On 2–d Euler equations with partial damping and some related model problems.

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The 2–d Navier–Stokes equation on a torus.

- $$\begin{cases} u_t + u\nabla u + \nabla p - \nu\Delta u & = & f & , \\ \mathrm{div} u & = & 0 & , \\ u\big|_{t=0} & = & u_0 & . \end{cases}$$
- $u = u(x,t) = (u_1(x,t), u_2(x,t))$,
- $p = p(x,t)$,
- $x \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$.

The 2–d Navier–Stokes equation on a torus.

- Solution $u(x, t)$ can be written as a Fourier series

$$u(x,t) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}^2} \hat{u}(k,t) e^{ikx} \ .$$

- The Laplace operator Δ is a "damping" term if we view it from the Fourier space

$$(\widehat{\Delta u})(k) = -|k|^2 \widehat{u}(k) \text{ for all } k \in \mathbb{Z}^2 \ .$$

Modification of the 2–d Navier–Stokes equation on a torus.

- Introduce the "partial damping" operator Y :

$$(\widehat{Yu})(k) = \begin{cases} -|k|^2\widehat{u}(k) & , k \notin K \ , \\ 0 & , k \in K \ . \end{cases}$$

- We will assume that K is symmetric (i.e. invariant under $k \to -k$) to keep the solution real–valued.
- Ultimate goal is to study

$$\begin{cases} u_t + u\nabla u + \nabla p - \nu Yu & = & f & , \\ \mathrm{div} u & = & 0 & , \\ u\big|_{t=0} & = & u_0 & . \end{cases}$$

Background.

- Canonical (conjectural) picture due to Kraichnan [2] : the energy and enstrophy cascades which spread the excitations to other Fourier modes through the nonlinearity.
- Nonlinear interactions should tend to distribute energy uniformly between all degrees of freedom, and hence a system for which some of the degrees of freedom are forced while some other degrees of freedom are damped should still reach some kind of dynamical equilibrium. [3]
- Kraichnan's downward cascade of energy : if in the case $K = \{k_0, -k_0\}$ where k_0 is one of the lowest non–trivial frequencies, the solution will presumably not stay bounded generically, even if the forcing acts "far away" in the Fourier space.

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2. Kraichnan, R.H., Inertial ranges in two dimensional turbulence, *Physics of Fluids*, **10**(7), pp. 1417–1423, 1967.

3. At least if there is enough interaction between the damped and forced parts of the system.

Background.

- The simplest way to establish "dynamical equilibrium" is to assume that f is random, and validate the fact that the system admits a unique invariant measure.[4]
- "Asymptotic Strong Feller" property.
- It does not work in the case of partial damping.

4. Hairer, M., Mattingly, J.C., Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing, *Annals of Mathematics*, (2) **164** (2006), no. 3., pp. 993–1032.

Our problem.

- We study the modest case

$$\begin{cases} u_t + u\nabla u + \nabla p - \nu Y u &= 0 \quad , \\ \mathrm{div} u &= 0 \quad , \\ u|_{t=0} &= u_0 \quad . \end{cases}$$

- Case 1 : K is finite (remove damping from finitely many modes) ;
- Case 2 : K is co–finite (i.e. $\mathbb{Z}^2 \backslash K$ is finite, leave damping at finitely many modes) ;
- Case 3 : Both K and $\mathbb{Z}^2 \backslash K$ are infinite.

General Result.

- Write
$$(\widehat{Zu})(k) = \begin{cases} -|k|^2\widehat{u}(k) & , k \in K \ , \\ 0 & , k \notin K \ . \end{cases}$$
- $Y + Z = \Delta$.
- "Regularization".

General Result.

- **Theorem.** *Let K be any symmetric subset of $\mathbb{Z}^2$. For each divergence–free vector field $u_0 \in L^2$ on the torus $\mathbb{T}^2$ the initial–value problem*

$$\begin{aligned} (u_\varepsilon)_t + u_\varepsilon \nabla u_\varepsilon + \nabla p_\varepsilon - \nu Y u_\varepsilon - \varepsilon Z u_\varepsilon &= 0 \ , \\ \operatorname{div} u_\varepsilon &= 0 \ , \\ u_\varepsilon\big|_{t=0} &= u_0 \end{aligned}$$

has a unique solution $u_\varepsilon \in C([0,\infty), L^2_x) \cap L^2_t \dot{H}^2_x$. The solution u_ε is smooth in $\mathbb{T}^2 \times (0,\infty)$ and satisfies the energy identity

$$\int_{\mathbb{T}^2} \frac{1}{2} |u_\varepsilon(x,t)|^2 dx + \int_0^t \int_{\mathbb{T}^2} (-\nu (Y u_\varepsilon) u_\varepsilon - \varepsilon (Z u_\varepsilon) u_\varepsilon) dx dt' = \int_{\mathbb{T}^2} \frac{1}{2} |u_0(x)|^2 dx$$

for each $t \geq 0$.

General Result.

- **Theorem.** (continued) *Moreover, if* $\omega_0 = curl u_0 \in L^2$, *then* $u_\varepsilon \in L^2_t \dot{H}^2_x$ *and satisfies*

$$\int_{\mathbb{T}^2} \frac{1}{2}\omega_\varepsilon^2(x,t)dx + \int_0^t \int_{\mathbb{T}^2} (-\nu(Yu_\varepsilon)u_\varepsilon - \varepsilon(Zu_\varepsilon)u_\varepsilon)dxdt' = \int_{\mathbb{T}^2} \frac{1}{2}\omega_0^2(x)dx$$

for each $t \geq 0$.

General Result.

- **Weak Solution.**

$$u \in C([0,\infty), L^2) \cap L^\infty_t \dot{H}^1_x \cap \{v, Yv \in L^2_t L^2_x\} \ .$$

- **Corollary.** *For any symmetric $K \subset \mathbb{Z}^2$ and any initial datum $u_0 \in H^1$ our problem has at least one weak solution.*
- Leray–Hopf argument.

Case 1. Finitely many undamped frequencies.

- Let $\kappa = \max_{k \in K} |k|$.
- $\|\nabla v\|^2 \leq -(Yv, v) + \kappa^2 \|v\|^2_{L^2_x}$.
- Uniform control of the solutions u_ε in $L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x$.
- Standard embedding argument gives existence and uniqueness of solutions to the initial–value problem with $u_0 \in L^2$ in the class $C([0, \infty), L^2) \cap L^2_t \dot{H}^1_x$.

Case 1. Finitely many undamped frequencies.

- Long–time behavior :

$$\mathcal{E}_{K,E,I} = \left\{ v : \mathbb{T}^2 \to \mathbb{R}^2, \begin{array}{l} v \text{ solves the } 2d \text{ incompressible} \\ \text{Euler equation and, in addition} \\ \hat{v}(k) = 0 \\ \text{for each } k \notin K \ , \\ \int_{\mathbb{T}^2} |v|^2 = 2E \ , \int_{\mathbb{T}^2} |\mathrm{curl} v|^2 = I \ . \end{array} \right\} .$$

- Krasovskii–LaSalle principle.
- Question : What is the structure of the solution of 2–d Euler on $\mathbb{T}^2$ that is supported on finitely many Fourier modes ?
- Answer : Independent of time and is supported either on a line passing through the origin or on a circle centered at the origin. (Will come back to this later.)
- So in the case finitely many undamped modes solutions converge to steady state !

Case 2. Finitely many damped frequencies.

- Vorticity formulation : $\omega_t + (u \cdot \nabla)\omega = \nu Y \omega$.
- Bound $\|Y\omega\|_{L^\infty} \leq c_K \|\omega\|_{L^2}$ due to the finiteness of $\mathbb{Z}^2 \backslash K$.
- $\|\omega(t)\|_{L^2}$ is not increasing.
- Thus $\frac{d}{dt}\|\omega(t)\|_{L^\infty} \leq c_K \|\omega_0\|_{L^2}$.
- We arrive at the a–priori estimate

$$\|\omega(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty} + c_K \|\omega_0\|_{L^2} t \ , t > 0 \ .$$

- If $\omega_0 \in L^\infty$, uniqueness follows from Yudovich theory.

The structure of Euler solution supported on finitely many Fourier modes.

- Question : What is the structure of the solution of 2–d Euler on $\mathbb{T}^2$ that is supported on finitely many Fourier modes ?
- Answer : Independent of time and is supported either on a line passing through the origin or on a circle centered at the origin.
- It must be steady state !

The structure of Euler solution supported on finitely many Fourier modes.

- 2–d Euler can be written in Fourier components as

$$\frac{d}{dt}\widehat{\omega}(m,t) = -\frac{1}{4\pi}\sum_{k+l=m}(k_1 l_2 - k_2 l_1)\left(\frac{1}{|k|^2} - \frac{1}{|l|^2}\right)\widehat{\omega}(k,t)\widehat{\omega}(l,t)\,.$$

- Suppose the support is a finite set of Fourier modes S.

The structure of Euler solution supported on finitely many Fourier modes.

- An (unordered) pair $\{k, l\}$ of two distinct points $k = (k_1, k_2)$, $l = (l_1, l_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ is called degenerate if either k, l lie on the same circle centered at the origin (i.e. $k_1^2 + k_2^2 = l_1^2 + l_2^2$), or k, l lie on the same line passing through the origin (i. e. $k_1 l_2 - k_2 l_1 = 0$). In the former case we call the pair to be c–degenerate, and in the latter case we call the pair to be l–degenerate. A pair which is not degenerate is called non–degenerate.
- If the set S is not degenerate, then there exists a non–zero element $m \in \mathbb{Z}^2 \backslash S$ such that $m = k + l$ for exactly one non–degenerate (unordered) pair $\{k, l\} \in S$.
- S "wakes up" those modes that are not in S.

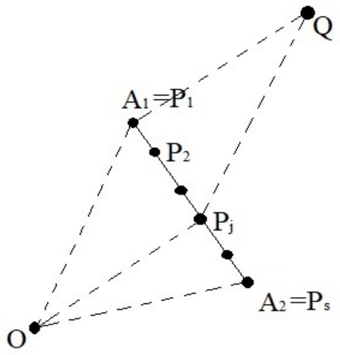

FIGURE: Proof that S is symmetric w.r.t. O (if not a line).

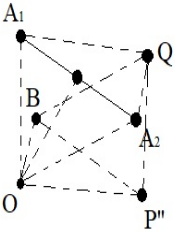

FIGURE: Proof that S is degenerate.

Reference.

Elgindi, T., **Hu, W.**, Sverak, V., On 2d incompressible Euler equations with partial damping. *Communications in Mathematical Physics*, **355**, Issue 1, October 2017, pp. 145-159.

Model problem : AB–model.

- We consider here a model problem

$$\begin{cases} dx_t = -x_t y_t dt \ , \\ dy_t = x_t^2 dt \ . \end{cases}$$

- A phase picture is shown in the next Figure.
- We see that the whole line Oy_A contains stable equilibriums and the whole line Oy_B contains unstable equilibriums. This is different from the cases considered in the classical Freidlin-Wentzell theory.
- In this case we can understand the symmetry of our model in a more rough way : the stable and unstable equilibriums are symmetric with respect to shifts in the directions of Oy_A and Oy_B, respectively.
- Our model preserves the energy $E(x,y) = x^2 + y^2$. The driving vector field $b(x,y) = (-xy, x^2)$ is degenerate on $x = 0$.

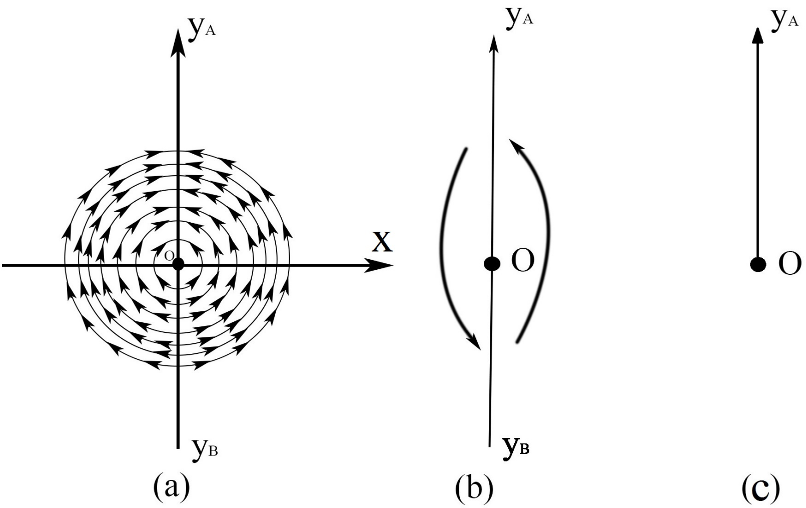

FIGURE: The *AB* model.

Randomly perturbed AB–model.

- We add friction and random perturbation to this system

$$\left\{ \begin{array}{l} d\mathcal{X}_t^\varepsilon = -\mathcal{X}_t^\varepsilon \mathcal{Y}_t^\varepsilon dt - \varepsilon \mathcal{X}_t^\varepsilon dt + \sqrt{\varepsilon} dW_t^1 \ , \ \mathcal{X}_0^\varepsilon = x_0 \ , \\ d\mathcal{Y}_t^\varepsilon = (\mathcal{X}_t^\varepsilon)^2 dt - \varepsilon \mathcal{Y}_t^\varepsilon dt + \sqrt{\varepsilon} dW_t^2 \ , \ \mathcal{Y}_0^\varepsilon = y_0 \ . \end{array} \right.$$

- Here W_t^1 and W_t^2 are two independent standard 1–dimensional Brownian motions;
- The small parameter $\varepsilon > 0$ is the intensity of the friction, and the small parameter $\sqrt{\varepsilon} > 0$ is the intensity of the noise.

Randomly perturbed AB–model.

- Question : What is the long–time behavior of the system $(\mathcal{X}_t^\varepsilon, \mathcal{Y}_t^\varepsilon)$ as $t \to \infty$ and $\varepsilon \downarrow 0$?

Randomly perturbed AB–model : Background.

- Our model problem here differs from the set–up in the classical Freidlin–Wentzell theory in that the point–like asymptotically stable attractor is replaced by a manifold. We can view our limiting process as a "process–level attractor" of our system.

Randomly perturbed AB–model : Background.

- Finite dimensional models for the inviscid stochastic 2–d Navier–Stokes equations of the form

$$\frac{\partial \omega}{\partial t}+u\cdot\nabla\omega-\nu\Delta\omega=\sqrt{\nu}\xi(t,x)\,,\,u=\nabla^{\top}\Delta^{-1}\omega\,,\omega(x,0)=\omega_0(x)\,,$$

  in which $\xi(t,x)$ is a noise, and positive parameter $\nu \to 0$.
- Invariant measure for Euler equations.
- Our consideration of the unperturbed system mimics the attractor for the 2–d Euler system, that has continuous sets of steady states.
- In fact, systems that arise in hydrodynamics, such as in the context of Euler's equation, typically possess equilibrium points that belong to an infinite dimensional "manifold" of other equilibria.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- We come back to our system

$$\begin{cases} d\mathcal{X}_t^\varepsilon = -\mathcal{X}_t^\varepsilon \mathcal{Y}_t^\varepsilon dt - \varepsilon \mathcal{X}_t^\varepsilon dt + \sqrt{\varepsilon} dW_t^1 \ , \ \mathcal{X}_0^\varepsilon = x_0 \ , \\ d\mathcal{Y}_t^\varepsilon = (\mathcal{X}_t^\varepsilon)^2 dt - \varepsilon \mathcal{Y}_t^\varepsilon dt + \sqrt{\varepsilon} dW_t^2 \ , \ \mathcal{Y}_0^\varepsilon = y_0 \ . \end{cases}$$

- We do a time rescaling $t \to \dfrac{t}{\varepsilon}$ and we let

$$(X_t^\varepsilon, Y_t^\varepsilon) = (\mathcal{X}_{t/\varepsilon}^\varepsilon, \mathcal{Y}_{t/\varepsilon}^\varepsilon) \ .$$

- Then

$$\begin{cases} dX_t^\varepsilon = -\dfrac{1}{\varepsilon} X_t^\varepsilon Y_t^\varepsilon dt - X_t^\varepsilon dt + dW_t^1 \ , \ X_0^\varepsilon = x_0 \ , \\ dY_t^\varepsilon = \dfrac{1}{\varepsilon}(X_t^\varepsilon)^2 dt - Y_t^\varepsilon dt + dW_t^2 \ , \ Y_0^\varepsilon = y_0 \ . \end{cases}$$

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- System $(X_t^\varepsilon, Y_t^\varepsilon)$:

$$\begin{cases} dX_t^\varepsilon = -\dfrac{1}{\varepsilon} X_t^\varepsilon Y_t^\varepsilon dt - X_t^\varepsilon dt + dW_t^1 \ , \ X_0^\varepsilon = x_0 \ , \\ dY_t^\varepsilon = \dfrac{1}{\varepsilon} (X_t^\varepsilon)^2 dt - Y_t^\varepsilon dt + dW_t^2 \ , \ Y_0^\varepsilon = y_0 \ . \end{cases}$$

- Separation of slow and fast motions.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- In the limit as $\varepsilon \downarrow 0$, the process $(X_t^\varepsilon, Y_t^\varepsilon)$ is pushed by the flow onto Oy_A, and will be close to $\pi(x_0, y_0)$ in short time.
- There, the Y–component Y_t^ε behaves as a 2–dimensional linearly damped radial Bessel process (*damped–BES(2)*) on Oy_A :

$$dY_t = \left(\frac{1}{2Y_t} - Y_t\right) dt + dW_t^2 \ , Y_0 = y^\pi(x_0, y_0) \ .$$

- What is the heuristic ?

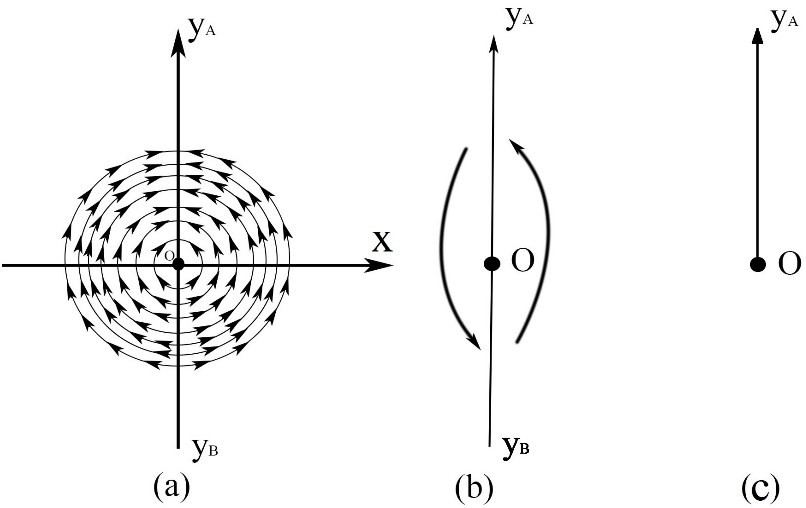

FIGURE: The AB model.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- The radial process $r_t^\varepsilon = \sqrt{(X_t^\varepsilon)^2 + (Y_t^\varepsilon)^2}$.
- By applying Itô's formula we see that

$$dr_t^\varepsilon = \left(\frac{1}{2r_t^\varepsilon} - r_t^\varepsilon\right) dt + dW_t^r \ , \ r_0^\varepsilon = \sqrt{(X_0^\varepsilon)^2 + (Y_0^\varepsilon)^2}$$

- As $\varepsilon \downarrow 0$ the process X_t^ε is pushed by the fast flow to be close to 0 when $Y_t^\varepsilon \geq \delta > 0$.
- $\delta = \varepsilon^{1/10}$.
- Near the Oy_A axis we have $r_t^\varepsilon \approx Y_t^\varepsilon$ as $\varepsilon \downarrow 0$.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- What happened when $Y_t^\varepsilon < \delta$?
- The above comparison with the radial process will not work.
- If $(X_t^\varepsilon, Y_t^\varepsilon)$ is close to the origin $O = (0,0)$, we look at our system

$$\begin{cases} dX_t^\varepsilon = -\frac{1}{\varepsilon} X_t^\varepsilon Y_t^\varepsilon dt - X_t^\varepsilon dt + dW_t^1 \ , \ X_0^\varepsilon = x_0 \ , \\ dY_t^\varepsilon = \frac{1}{\varepsilon}(X_t^\varepsilon)^2 dt - Y_t^\varepsilon dt + dW_t^2 \ , \ Y_0^\varepsilon = y_0 \ . \end{cases}$$

- In the limit as $\varepsilon \downarrow 0$ the positive drift $\frac{1}{\varepsilon}(X_t^\varepsilon)^2 \to \frac{1}{2Y_t}$.
- Recall that the damped BES(2) process takes the form

$$dY_t = \left(\frac{1}{2Y_t} - Y_t\right) dt + dW_t^2 \ , Y_0 = y^\pi(x_0, y_0) \ .$$

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- This suggests that the origin O is inaccessible and as $\varepsilon \to 0$ the limit process $Y_t^\varepsilon \to Y_t$ lives only on Oy_A axis.
- Also supported by simulation results.

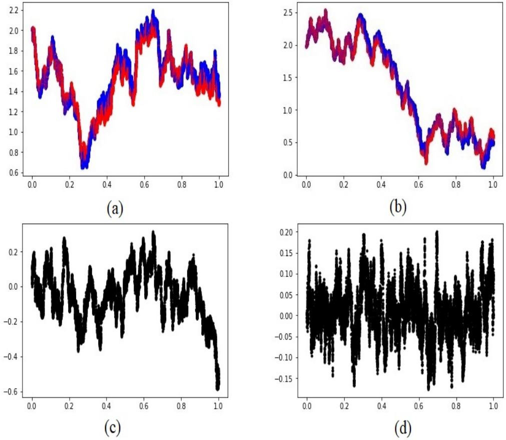

Figure: Sample pathes of the X_t^ε and Y_t^ε processes, as well as the limiting Y–process (driven by W_t^2) starting from $(X, Y) = (0, 2)$ in 15000 steps for stepsize$= 0.0001$, that is rescaled to $[0, 1]$. (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.01$; the red curves are the sample pathes for Y_t, the blue curves are the sample pathes for Y_t^ε. (c) $\varepsilon = 0.1$; (d) $\varepsilon = 0.01$; the black curves are the sample pathes for X_t^ε.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- To prove this we have to carefully analyze the behavior of the process $(X_t^\varepsilon, Y_t^\varepsilon)$ near the origin $O = (0,0)$.
- We introduce the angular variable $\theta_t^\varepsilon = \arctan\left(\frac{Y_t^\varepsilon}{X_t^\varepsilon}\right)$.
-

$$\begin{cases} d\theta_t^\varepsilon = \frac{1}{\varepsilon} X_t^\varepsilon dt + \frac{1}{(r_t^\varepsilon)^2} dW_t^\theta \ , \theta_0^\varepsilon = \arctan\left(\frac{Y_0^\varepsilon}{X_0^\varepsilon}\right) \ , \\ dr_t^\varepsilon = \left(\frac{1}{2r_t^\varepsilon} - r_t^\varepsilon\right) dt + dW_t^r \ , r_0^\varepsilon = \sqrt{(X_0^\varepsilon)^2 + (Y_0^\varepsilon)^2} \ . \end{cases}$$

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- $\delta = \varepsilon^{1/10}$.
- Set the slow time clock $\mathrm{t} = (\delta/\varepsilon)t$ and let us consider a time–rescaled pair of processes $\Theta^\varepsilon_\mathrm{t} = \theta^\varepsilon_{(\varepsilon/\delta)\mathrm{t}}$ and $R^\varepsilon_\mathrm{t} = r^\varepsilon_{(\varepsilon/\delta)\mathrm{t}}$.
- Then the stochastic differential equations satisfied by $(\Theta^\varepsilon_\mathrm{t}, R^\varepsilon_\mathrm{t})$ are given by

$$\begin{cases} d\Theta^\varepsilon_\mathrm{t} = \dfrac{X^\varepsilon_{(\varepsilon/\delta)\mathrm{t}}}{\delta} d\mathrm{t} + \sqrt{\dfrac{\varepsilon}{\delta}} \cdot \dfrac{1}{(R^\varepsilon_\mathrm{t})^2} dW^\theta_\mathrm{t} \ , \ \Theta^\varepsilon_0 = \theta^\varepsilon_0 \\ dR^\varepsilon_\mathrm{t} = \dfrac{\varepsilon}{\delta} \left(\dfrac{1}{2R^\varepsilon_\mathrm{t}} - R^\varepsilon_\mathrm{t} \right) d\mathrm{t} + \sqrt{\dfrac{\varepsilon}{\delta}} dW^r_\mathrm{t} \ , R^\varepsilon_0 = r^\varepsilon_0 \ . \end{cases}$$

- $\Theta^\varepsilon_\mathrm{t}$ is fast motion and R^ε_t is slow motion.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- This analysis enables us to conclude that for any initial condition $|X_0^\varepsilon| \geq 2\delta$, the flow will quickly bring the particle back to the region $Y \geq \delta$, and during this process the $|X|$–value is less or equal than 3δ.
- In particular, this implies that

$$\mathbf{P}\left(|X_t^\varepsilon| \leq 3\delta \text{ for } 0 \leq t \leq T\right) \to 1$$

  as $\varepsilon \downarrow 0$.
- As $\varepsilon \to 0$ the process X_t^ε will be localized near 0, and the process Y_t^ε lives on $\{Y \geq \delta\}$.
- We also need to do some exit time analysis and the proof of tightness for $\{Y_t^\varepsilon : 0 \leq t \leq T\}_{\varepsilon > 0}$.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

Theorem

Let $T > 0$ and initial condition $(x_0, y_0) \in \mathbb{R}^2$. Then
(a) For any bounded continuous function $F : \mathbb{R}^2 \to \mathbb{R}$ that is uniformly Lipschitz continuous with a Lipschitz constant $Lip(F) < \infty$ we have

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}\left[F(X_T^\varepsilon, Y_T^\varepsilon) - F(0, Y_T^\varepsilon)\right] = 0 \ .$$

(b) The measures on $\mathbf{C}_{[0,T]}(\mathbb{R})$ induced by the process Y_t^ε converge weakly as $\varepsilon \downarrow 0$ to the measure induced by Y_t with $Y_0 = y^\pi(x_0, y_0)$.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- The proof makes use of the martingale problem framework for Markov processes.

Metastable behavior.

- For fixed $\varepsilon > 0$, at a subexponential time scale, excursions of the process $(X_t^\varepsilon, Y_t^\varepsilon)$ moving from Oy_A towards a level set $y = -a$ will be observed.
- This induces jumps from points in Oy_B to points in Oy_A.
- As ε becomes smaller, motions of the process $(X_t^\varepsilon, Y_t^\varepsilon)$ to Oy_B and jumping back become more and more rare, and in the limit no more such jumps appear, so that we come to the limiting process Y_t which cannot penetrate through O.
- Thus as $\varepsilon > 0$ is close to 0, the description of the "metastable" behavior of system involves both a diffusion part and a jump part.

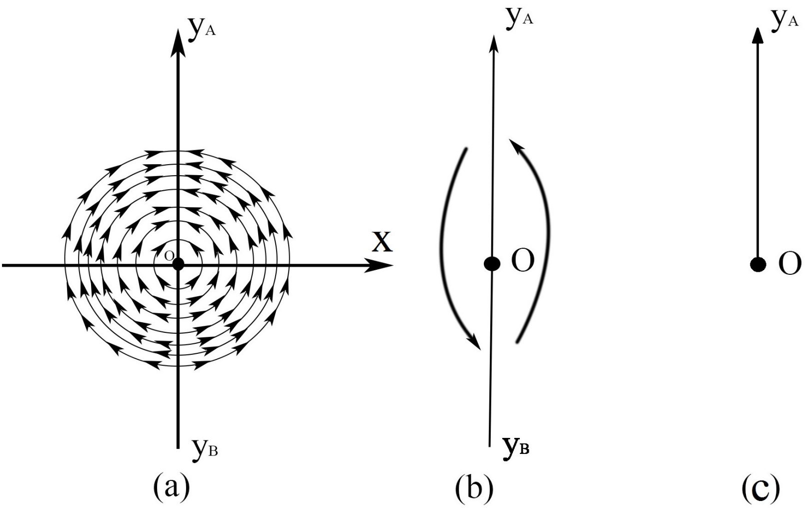

FIGURE: The *AB* model.

Relation with the partial damping model.

- System $(X_t^\varepsilon, Y_t^\varepsilon)$ (with rescaled time) :

$$\begin{cases} dX_t^\varepsilon = -\dfrac{1}{\varepsilon} X_t^\varepsilon Y_t^\varepsilon dt - X_t^\varepsilon dt + dW_t^1 \ , \ X_0^\varepsilon = x_0 \ , \\ dY_t^\varepsilon = \dfrac{1}{\varepsilon}(X_t^\varepsilon)^2 dt - \ Y_t^\varepsilon dt + dW_t^2 \ , \ Y_0^\varepsilon = y_0 \ . \end{cases}$$

- Limit of the Y–process is given by

$$dY_t = \left(\frac{1}{2Y_t} - Y_t \right) dt + dW_t^2 \ , Y_0 = y^\pi(x_0, y_0) \ .$$

Relation with the partial damping model.

- Remove the damping in Y–direction

$$\begin{cases} dX_t^\varepsilon = -\frac{1}{\varepsilon} X_t^\varepsilon Y_t^\varepsilon dt - X_t^\varepsilon dt + dW_t^1 \ , \ X_0^\varepsilon = x_0 \ , \\ dY_t^\varepsilon = \frac{1}{\varepsilon} (X_t^\varepsilon)^2 dt + dW_t^2 \ , \ Y_0^\varepsilon = y_0 \ . \end{cases}$$

- Limit of the Y–process is given by

$$dY_t = \frac{1}{2Y_t} dt + dW_t^2 \ , Y_0 = y^\pi(x_0, y_0) \ .$$

- A positive drift in the Y–direction !

A few more remarks.

- Motion on the cone of invariant measures of the unperturbed system.
- One can use this result to analyze behavior of elliptic operator $L = \frac{1}{\varepsilon} L_0 + L_1$, where

$$L_0 = -xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$$

  and

$$L_1 = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\partial^2}{\partial y^2} \ .$$

- The underlying 2d system is related to Euler equation - it is, in fact, the Euler-Arnold equation on a Lie algebra of the 2d Lie group of affine transformations of the line.

Reference.

Hu, W., On the long time behavior of a perturbed conservative system with degeneracy. *Journal of Theoretical Probability*, in revision.

Open Problem : Invariant measure for 2–d Euler equations.

- Kuksin, S., Shirikyan, A., Rigorous results in space–periodic two–dimensional turbulence, *Physics of Fluids*, 29 :125106, 2017.
- 2–d Euler equation with damping and noise :

$$\omega_t+(u\cdot\nabla)\omega-\nu\Delta\omega=\sqrt{\nu}\eta(x,t)\,,\, u=\nabla^{\perp}\Delta^{-1}\omega\,,\,\omega(x,0)=\omega_0(x)\,.$$

- "random force" $\eta(x,t)=\sum_{k\in\Lambda^*} c_k e^{ikx} dw_k(t)$ where Λ^* is the lattice $\Lambda^*=\frac{2\pi}{a}\mathbb{Z}\oplus\frac{2\pi}{b}\mathbb{Z}$.
- invisid scaling $\nu\to 0$.
- Change of variable $t\to\nu t$:

$$\omega_t+\frac{1}{\nu}(u\cdot\nabla)\omega-\Delta\omega=\eta(x,t)\,,\, u=\nabla^{\perp}\Delta^{-1}\omega\,,\,\omega(x,0)=\omega_0(x)\,.$$

Open Problem : Invariant measure for 2–d Euler equations.

- Under some natural (and very mild) assumptions on the noise η, a well known result of Hairer–Mattingly [5] asserts that there is a unique stationary measure μ_ν for the stochastic process ω_t.
- As $\nu \to 0$ one can show that one has sufficient control over the measures μ_ν to establish that one might choose a subsequence μ_{ν_n} of these measures which converges to a (nontrivial) probability measure μ.
- μ is an invariant measure for the system with $\mu = 0$: the invariant measure for the 2d Euler equations.
- The more sets the measure charges, the more possibilities we have for the long–time behavior of 2d Euler equations.
- The simplest situation would be for μ to be supported on a subset of the steady–state solutions of the Euler equation.

5. Hairer, M., Mattingly, J.C., Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing, *Annals of Mathematics*, (2) **164** (2006), no. 3., pp. 993–1032.

Open Problem : Invariant measure for 2–d Euler equations.

- Finite–dimensional Hamiltonian system

$$\dot{x} + \varepsilon \Gamma x = J \nabla H + \sqrt{\varepsilon} B \dot{w}$$

  where w is the m–dimensional Brownian motion and B is a suitable matrix, and Γ is the matrix describing damping.
- Change of variable $t \to \varepsilon t$:

$$\dot{x} + \Gamma x = \frac{1}{\varepsilon} J \nabla H + \sqrt{\varepsilon} B \dot{w} \ .$$

- Invariant measure μ_ε.
- Assuming that the limit $\mu = \lim_{\varepsilon \to 0} \mu_\varepsilon$ exists (at least for a subsequence), it is easy to see that μ is an invariant measure for the original Hamiltonian system $\dot{x} = J \nabla H$: Liouville measure or Gibbs measure.

Open Problem : Invariant measure for 2–d Euler equations.

- At the infinite–dimensional level one does not have a reference measure such as the Liouville or Gibbs measure.
- Situation can be quite different from finite–dimensional case.
- "inviscid damping" effects cause solution to approach (in the weak–$*$) sense the solution which no longer exhibit inviscid damping (such as steady–state or a stable periodic solution).
- The long–time behavior can be in some sense a mixture between the behavior of a dissipative system (with the dissipation replaced by the inviscid damping) and the finite–dimensional Hamiltonian behavior (for which one does not generically expect to converge to an equilibrium, but rather motion along an ergodic component).
- Better finite–dimensional models...

Thank you for your attention !