leq \mathbf{E}\tau_\varepsilon^\varepsilon - \mathbf{E}t_\varepsilon \leq C$ for some constant $C > 0$ independent of ε when ε is small. This, together with (25), (30), prove part (i) of this Lemma.

Regarding part (ii) of this Lemma, it is readily checked from (26) and (27), in parallel to (8.18) of [4], that as $\varepsilon \rightarrow 0$,

$$\sup_{t \leq \bar{\tau}_\varepsilon} |\bar{\mathcal{Y}}_\varepsilon(t) - S_\Lambda^t(\varepsilon^a v)| \xrightarrow{\mathbf{P}} 0 \quad (31)$$

where $v \in \text{span}(v_1, \dots, v_q)$ with v_1, \dots, v_q being normal unit vectors in the eigenvector directions corresponding to $\lambda_1, \dots, \lambda_q$. This, together with the fact that as $\varepsilon \rightarrow 0$,

$$\sup_{t \in [0, T]} |\mathcal{Y}_\varepsilon(t) - S'_\Lambda y_0| \xrightarrow{\mathbf{P}} 0, \quad (32)$$

show part (ii) of this Lemma. \square

Lemma 2.4.

- (i) For any $r > 0$, there exist some $k_0 = k_0(r) \in \mathbb{N}$ such that, there exist some $\varepsilon_0 = \varepsilon_0(r, k_0)$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $x \in U \cup \partial U$ with $\text{dist}(x, A_1) \geq \varepsilon^{1/2^{k_0}}$ we have

$$\mathbf{E}\tau_x^\varepsilon \leq 2r \ln(\varepsilon^{-1}).$$

- (ii) For any $\rho > 0$, any $\mu > 0$, there exist some $k_0 = k_0(\rho, \mu) \in \mathbb{N}$ such that, there exist some $\varepsilon_0 = \varepsilon_0(\rho, \mu, k_0)$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $x \in U \cup \partial U$ with $\text{dist}(x, A_1) \geq \varepsilon^{1/2^{k_0}}$ we have

$$\mathbf{P}(Y_{\tau_x^\varepsilon}^\varepsilon \in Q^\mu) \geq 1 - \rho.$$

Proof. For some $k_0 \in \mathbb{N}$ let us consider the domain $D = \{x \in G : \text{dist}(x, (A_1 \cup O)) > \varepsilon^{1/2^{k_0-1}}\}$. For any initial point $x \in D$, the deterministic flow $S^t x$ hits ∂G in a penetrating manner (see Definition 3.1) within time $t(x, \partial G) \leq \frac{C}{2^{k_0}} \ln(\varepsilon^{-1})$, where $C > 0$ is a constant independent of ε . From here, by using an ε -dependent version of the arguments in Lemma 3.2, we conclude part (i) of this Lemma. Part (ii) of this lemma follows the same way as (37) in Lemma 3.2. \square

3. The general case: Finitely many saddles

Let us consider the case when the Morse function F has finitely many strong saddles O_1, \dots, O_k (according to Definition 1.3). Since the problem formulated in Section 1 is only about function value decay so that the process Y_t^ε hits a local minimum point of F , we can assume, without loss of generality, that the saddles O_1, \dots, O_k are ordered in such a way that $F(O_1) > F(O_2) > \dots > F(O_k)$, and the point x^* is a local minimum point of F with $F(x^*) < F(O_k)$. We will work within the basin $U(x^*)$ of x^* , so that O_1, \dots, O_k are the only critical points besides x^* in $U(x^*)$. Let us work with the perturbed process Y_t^ε in (1). Let $Y_0^\varepsilon = x \in U(x^*)$ and $F(x) > F(O_1)$. Given a fixed small number $e > 0$, such that $F(O_k) > F(x^*) + e$, let

$$T_x^\varepsilon = \inf\{t \geq 0 : Y_0^\varepsilon = x, F(Y_t^\varepsilon) \leq F(x^*) + e\}.$$

Our goal is to give an asymptotic estimate of T_x^ε .

For each saddle point O_i , we consider a nested pair of open neighborhoods $U_i \subsetneq V_i$ containing O_i . Let us first pick all $V_i, i = 1, 2, \dots, k$ in such a way that $O_i \in V_i$ is the only stationary point inside V_i , and for any $i \neq j$, we have $V_i \cap V_j = \emptyset$. Let us then pick $U_i \subset V_i$ such that $\text{dist}(\partial U_i, \partial V_i) > 0$ for all $i = 1, 2, \dots, k$. For each saddle point O_i , let us denote by $\Gamma_{i, \max}$ the eigenspace of $\nabla^2 F(O_i)$

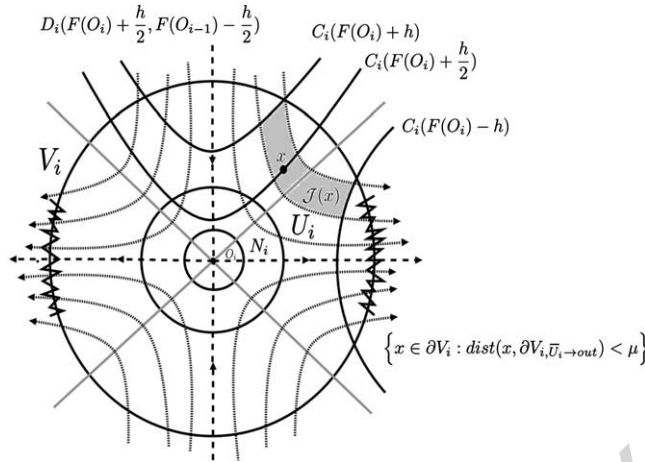


Fig. 1. Geometry of gradient flow dynamics near a saddle point.

which corresponds to the negative eigenvalues of $\nabla^2 F(O_i)$ with largest absolute value. For each open neighborhood V_i , as in Section 2, we can construct a submanifold $W_{i,\max}$ that is tangent to $\Gamma_{i,\max}$ at O_i and is invariant with respect to S^t . Let $Q_{i,\max} = W_{i,\max} \cap \partial V_i$. By the classification of points in $V_i \cup \partial V_i$ as O_i, A_1, A_2, A_3 as in Section 2, we see that for any point $x \in U_i \cup \partial U_i$, either there exist some finite $t(x)$ such that $S^{t(x)}x \in \partial V_i$, or $S^t x \rightarrow O_i$ as $t \rightarrow \infty$. Let

$$\partial V_{i,U_i \cup \partial U_i \rightarrow \text{out}} = \{S^{t(x)}x \text{ for some } x \in U_i \cup \partial U_i \text{ with finite } t(x)\} \cup Q_{i,\max}.$$

For any fixed positive number $h > 0$, we introduce the level curves

$$C_i(F(O_i) \pm h) = \{x \in \mathbb{R}^n : F(x) = F(O_i) \pm h\}.$$

Given $h > 0$ sufficiently small, let us introduce the open domain

$$D((F(O_i) + h, F(O_{i-1}) - h)) = \{x \in \mathbb{R}^n : F(O_i) + h < F(x) < F(O_{i-1}) - h\}.$$

If $i = 1$, we can symbolically let $F(O_0) = \infty$, so that

$$D((F(O_1) + h, F(O_0) - h)) = D((F(O_1) + h, \infty)) = \{x \in \mathbb{R}^n : F(O_1) + h < F(x)\}.$$

Note that the curve $C_i(F(O_i) \pm h)$ and the domain $D((F(O_i) + h, F(O_{i-1}) - h))$ may have several disconnected components, and only some of these components have points that are close to O_i (O_{i-1}). Also note that $C_i(F(O_i) - h)$ ($C_{i-1}(F(O_{i-1}) + h)$) are the lower (upper) boundaries of $D((F(O_i) + h, F(O_{i-1}) - h))$.

We will now present a geometric lemma, that describes the geometry (see Fig. 1) of the gradient flow in a neighborhood of a non-degenerate strong saddle.

Lemma 3.1 (Geometric lemma). *Uniformly for all $i = 1, 2, \dots, k$, we have the following.*

- (i) Besides the basic assumptions of the nested neighborhoods U_i, V_i that we imposed before, we can pick U_i, V_i in such a way that there exist some constant $\mu > 0$ independent of i , and some constant $h > 0$ that may depend on the choice of U_i and V_i , such that for any $x \in \partial V_i$ with $\text{dist}(x, \partial V_{i, U_i \cup \partial U_i \rightarrow \text{out}}) < \mu$, we have $F(x) < F(O_i) - h$;
- (ii) One can pick the constant $h > 0$ so small that $C_i(F(O_i) + \frac{h}{2}) \cap U_i \neq \emptyset$;
- (iii) For fixed $h > 0$ as in (ii), there exist some $\kappa > 0$ such that $|\nabla F(x)| \geq \kappa$ for any point $x \in D(F(O_i) + \frac{h}{2}, F(O_{i-1}) - \frac{h}{2})$;
- (iv) One can pick the constant $\kappa > 0$ so small that for any point $x \in C_i(F(O_i) + \frac{h}{2})$, and $x \notin U_i$, there exist a tubular neighborhood $\mathcal{T}(x)$, such that $\partial \mathcal{T}(x) \cap C_i(F(O_i) - h) \neq \emptyset$, $\partial \mathcal{T}(x) \cap C_i(F(O_i) + h) \neq \emptyset$, and all other boundaries of $\mathcal{T}(x)$ are parallel to flow lines of (2). Moreover, for any $y \in \mathcal{T}(x)$, we have $|\nabla F(y)| \geq \kappa$.

Proof. By the classical Hartman–Grobman Theorem ([1, §13]), within a sufficiently small neighborhood of the strong saddle point O_i , the flow $S^t x$ in (2) is $\mathbf{C}^{(0)}$ -conjugate to the linear flow with matrix $-\nabla^2 F(O_i)$. Since the claims stated in this Lemma are all invariant under $\mathbf{C}^{(0)}$ deformation, we just have to prove this lemma under the framework of linear dynamics (see Fig. 1).

(i) By the so-called Inclination Lemma [9, Theorem 5.7.2] (also see Lemma 5.7.1 in the same reference), one can pick U_i small enough so that all flows starting from $U_i \cup \partial U_i$, if not attracted to O_i , will approach the unstable manifold of the saddle point O_i , before they hit ∂V_i . In fact, the Inclination Lemma indicates that the flow $S^t x$ will map any small disk that is intersecting transversally with the stable manifold, with the dimension equal to that of the unstable manifold, to be close to the unstable manifold in $\mathbf{C}^{(1)}$ -norm. Moreover, this disk is stretched in the unstable directions while it is being attracted to the unstable manifold. Due to non-degeneracy of the saddle point O_i , by picking V_i relatively large with respect to U_i we see that the statement follows;

(ii) Note that if $h > 0$ satisfy (i), then any $\tilde{h} > 0$ such that $\tilde{h} < h$ also satisfy (i). Thus by picking \tilde{h} sufficiently small we have (ii);

(iii) Fix $h > 0$ as in (ii), since one can find a neighborhood N_i of O_i inside U_i , such that $C_i(F(O_i) + \frac{h}{2})$ does not touch this neighborhood, and O_i 's are isolated zeros of $|\nabla F(x)|$, we see that we have (iii);

(iv) We can take the neighborhood N_i of O_i inside U_i to be so small that for any point $x \in C_i(F(O_i) + \frac{h}{2})$ and $x \notin U_i$, the flow line of (2) passing through x and connecting $C_i(F(O_i) + h)$ to $C_i(F(O_i) - h)$, does not touch N_i . In fact, since by (ii) we know that $C_i(F(O_i) + \frac{h}{2}) \cap U_i \neq \emptyset$, we have, that for any $x \notin U_i$ and $x \in C_i(F(O_i) + \frac{h}{2})$, the point x lies outside of some invariant stable cone K_δ^s (see Fig. 1). The Inclination Lemma again implies that the forward flow $S^t x$ stretches x along the unstable direction, thus keeps the resulting flow away from N_i . Thus the flow line of (2) passing through x and connecting $C_i(F(O_i) + h)$ to $C_i(F(O_i) - h)$, does not touch N_i . Further, the family of flow lines in a tubular neighborhood $\mathcal{T}(x)$ of this flow line are connecting $C_i(F(O_i) + h)$ to $C_i(F(O_i) - h)$, and they do not touch N_i . For any y on this $\mathcal{T}(x)$, we will have $|\nabla F(y)| \geq \kappa > 0$. \square

Note that as $h \rightarrow 0$, we have $\kappa \rightarrow 0$, and it might happen that $\frac{h}{\kappa^2} \rightarrow \infty$. However, in what follows we will pick some fixed $h > 0$ and $\kappa > 0$ as well as the neighborhoods of the saddle points as in Lemma 3.1, and we let $\varepsilon \rightarrow 0$.

Let us provide now an auxiliary lemma, which is essentially adapted from [11, Chapter 2, Lemma 3.1], that will be used frequently in our subsequent arguments. Before we state the lemma, let us first introduce some definition regarding exit behavior of the flow $\{S^t x : t \geq 0\}$ in (2).

Let $D \subset \mathbb{R}^n$ be an open domain in \mathbb{R}^n . Let $x \in D$ and we consider the deterministic trajectory $\{S^t x : t \geq 0\}$.

Definition 3.1. We say that the trajectory $\{S^t x : t \geq 0\}$ starting from initial condition $x \in D$, exits D in a “regular manner” if

$$t(x, \partial D) = \inf\{t \geq 0 : S^t x \notin D\} < \infty,$$

and for some sufficiently small $\delta > 0$ we have

$$S^{t(x, \partial D) + \delta}(x) \notin D \cup \partial D. \tag{33}$$

We say that the trajectories $\{S^t x : t \geq 0, x \in D\}$ exit D in a “penetrating manner” if for any $x \in D$, the trajectory $\{S^t x : t \geq 0\}$ exits D in a regular manner, such that there exist some constant $T_0 > 0$ with $t(x, \partial D) \leq T_0$ for all $x \in D$, and there exist some constant $c > 0$, such that for some sufficiently small $\delta > 0$, we have

$$\max_{t(x, \partial D) \leq t \leq t(x, \partial D) + \delta} \text{dist}(S^t x, D \cup \partial D) \geq c > 0, \tag{34}$$

uniformly for all $x \in D$.

We say that the part of the boundary $\Gamma \subseteq \partial D$ is the “exit piece” of $\{S^t x, x \in D, t \geq 0\}$ if $\{S^t x : x \in D, t \geq 0\}$ exit D in a penetrating manner, and for any point $y \in \Gamma$, there exist some $x \in D$ and $t(x, \partial D) \leq T_0$ such that $S^{t(x, \partial D)} x = y$.

Lemma 3.2. Let the domain $D \subset \mathbb{R}^n$ and initial point $x \in D$ be stated as in Definition 3.1. Let

$$t(x, \partial D) = \inf\{t \geq 0 : S^t x \notin D\}.$$

Let

$$\tau_x^\varepsilon(\partial D) = \inf\{t \geq 0 : Y_0 = x, Y_t^\varepsilon \in \partial D\}.$$

If $t(x, \partial D)$ is finite for some choice of x and D , and $\{S^t x : t \geq 0\}$ exits D in a regular manner, then $\tau_x^\varepsilon(\partial D) \rightarrow t(x, \partial D)$ in probability as $\varepsilon \downarrow 0$, i.e., for any given $\delta > 0$,

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}_x(|\tau_x^\varepsilon(\partial D) - t(x, \partial D)| > \delta) = 0. \tag{35}$$

Moreover, if $\{S^t x, t \geq 0, x \in D\}$ exits D in a penetrating manner, and for some small $r > 0$ we have $x \in D^r \equiv \{x \in D, \text{dist}(x, \partial D) \geq r\}$, then there exist some $\varepsilon_0 > 0$ that may depend on D and c , and some constant C that may depend on T_0 , but is independent of ε_0 , such that for any $0 < \varepsilon < \varepsilon_0$ we have

$$\sup_{x \in D^r} \mathbf{E}_x \tau_x^\varepsilon(\partial D) \leq C. \tag{36}$$

Finally, suppose $\{S^t x, t \geq 0, x \in D\}$ exits D in a penetrating manner, and $\Gamma \subseteq \partial D$ is the exit piece as in Definition 3.1. Then for any open set Q of ∂D and $Q \supset \Gamma$, for any $\rho > 0$ sufficiently small, there exist some $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, we have

$$\mathbf{P}_x(Y_{\tau_x^\varepsilon(\partial D)}^\varepsilon \in Q) \geq 1 - \rho, \quad (37)$$

for all $x \in D'$.

Proof. If $\{S^t x, t \geq 0\}$ exits D in a regular manner, then for every sufficiently small $\delta > 0$ we have

$$S^{t(x, \partial D) - \delta} x \in D, \quad S^{t(x, \partial D) + \delta} x \notin D \cup \partial D.$$

Let $\delta_1 > 0$ be the distance of the trajectory segment $\{S^s x, s \in [0, t(x, \partial D) - \delta]\}$ from ∂D , and $\delta_2 > 0$ be the distance of $S^{t(x, \partial D) + \delta} x$ from ∂D , and let $\bar{\delta} = \min(\delta_1, \delta_2) > 0$. By [11, Chapter 2, Theorem 1.2] we see that

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}_x \left\{ \sup_{0 \leq s \leq t(x, \partial D) + \delta} |Y_s^\varepsilon(x) - S^s x| > \bar{\delta} \right\} = 0.$$

Here $Y_t^\varepsilon(x)$ denotes the process Y_t^ε in (1) with $Y_0^\varepsilon = x$. This implies that

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}_x(\tau_x^\varepsilon(\partial D) \in [t(x, \partial D) - \delta, t(x, \partial D) + \delta]) = 1, \quad (38)$$

which is (35), and from here we see $\tau_x^\varepsilon(\partial D) \rightarrow t(x, \partial D)$ in probability as $\varepsilon \downarrow 0$.

Now let us assume that $\{S^t x, t \geq 0, x \in D\}$ exit D in a penetrating manner. Let $t(x, \partial D) \leq T_0$ for all $x \in D$ and $\max_{t(x, \partial D) \leq t \leq t(x, \partial D) + \delta} \text{dist}(S^t x, D \cup \partial D) \geq c > 0$ uniformly for all $x \in D$ and some $\delta > 0$. This implies that for every $\delta > 0$, there exist some $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we have

$$\mathbf{P}_x(|\tau_x^\varepsilon(\partial D) - t(x, \partial D)| > \delta) \leq \delta \quad (39)$$

for all $x \in D'$. Therefore

$$\sup_{x \in D'} \mathbf{P}_x(\tau_x^\varepsilon(\partial D) > 2T_0) \leq \delta.$$

By strong Markov property of the process Y_t^ε we see that

$$\begin{aligned} & \sup_{x \in D'} \mathbf{P}_x(\tau_x^\varepsilon(\partial D) > n \cdot 2T_0) \\ &= \sup_{x \in D'} [\mathbf{P}_x(\tau_{Y_{(n-1)2T_0}^\varepsilon}^\varepsilon(\partial D) > 2T_0 \mid Y_{(n-1)2T_0}^\varepsilon \in D) \mathbf{P}_x(\tau_x^\varepsilon(\partial D) > (n-1)2T_0)] \\ &\leq \delta \cdot \sup_{x \in D'} \mathbf{P}_x(\tau_x^\varepsilon(\partial D) > (n-1)2T_0). \end{aligned}$$

Therefore for every integer $n \geq 0$ we have, for every $x \in D'$,

$$\mathbf{P}_x(\tau_x^\varepsilon(\partial D) > n \cdot 2T_0) \leq \delta^n.$$

Therefore we have

$$\begin{aligned} \mathbf{E}_x \tau_x^\varepsilon(\partial D) &\leq \sum_{n=0}^{\infty} (n+1) \cdot 2T_0 \cdot \mathbf{P}_x(n \cdot 2T_0 < \tau_x^\varepsilon(\partial D) \leq (n+1) \cdot 2T_0) \\ &\leq \sum_{n=0}^{\infty} (n+1) \cdot 2T_0 \cdot \mathbf{P}_x(\tau_x^\varepsilon(\partial D) > n \cdot 2T_0) \\ &\leq \sum_{n=0}^{\infty} (n+1) \cdot 2T_0 \cdot \delta^n \\ &\leq C \end{aligned}$$

since we can choose $\delta > 0$ to be sufficiently small. This implies (36).

Finally, let us prove (37). For any $\delta > 0$ and any $\mu > 0$, we have

$$\begin{aligned} &\mathbf{P}_x(|Y_{\tau_x^\varepsilon(\partial D)}^\varepsilon - S^{t(x, \partial D)}x| > \delta) \\ &= \mathbf{P}_x(|Y_{\tau_x^\varepsilon(\partial D)}^\varepsilon - S^{t(x, \partial D)}x| > \delta, |\tau_x^\varepsilon(\partial D) - t(x, \partial D)| > \mu) \\ &\quad + \mathbf{P}_x(|Y_{\tau_x^\varepsilon(\partial D)}^\varepsilon - S^{t(x, \partial D)}x| > \delta, |\tau_x^\varepsilon(\partial D) - t(x, \partial D)| \leq \mu) \\ &\leq \mathbf{P}_x(|\tau_x^\varepsilon(\partial D) - t(x, \partial D)| > \mu) \\ &\quad + \mathbf{P}_x(|Y_{\tau_x^\varepsilon(\partial D)}^\varepsilon - S^{t(x, \partial D)}x| > \delta, |\tau_x^\varepsilon(\partial D) - t(x, \partial D)| \leq \mu) \\ &\leq \mathbf{P}_x(|\tau_x^\varepsilon(\partial D) - t(x, \partial D)| > \mu) \\ &\quad + \mathbf{P}_x\left(\sup_{t(x, \partial D) - \mu \leq t \leq t(x, \partial D) + \mu} |Y_t^\varepsilon - S^{t(x, \partial D)}x| > \delta\right) \\ &\leq \mathbf{P}_x(|\tau_x^\varepsilon(\partial D) - t(x, \partial D)| > \mu) \\ &\quad + \mathbf{P}_x\left(\sup_{t(x, \partial D) - \mu \leq t \leq t(x, \partial D) + \mu} (|Y_t^\varepsilon - S^t x| + |S^t x - S^{t(x, \partial D)}x|) > \delta\right) \\ &\leq \mathbf{P}_x(|\tau_x^\varepsilon(\partial D) - t(x, \partial D)| > \mu) \\ &\quad + \mathbf{P}_x\left(\sup_{t(x, \partial D) - \mu \leq t \leq t(x, \partial D) + \mu} |Y_t^\varepsilon - S^t x| + \sup_{t(x, \partial D) - \mu \leq t \leq t(x, \partial D) + \mu} |S^t x - S^{t(x, \partial D)}x| > \delta\right). \end{aligned}$$

Due to the continuity of the flow $S^t x$, one can pick $\mu > 0$ small enough such that for all $x \in D$ we have $\sup_{t(x, \partial D) - \mu \leq t \leq t(x, \partial D) + \mu} |S^t x - S^{t(x, \partial D)}x| < \frac{\delta}{2}$. Thus we have

$$\begin{aligned} &\mathbf{P}_x(|Y_{\tau_x^\varepsilon(\partial D)}^\varepsilon - S^{t(x, \partial D)}x| > \delta) \\ &\leq \mathbf{P}_x(|\tau_x^\varepsilon(\partial D) - t(x, \partial D)| > \mu) + \mathbf{P}_x\left(\sup_{t(x, \partial D) - \mu \leq t \leq t(x, \partial D) + \mu} |Y_t^\varepsilon - S^t x| > \frac{\delta}{2}\right). \end{aligned}$$

By (39) as well as the fact that $t(x, \partial D) \leq T_0$ for all $x \in D'$, we see that the above estimate implies (37). \square

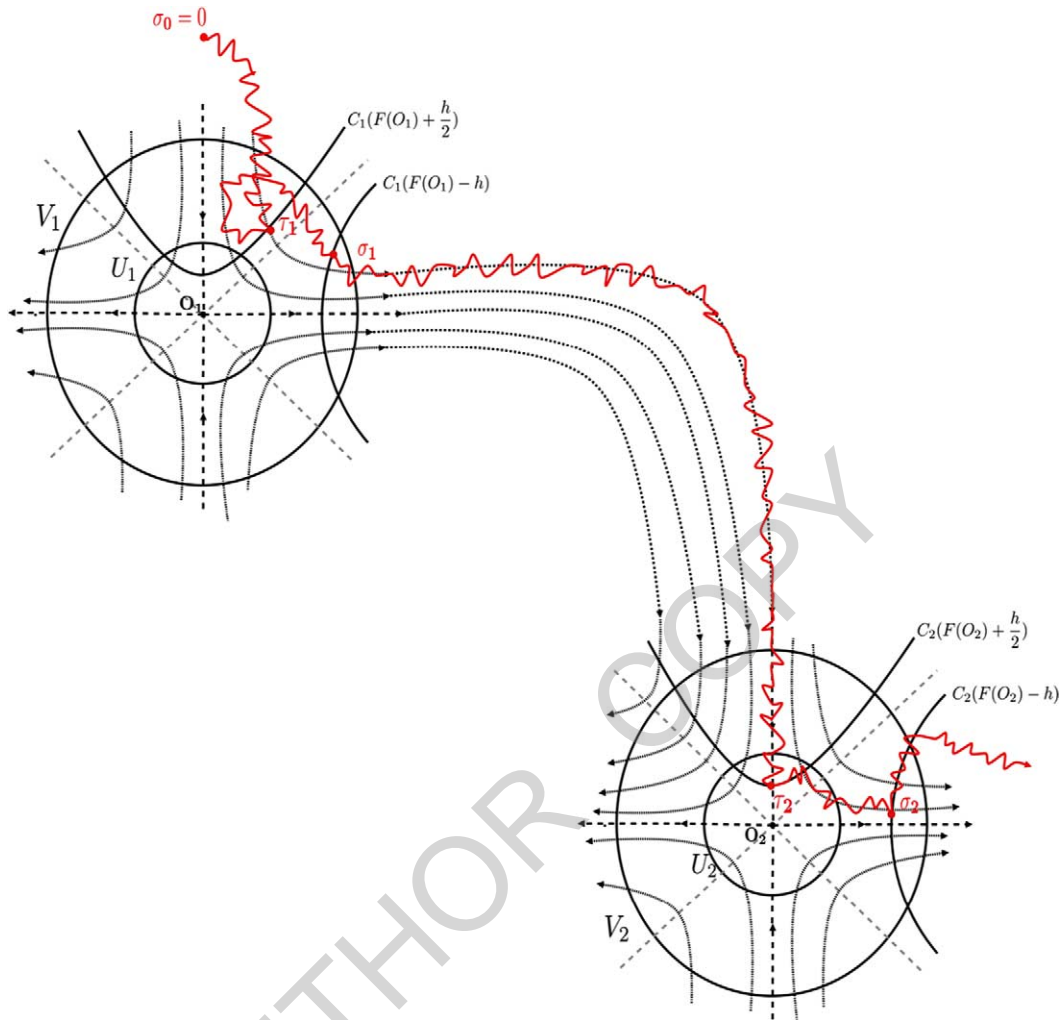


Fig. 2. Sequence of stopping times.

Let us define

$$C(\pm h) = \bigcup_{i=1}^k C_i(F(O_i) \pm h).$$

For simplicity of notations let us also define $C_i(\pm h) = C_i(F(O_i) \pm h)$.

Let us define a sequence of stopping times (see Fig. 2, one can also find similar constructions in [11, Chapter 6], also see [17])

$$0 = \sigma_0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq \dots \tag{40}$$

such that

$$\tau_j = \inf\{t > \sigma_{j-1} : Y_t^\varepsilon \in C(h/2)\}, \tag{41}$$

$$\sigma_j = \inf\{t > \tau_j : Y_t^\varepsilon \in C(-h)\}. \tag{42}$$

Let us start the process Y_t^ε from $Y_0^\varepsilon = x \in \mathbb{R}^n$, such that $F(x) > F(O_1)$, and at the same time $x \notin V_1 \cup \partial V_1$. Let us also assume that we have some $H > 0$ such that for some small $r > 0$ we have $H > F(x) + r$, and for all points $y \in \partial V_1$, we have $H > F(y) + r$.

From our geometric lemma Lemma 3.1, we see that for any x_1 such that $F(x_1) \geq F(O_1) + h/2$, we have $|\nabla F(x_1)| \geq \kappa > 0$. By the fact that $\frac{d}{dt}F(S^t x) = -|\nabla F(S^t x)|^2 \leq -\kappa^2$, this implies that

$$t(x, C_1(h/2)) \leq \frac{F(x) - F(O_1) - h/2}{\kappa^2} < \frac{H - r - F(O_1) - h/2}{\kappa^2}. \tag{43}$$

Moreover, since $S^t x$ is a gradient flow, the flow lines of $S^t x$ will be perpendicular to $C_1(h/2)$ when $x \in C_1(h/2)$. Due to the fact that $|\nabla F(x_1)| \geq \kappa > 0$ when $x_1 \in C_1(h/2)$, we see that the flow $S^t x$ will hit $C_1(h/2)$ in a penetrating manner. Therefore by Lemma 3.2 we see that for such initial point x we have the following.

Lemma 3.3. *Given a small $r > 0$, there exists some $\varepsilon_0 > 0$ uniformly for all x with $H - r > F(x) \geq F(O_1) + h/2$, such that for all $0 < \varepsilon < \varepsilon_0$, there exist some finite $C > 0$ independent of ε such that*

$$\mathbf{E}_x \tau_1 \leq C. \tag{44}$$

Now we are ready to state and prove our main Theorem.

Theorem 3.4. *Consider the process Y_t^ε defined as in (1). Suppose the Morse function $F(x)$ have k -strong saddle points (according to Definition 1.3) O_1, \dots, O_k inside the basin $U(x^*)$, such that $F(O_1) > \dots > F(O_k)$. Let $\gamma_1 > 0$ be chosen as in Definition 1.2. Let the initial condition $Y_0 = x \in U(x^*)$ be such that $F(x) > F(O_1)$. Let x^* be the unique local minimum of F within $U(x^*)$ such that $F(x^*) < F(O_k)$. Then we have*

- (i) *For any small $\rho > 0$, with probability at least $1 - \rho$, the process Y_t^ε in (1) converges to the minimizer x^* for sufficiently small ε after passing through all k saddle points O_1, \dots, O_k ;*
- (ii) *Consider the stopping time T_x^ε defined in (3). Then as $\varepsilon \downarrow 0$, conditioned on the above convergence of Y_t^ε to x^* , we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbf{E}T_x^\varepsilon}{\ln(\varepsilon^{-1})} \leq \frac{k}{\gamma_1}. \tag{45}$$

Proof. By the strong Markov property of the process Y_t^ε , we see that the process $\tilde{Y}_s^\varepsilon = Y_{\tau_1+s}^\varepsilon$ can be viewed as an independent copy of the process Y_s^ε starting from $Y_0^\varepsilon = Y_{\tau_1}^\varepsilon$. By part (ii) of our Lemma 3.1, we can consider two cases, that either $Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1$ or $Y_{\tau_1}^\varepsilon \notin U_1 \cup \partial U_1$ but $Y_{\tau_1}^\varepsilon \in C_1(h/2)$.

Let us consider first the case when $Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1$. Let $\mu > 0$ be chosen as in Lemma 3.1. Let

$$Q_1^\mu = \{x \in \partial V_1, \text{dist}(x, \partial V_{1, U_1 \cup \partial U_1 \rightarrow \text{out}}) < \mu\}.$$

Let $\tilde{\sigma}_1 = \inf\{t \geq \tau_1 : Y_t^\varepsilon \in \partial V_1\}$. Then by Theorem 2.1 equation (10) we see that

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}_x(Y_{\tilde{\sigma}_1}^\varepsilon \in Q_1^\mu \mid Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1) = 1.$$

From the above we see that for any small $q > 0$ there exist some $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we have

$$\mathbf{P}_x(Y_{\tilde{\sigma}_1}^\varepsilon \in Q_1^\mu \mid Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1) \geq 1 - q.$$

By part (c) of our Lemma 3.1 we know that for all $x \in Q_1^\mu$, we have $F(x) \leq F(O_1) - h$. This implies that $\{Y_{\tilde{\sigma}_1}^\varepsilon \in Q_1^\mu\} \subseteq \{\tilde{\sigma}_1 \geq \sigma_1\}$. Therefore we have

$$\mathbf{P}_x(\tilde{\sigma}_1 \geq \sigma_1 \mid Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1) \geq 1 - q.$$

This indicates

$$\lim_{\varepsilon \rightarrow 0} \mathbf{P}_x(\tilde{\sigma}_1 \geq \sigma_1 \mid Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1) = 1. \quad (46)$$

From Theorem 2.1 equation (9) we see that

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbf{E}_x(\tilde{\sigma}_1 - \tau_1 \mid Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1)}{\ln(\varepsilon^{-1})} \leq \frac{1}{\gamma_1}. \quad (47)$$

The above (46) and (47) together indicate that

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbf{E}_x(\sigma_1 - \tau_1 \mid Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1)}{\ln(\varepsilon^{-1})} \leq \frac{1}{\gamma_1}. \quad (48)$$

Let us then turn to the case when $Y_{\tau_1}^\varepsilon \notin U_1 \cup \partial U_1$. Notice that by definition of τ_1 , in this case we have $Y_{\tau_1}^\varepsilon \in C_1(h/2)$. Thus by part (iv) of Lemma 3.1, we can construct a tubular neighborhood containing $Y_{\tau_1}^\varepsilon$ and connecting $C_1(h/2)$ to $C_1(-h)$. The neighborhood can be chosen so that its boundaries are of a positive distance to $Y_{\tau_1}^\varepsilon$. Hence we can then apply Lemma 3.2 again so that for any $\nu > 0$, there exist some $\varepsilon_0 > 0$, for any $0 < \varepsilon < \varepsilon_0$ we have

$$\frac{\sup_{H-r > F(y) \geq F(O_1) + h/2} \mathbf{E}_y(\sigma_1 - \tau_1 \mid Y_{\tau_1}^\varepsilon \notin U_1 \cup \partial U_1)}{\ln(\varepsilon^{-1})} \leq \nu. \quad (49)$$

Combining equations (48), (49) we see that, for any $\nu > 0$, there exist some $\varepsilon_0 > 0$ so that for any $0 < \varepsilon < \varepsilon_0$ we have

$$\begin{aligned} & \frac{\sup_{H-r>F(y)\geq F(O_1)+h/2} \mathbf{E}_y(\sigma_1 - \tau_1)}{\ln(\varepsilon^{-1})} \\ &= \frac{\sup_{H-r>F(y)\geq F(O_1)+h/2} \mathbf{E}_y(\sigma_1 - \tau_1 \mid Y_{\tau_1}^\varepsilon \notin U_1 \cup \partial U_1) \cdot \mathbf{P}_y(Y_{\tau_1}^\varepsilon \notin U_1 \cup \partial U_1)}{\ln(\varepsilon^{-1})} \\ & \quad + \frac{\sup_{H-r>F(y)\geq F(O_1)+h/2} \mathbf{E}_y(\sigma_1 - \tau_1 \mid Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1) \cdot \mathbf{P}_y(Y_{\tau_1}^\varepsilon \in U_1 \cup \partial U_1)}{\ln(\varepsilon^{-1})} \\ & \leq \frac{1}{\gamma_1} + \nu. \end{aligned} \tag{50}$$

Since $\nu > 0$ can be picked arbitrarily small as $\varepsilon > 0$ is small, we see from (50) that we finally have

$$\lim_{\varepsilon \rightarrow 0} \frac{\sup_{H-r>F(y)\geq F(O_1)+h/2} \mathbf{E}_y(\sigma_1 - \tau_1)}{\ln(\varepsilon^{-1})} \leq \frac{1}{\gamma_1}. \tag{51}$$

Notice that we have $F(Y_{\sigma_1}^\varepsilon) \leq F(O_1) - h$. Due to strong Markov property, given $Y_{\sigma_1}^\varepsilon$, the process $\tilde{Y}_s^\varepsilon = Y_{\sigma_1+s}^\varepsilon$ is an independent copy of Y_t^ε . Therefore we can iteratively make use of (51). From Lemma 3.2 we know that, given any small $\rho > 0$, as $\varepsilon > 0$ is small, with probability greater or equal than $1 - \rho$, $Y_{\tau_2}^\varepsilon$ lie on $C_2(h/2)$, i.e., $F(Y_{\tau_2}^\varepsilon) = F(O_2) + h/2$. Similarly, given any small $\rho > 0$, as $\varepsilon > 0$ is small, with probability greater or equal than $1 - \rho$, $Y_{\sigma_2}^\varepsilon$ lie on $C_2(-h)$, i.e., $F(Y_{\sigma_2}^\varepsilon) = F(O_2) - h$.

Thus by running the above argument iteratively, we see that for any $\rho > 0$ sufficiently small, there exist some $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, with probability greater or equal than $1 - \rho$, the random variable $Y_{\sigma_k}^\varepsilon$ satisfy $F(Y_{\sigma_k}^\varepsilon) = F(O_k) - h$. Conditioned on this event, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbf{E}_x T_x^\varepsilon}{\ln(\varepsilon^{-1})} \leq \frac{k}{\gamma_1}.$$

So we arrive at the conclusion of this Theorem. \square

Taking into account that $Y_t^{\sqrt{\beta}} = X_{t/\beta}$ for the process X_t defined in (5) and the process Y_t^ε defined in (1), we conclude that τ^β in (6) satisfies $\tau^\beta = T_x^{\sqrt{\beta}}/\beta$. Thus we see that we have the following corollary in regards to the diffusion approximation of the stochastic gradient descent (Section 1.3).

Corollary 3.5. *Consider the process X_t defined as in (5) and let $F(\bullet)$ satisfy the landscape conditions as in Theorem 3.4, and also suppose the noise in (5) is sufficiently omnidirectional in the sense that $\sigma(x)\sigma^T(x)$ is uniformly positive definite. Let x^* be the unique local minimum of F as in Theorem 3.4, then*

- (i) *For any small $\rho > 0$, with probability at least $1 - \rho$, SGD diffusion process X_t in (5) converges to the minimizer x^* for sufficiently small β after passing through all k saddle points O_1, \dots, O_k ;*

- (ii) Consider the stopping time τ^β defined in (6). Then as $\beta \downarrow 0$, conditioned on the above convergence of SGD diffusion process X_t , we have

$$\lim_{\beta \rightarrow 0} \frac{\mathbf{E}\tau^\beta}{\beta^{-1} \ln \beta^{-1}} \leq \frac{k}{2\gamma_1}. \quad (52)$$

Acknowledgements

Jiaojiao Yang was supported by National Natural Science Foundation of China (Grant No.11801199), Natural Science Foundation of Anhui Province (CN)(1908085QA30) and Anhui Normal University Foundation (CN)(903-751819). Wenqing Hu was supported by a University of Missouri Research Board Grant. The authors thank the anonymous referee for valuable comments that help to improve a preliminary version of this paper.

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