On 2–d incompressible Euler equations with partial damping.

Wenqing Hu [1].
(Joint work with Tarek Elgindi [2] and Vladimir Šverák [3].)

1. Department of Mathematics and Statistics, Missouri S&T.
2. Department of Mathematics, Princeton University.
3. School of Mathematics, University of Minnesota, Twin Cities.

The 2–d Navier–Stokes equation on a torus.

- $$\begin{cases} u_t + u\nabla u + \nabla p - \nu\Delta u & = & f & , \\ \text{div} u & = & 0 & , \\ u\big|_{t=0} & = & u_0 & . \end{cases}$$
- $u = u(x,t) = (u_1(x,t), u_2(x,t))$,
- $p = p(x,t)$,
- $x \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$.

The 2–d Navier–Stokes equation on a torus.

- Solution $u(x,t)$ can be written as a Fourier series

$$u(x,t) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2} \hat{u}(k,t) e^{ikx} \ .$$

- The Laplace operator Δ is a "damping" term if we view it from the Fourier space

$$(\widehat{\Delta u})(k) = -|k|^2 \widehat{u}(k) \text{ for all } k \in \mathbb{Z}^2 \ .$$

Modification of the 2–d Navier–Stokes equation on a torus.

- Introduce the "partial damping" operator Y :

$$(\widehat{Yu})(k) = \begin{cases} -|k|^2 \widehat{u}(k) & , k \notin K \ , \\ 0 & , k \in K \ . \end{cases}$$

- We will assume that K is symmetric (i.e. invariant under $k \to -k$) to keep the solution real–valued.
- Ultimate goal is to study

$$\begin{cases} u_t + u\nabla u + \nabla p - \nu Yu &= f \quad , \\ \mathrm{div} u &= 0 \quad , \\ u|_{t=0} &= u_0 \quad . \end{cases}$$

Background.

- Canonical (conjectural) picture due to Kraichnan [4] : the energy and enstrophy cascades which spread the excitations to other Fourier modes through the nonlinearity.
- Nonlinear interactions should tend to distribute energy uniformly between all degrees of freedom, and hence a system for which some of the degrees of freedom are forced while some other degrees of freedom are damped should still reach some kind of dynamical equilibrium. [5]
- Kraichnan's downward cascade of energy : if in the case $K = \{k_0, -k_0\}$ where k_0 is one of the lowest non–trivial frequencies, the solution will presumably not stay bounded generically, even if the forcing acts "far away" in the Fourier space.

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4. Kraichnan, R.H., Inertial ranges in two dimensional turbulence, *Physics of Fluids*, **10**(7), pp. 1417–1423, 1967.

5. At least if there is enough interaction between the damped and forced parts of the system.

Background.

- The simplest way to establish "dynamical equilibrium" is to assume that f is random, and validate the fact that the system admits a unique invariant measure.[6]
- "Asymptotic Strong Feller" property.
- It does not work in the case of partial damping.

6. Hairer, M., Mattingly, J.C., Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing, *Annals of Mathematics*, (2) **164** (2006), no. 3., pp. 993–1032.

Our problem.

- We study the modest case

$$\left\{ \begin{array}{rcll} u_t + u\nabla u + \nabla p - \nu Y u & = & 0 & , \\ \mathrm{div} u & = & 0 & , \\ u|_{t=0} & = & u_0 & . \end{array} \right. \tag{1}$$

- Case 1 : K is finite (remove damping from finitely many modes) ;
- Case 2 : K is co–finite (i.e. $\mathbb{Z}^2 \backslash K$ is finite, leave damping at finitely many modes) ;
- Case 3 : Both K and $\mathbb{Z}^2 \backslash K$ are infinite.

General Result : energy identity.

- Write
$$(\widehat{Zu})(k) = \begin{cases} -|k|^2 \widehat{u}(k) & , k \in K \ , \\ 0 & , k \notin K \ . \end{cases}$$
- $Y + Z = \Delta$.
- "Regularization".

General Result : energy identity.

- **Theorem 1.** *Let K be any symmetric subset of $\mathbb{Z}^2$. For each divergence–free vector field $u_0 \in L^2$ on the torus $\mathbb{T}^2$ the initial–value problem*

$$\begin{aligned} (u_\varepsilon)_t + u_\varepsilon \nabla u_\varepsilon + \nabla p_\varepsilon - \nu Y u_\varepsilon - \varepsilon Z u_\varepsilon &= 0 \;, \\ \mathrm{div} u_\varepsilon &= 0 \;, \\ u_\varepsilon \big|_{t=0} &= u_0 \end{aligned}$$

has a unique solution $u_\varepsilon \in C([0,\infty), L^2_x) \cap L^2_t \dot{H}^2_x$. The solution u_ε is smooth in $\mathbb{T}^2 \times (0,\infty)$ and satisfies the energy identity

$$\begin{aligned} \int_{\mathbb{T}^2} \frac{1}{2} |u_\varepsilon(x,t)|^2 dx + \int_0^t \int_{\mathbb{T}^2} (-\nu (Y u_\varepsilon) u_\varepsilon - \varepsilon (Z u_\varepsilon) u_\varepsilon) dx dt' \\ = \int_{\mathbb{T}^2} \frac{1}{2} |u_0(x)|^2 dx \end{aligned}$$

for each $t \geq 0$.

General Result : energy identity.

- **Theorem 1 (cont'd).** *Moreover, if* $\omega_0 = curl u_0 \in L^2$*, then* $u_\varepsilon \in L^2_t \dot{H}^2_x$ *and satisfies*

$$\int_{\mathbb{T}^2} \frac{1}{2}\omega_\varepsilon^2(x,t)dx + \int_0^t \int_{\mathbb{T}^2} (-\nu(Yu_\varepsilon)u_\varepsilon - \varepsilon(Zu_\varepsilon)u_\varepsilon)dxdt' = \int_{\mathbb{T}^2} \frac{1}{2}\omega_0^2(x)dx$$

for each $t \geq 0$.

General Result : energy identity.

- **Proof of Theorem 1.** First show that the corresponding linear problem

$$\left\{\begin{array}{rcll} u_{\varepsilon t} + \nabla p_\varepsilon - \nu Y u_\varepsilon - \varepsilon Z u_\varepsilon & = & \mathrm{div} f & , \\ \mathrm{div} u_\varepsilon & = & 0 & , \\ u_\varepsilon\big|_{t=0} & = & u_0 & . \end{array}\right. \qquad (2)$$

  is uniquely solvable in $C([0,\infty), L^2) \cap L^2_t \dot{H}^1_x$ for $u_0 \in L^2$ and $f \in L^2_t L^2_x$ with a bound on the norm by $C(\|u_0\|_{L^2_x} + \|f\|_{L^2_t L^2_x})$.

General Result : energy identity.

- **Proof of Theorem 1 (cont'd).** Then set $f = -u \otimes u$, and formally
$$u = U + B(u, u)$$
where U is the solution to (2) for $f = 0$ and $B(u, u)$ is the solution to (2) with $u_0 = 0$ and $f = -u \otimes u$.
- Embedding $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x \subset L^4_t L^4_x$.
- Contration mapping in $L^4(\mathbb{T}^2 \times (0, T))$.
- As the L^2_x–norm for the solution is non–increasing, this procedure gives a global solution by routine arguments.

General Result : Leray–Hopf weak solution.

- We have $u_\varepsilon \in C([0,\infty), L^2_x) \cap L^2_t \dot{H}^2_x$ for each $\varepsilon > 0$.
- Pass to $\varepsilon \downarrow 0$ and choosing a subsequence u_ε which converges strongly in $L^2_t L^2_x$.
- Precompactness in $L^2_t L^2_x$ of the family u_ε for small $\varepsilon > 0$ follows from energy identity, standard imbeddings and the Aubin–Lions lemma.
- **Weak Solution.**

$$u \in C([0,\infty), L^2) \cap L^\infty_t \dot{H}^1_x \cap \{v, Yv \in L^2_t L^2_x\} \ .$$

- **Corollary.** *For any symmetric $K \subset \mathbb{Z}^2$ and any initial datum $u_0 \in H^1$ our problem has at least one weak solution.*
- The classical Leray–Hopf argument.
- Uniqueness is not known.

Case 1. Finitely many undamped frequencies.

- Let $\kappa = \max_{k \in K} |k|$.
- $\|\nabla v\|^2 \leq -(Yv, v) + \kappa^2 \|v\|^2_{L^2_x}$.
- Uniform control of the solutions u_ε in $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$.
- Standard embedding argument gives existence and uniqueness of solutions to the initial–value problem with $u_0 \in L^2$ in the class $C([0, \infty), L^2) \cap L^2_t \dot{H}^1_x$.

Case 1. Finitely many undamped frequencies.

- Long–time behavior :

$$\mathcal{E}_{K,E,I} = \left\{ v : \mathbb{T}^2 \to \mathbb{R}^2, \begin{array}{l} v \text{ solves the } 2d \text{ incompressible} \\ \text{Euler equation and, in addition} \\ \hat{v}(k) = 0 \\ \text{for each } k \notin K \ , \\ \int_{\mathbb{T}^2} |v|^2 = 2E \ , \int_{\mathbb{T}^2} |\mathrm{curl} v|^2 = I \ . \end{array} \right\} .$$

- Krasovskii–LaSalle principle.

Case 1. Finitely many undamped frequencies.

- **Theorem 2.** Assume the set K of the undamped frequencies is symmetric and finite. Then for each divergence–free vector field on the torus $u_0 \in L^2$ the initial value problem (1) has a unique solution u in the class $C([0,\infty), L^2) \cap L^2_t \dot{H}^1_x$ which satisfies the energy identity

$$\frac{1}{2}\int_{\mathbb{T}^2} |u(x,t)|^2 dx + \int_0^t \int_{\mathbb{T}^2} -\nu(Yu,u)\,dx\,dt' = \int_{\mathbb{T}^2} \frac{1}{2}|u_0(x)|^2 dx$$

for each $t \geq 0$. The solution is smooth for $t > 0$. As $t \to \infty$ the trajectory $u(t)$ stays in a compact subset of C^k for each $k = 1, 2, \ldots$ and its ω–limit set is a subset of a connected component of $\mathcal{E}_{K,E,I}$, where

$$E = \lim_{t\to\infty} \int_{\mathbb{T}^2} \frac{1}{2}|u(x,t)|^2 dx\ ,\quad I = \lim_{t\to\infty} \int_{\mathbb{T}^2} \omega^2(x,t)\,dx\ .$$

Case 1. Finitely many undamped frequencies.

- Question : What is the structure of the solution of 2–d Euler on $\mathbb{T}^2$ that is supported on finitely many Fourier modes ?
- Answer : Independent of time and is supported either on a line passing through the origin or on a circle centered at the origin. (Will come back to this later.)
- So in the case of finitely many undamped modes solutions converge to steady state !

Case 1. Finitely many undamped frequencies.

- The Kraichnan picture of 2d turbulence suggests that all the Euler equilibria other than the ones given by the lowest non–trivial modes should be unstable, and the lowest modes are stable only when the energy is fixed. It is therefore conceivable that a generic solution of (1) for finitely many undamped modes will first approach the lowest modes as a kind of meta-stable state, and–assuming these are not damped–then very slowly approach zero.

Case 2. Finitely many damped frequencies.

- What happens if K is co–finite (i.e. K^c is finite) ?
- Finitely many damped frequencies.

Case 2. Finitely many damped frequencies.

- Vorticity formulation : $\omega_t + (u \cdot \nabla)\omega = \nu Y \omega$.
- $Y\omega = \frac{1}{(2\pi^2)} \sum_{k \notin K} -|k|^2 \widehat{\omega}(k,t) e^{ikt}$ is only a finite sum.
- Bound $\|Y\omega\|_{L^\infty} \leq c_K \|\omega\|_{L^2}$ due to the finiteness of $\mathbb{Z}^2 \backslash K$.
- $\|\omega(t)\|_{L^2}$ is not increasing due to dissipative property of Y.
- Thus $\frac{d}{dt}\|\omega(t)\|_{L^\infty} \leq c_K \|\omega_0\|_{L^2}$.
- We arrive at the a–priori estimate

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + c_K \|\omega_0\|_{L^2} t \ , t > 0 \ .$$

- If $\omega_0 \in L^\infty$, uniqueness follows from Yudovich theory.

Case 2. Finitely many damped frequencies.

- This is indeed a weak–strong uniqueness result.
- The trajectory of $u(t)$ is pre–compact in L^2 and if $u(t_j) \to z$ in L^2 for some sequence $t_j \to \infty$ and $\mathrm{curl} z \in L^\infty$, then the solution $u_j(t) = u(t + t_j)$ converge as $j \to \infty$ in $L^\infty_t L^2_x$ on any finite time interval to a solution of the (incompressible) Euler equation whose Fourier coefficients are supported in K.

The structure of Euler solution supported on finitely many Fourier modes.

- Question : What is the structure of the solution of 2–d Euler on $\mathbb{T}^2$ that is supported on finitely many Fourier modes ?
- Answer : Independent of time and is supported either on a line passing through the origin or on a circle centered at the origin.
- It must be steady state !

The structure of Euler solution supported on finitely many Fourier modes.

- 2–d Euler can be written in Fourier components as

$$\frac{d}{dt}\widehat{\omega}(m,t) = -\frac{1}{4\pi}\sum_{k+l=m}(k_1 l_2 - k_2 l_1)\left(\frac{1}{|k|^2} - \frac{1}{|l|^2}\right)\widehat{\omega}(k,t)\widehat{\omega}(l,t)\,.$$

- Suppose the support is a finite set of Fourier modes S.

The structure of Euler solution supported on finitely many Fourier modes.

- An (unordered) pair $\{k, l\}$ of two distinct points $k = (k_1, k_2)$, $l = (l_1, l_2) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$ is called degenerate if either k, l lie on the same circle centered at the origin (i.e. $k_1^2 + k_2^2 = l_1^2 + l_2^2$), or k, l lie on the same line passing through the origin (i. e. $k_1 l_2 - k_2 l_1 = 0$). In the former case we call the pair to be c–degenerate, and in the latter case we call the pair to be l–degenerate. A pair which is not degenerate is called non–degenerate.

The structure of Euler solution supported on finitely many Fourier modes.

- Claim : If the set S is not degenerate, then there exists a non–zero element $m \in \mathbb{Z}^2 \backslash S$ such that $m = k + l$ for exactly one non–degenerate (unordered) pair $\{k, l\} \in S$.
- S "wakes up" those modes that are not in S.

The structure of Euler solution supported on finitely many Fourier modes.

- We just need to prove the previous claim !
- $S^{\text{conv}} = \{x \in \mathbb{R}^2, N(x) \leq 1\}$.
- $A_1, ..., A_r$ are extremal points of S^{conv}.

The structure of Euler solution supported on finitely many Fourier modes.

- Step 1 : The points $A_1, ..., A_r$ lie on a circle centered at the origin so that

$$\{x \in \mathbb{R}^2, N(x) = 1\} \cap S = \{A_1, ..., A_r\} \ .$$

- S is symmetric about O !

The structure of Euler solution supported on finitely many Fourier modes.

- L is a linear function on $\mathbb{R}^2$ such that $L(A_1) > L(A_2) > \max_{P \in (S \setminus [A_1, A_2]) \cup \{O\}} L(P)$.
- $A_1 = P_1, P_2, ..., P_s = A_2$.
- $L(P_1) > L(P_2) > ... > L(P_s)$

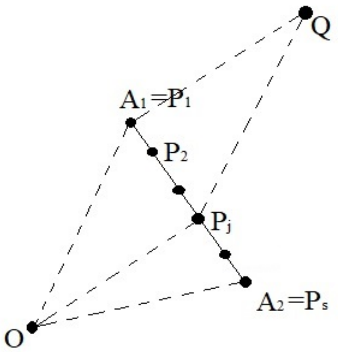

FIGURE 1: Proof that S is symmetric w.r.t. O (if not a line).

The structure of Euler solution supported on finitely many Fourier modes.

- The set $\{P_1, P_2, ..., P_s\}$ contains no l–degenerate points.
- If all pairs $\{P_1, P_j\}$, $j \geq 2$ are degenerate, they must always be c–degenerate and hence $|A_1| = |A_2|$.
- Set $1 < j \leq s$ be the smallest index j such that $\{P_1, P_j\}$ is not degenerate and $Q = P_1 + P_j$.

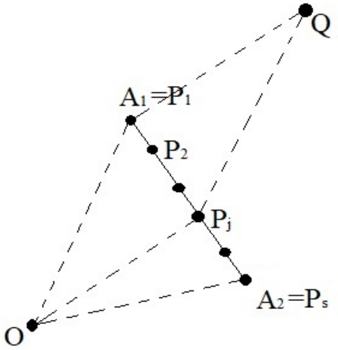

FIGURE 2: Proof that S is symmetric w.r.t. O (if not a line).

The structure of Euler solution supported on finitely many Fourier modes.

- $L(Q) = L(P_1) + L(P_j) \geq L(A_1) + L(A_2) > L(A_1)$.
- Hence $Q \notin S$.
- If $Q = P' + P''$ then $L(P') + L(P'') = L(P_1) + L(P_j)$.
- By definition of j we know $\{P', P''\} = \{P_1, P_j\}$ if we want $\{P', P''\}$ to be non–degenerate.
- Thus S should contain Q!

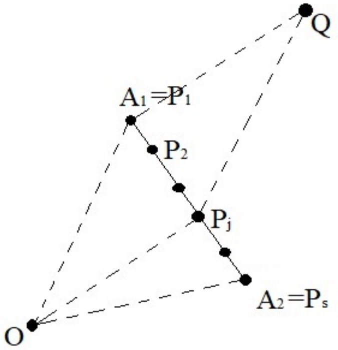

FIGURE 3: Proof that S is symmetric w.r.t. O (if not a line).

The structure of Euler solution supported on finitely many Fourier modes.

- Step 2 : $S = \{A_1, ..., A_r\}$.
- There is no multiple concentric layer in the structure of S !

The structure of Euler solution supported on finitely many Fourier modes.

- Assume this is not the case and consider $B \in S$ such that

$$0 < N(B) = \max_{P \in S \setminus \{A_1, \ldots, A_r\}} N(P) \ .$$

- Say $B \in OA_1A_2$.
- C is the center of $[A_1, A_2]$.
- M is a linear function on $\mathbb{R}^2$ such that $M = N$ on the triangle OA_1A_2.

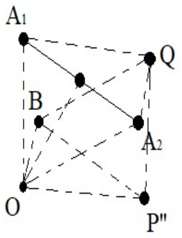

FIGURE 4: Proof that S is degenerate.

The structure of Euler solution supported on finitely many Fourier modes.

- $Q = A_2 + B$.
- $\{B, A_2\}$ is clearly non–degenerate and

$$1 = M(A_1) = M(A_2) > M(B) \geq \max_{P \in (S \setminus \{A_1, \ldots, A_r\}) \cup \{O\}} M(P) \ .$$

- Say $Q = A_2 + B = P' + P''$.
- $M(P') + M(P'') = M(B) + M(A_2)$.

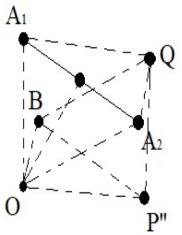

FIGURE 5: Proof that S is degenerate.

The structure of Euler solution supported on finitely many Fourier modes.

- $P' \in \{A_1, A_2\}$ and $M(P'') = M(B)$.
- If $P' = A_2$, then $P'' = B$.
- If $P' = A_1$, then $P'' = B + (A_2 - A_1)$.
- $N(P'') > N(B)$.
- $P'' \in \{A_1, ..., A_r\}$ so $\{P', P''\}$ is degenerate.

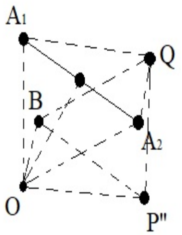

FIGURE 6: Proof that S is degenerate.

Thank you for your attention !