Large deviations and averaging for systems of slow–fast stochastic reaction–diffusion equations.

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Systems of slow–fast stochastic reaction–diffusion equations : Equation.

- $$\begin{cases} \dfrac{\partial X^{\varepsilon,\delta}}{\partial t}(t,x) = \mathcal{A}_1 X^{\varepsilon,\delta}(t,x) + b_1(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x)) \\ \qquad\qquad + \sqrt{\varepsilon}\sigma_1(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x)) \dfrac{\partial W^{Q_1}}{\partial t}(t,x) \ , \\ \dfrac{\partial Y^{\varepsilon,\delta}}{\partial t}(t,x) = \dfrac{1}{\delta^2}\left[\mathcal{A}_2 Y^{\varepsilon,\delta}(t,x) + b_2(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x))\right] \\ \qquad\qquad + \dfrac{1}{\delta}\sigma_2(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x)) \dfrac{\partial W^{Q_2}}{\partial t}(t,x) \ , \\ X^{\varepsilon,\delta}(0,x) = X_0(x) \ , \ Y^{\varepsilon,\delta}(0,x) = Y_0(x) \ , \ x \in D \ , \\ \mathcal{N}_1 X^{\varepsilon,\delta}(t,x) = \mathcal{N}_2 Y^{\varepsilon,\delta}(t,x) = 0 \ , \ t \geq 0 \ , \ x \in \partial D \ . \end{cases} \tag{1}$$

Systems of slow–fast stochastic reaction–diffusion equations : Equation.

- Here $\varepsilon > 0$ is a small parameter and $\delta = \delta(\varepsilon) > 0$ is such that $\delta \to 0$ as $\varepsilon \downarrow 0$.
- The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are two strictly elliptic operators and $b_i(x, X, Y)$, $i = 1, 2$ are the nonlinear terms.
- The noise processes W^{Q_1} and W^{Q_2} are two cylindrical Wiener processes with covariance matrices Q_1 and Q_2, and $\sigma_i(x, X, Y)$, $i = 1, 2$ give the corresponding multiplicative noises.
- The initial values X_0 and Y_0 are assumed to be in $L^2(D)$.
- The boundary conditions are given by operators $\mathcal{N}_i$, $i = 1, 2$ which may correspond to either Dirichlet or Neumann conditions.

Systems of slow–fast stochastic reaction–diffusion equations : Averaging and Large Deviations.

- Fast–slow system characterized by the small parameter $\delta > 0$;
- The noise term $\sqrt{\varepsilon}\sigma_1(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x))\dfrac{\partial W^{Q_1}}{\partial t}(t,x)$ in the equation for the slow process $X^{\varepsilon,\delta}(t,x)$ has a small parameter $\sqrt{\varepsilon}$, and both the deterministic as well as the noise term in the equation for the fast process $Y^{\varepsilon,\delta}(t,x)$ contain large parameters $\dfrac{1}{\delta^2}$ and $\dfrac{1}{\delta}$.
- In the limit, we expect an interplay between an averaging effect in the fast process $Y^{\varepsilon,\delta}(t,x)$ and a large deviation effect in the slow process $X^{\varepsilon,\delta}(t,x)$.

Systems of slow–fast stochastic reaction–diffusion equations : Previous work.

- Large deviations of stochastic partial differential equations of reaction–diffusion type has been considered in previous works, but without the effect of multiple scales.
- Results in the case of slow–fast systems of stochastic reaction–diffusion equations have been considered in dimension one, with additive noise in the fast motion and no noise component in the slow motion.
- In finite dimensions the large deviations problem for multiscale diffusions has been well studied.
- To the best of our knowledge, the problem of large deviations for multiscale stochastic reaction diffusion equations in multiple dimensions, with multiplicative noise is being considered for the first time in the present work.

Systems of slow–fast stochastic reaction–diffusion equations : Application background.

- Fast–slow stochastic partial differential equations of reaction–diffusion type have many applications in chemistry and biology.
- In the classical chemical kinetics, the evolution of concentrations of various components in a reaction is described by ordinary differential equations. Such a description turns out to be unsatisfactory in a number of applications, especially in biology.

Systems of slow–fast stochastic reaction–diffusion equations : Application background.

- There are several ways to construct a more adequate mathematical model. If the reaction is fast enough, one should take into account that the concentration is not constant in space in the volume where the reaction takes place. Then the change of concentration due to the spatial transport, as a role the diffusion, should be taken into consideration and the system of ordinary differential equations should be replaced by a system of partial differential equations of reaction–diffusion type.

Systems of slow–fast stochastic reaction–diffusion equations : Application background.

- In some cases, one should also take into account random change in time of the rates of reaction. Then the ordinary differential equation is replaced by a stochastic differential equation. If the rates change randomly not just in time but also in space, the evolution of concentrations can be described by a system of stochastic partial differential equations.
- On the other hand, the rates of chemical reactions in the system and the diffusion coefficients may have, and as a rule have, different orders. Some of them are much smaller than others and this leads to mathematical models based on slow–fast stochastic reaction–diffusion equations.

Systems of slow–fast stochastic reaction–diffusion equations : Problem Formulation.

- We want to have large deviation results for the slow motion $X^{\varepsilon,\delta}$ in terms of Laplace principle.
- For every bounded and continuous function $h : C([0,T];H) \to \mathbb{R}$ we have

$$\lim_{\varepsilon\downarrow 0} -\varepsilon \ln \mathbf{E}_{X_0,Y_0}\left[\exp\left(-\frac{1}{\varepsilon}h(X^{\varepsilon,\delta})\right)\right] = \inf_{\phi\in C([0,T];H)}[S_{X_0}(\phi)+h(\phi)] \ .$$

  (Laplace principle.)
- Calculate the action functional (rate function) $S_{X_0}(\phi)$.

Large deviations in terms of Laplace principle : Weak convergence method.

- One of the most effective methods in analyzing large deviation effects is the weak convergence method.
- Roughly speaking, by a variational representation of exponential functionals of Wiener processes, one can represent the exponential functional of the slow process $X^{\varepsilon,\delta}(t,x)$ that appears in the Laplace principle (which is equivalent to large deviations principle) as a variational infimum over a family of *controlled* slow processes $X^{\varepsilon,\delta,u}(t,x)$.

Large deviations in terms of Laplace principle : Weak convergence method.

- In particular, we have, for any bounded continuous function $h : C([0,T]; L^2(D)) \to \mathbb{R}$, that

$$-\varepsilon \ln \mathbf{E}\left[\exp\left(-\frac{1}{\varepsilon} h(X^{\varepsilon,\delta})\right)\right] = \inf_{u \in L^2([0,T];U)} \mathbf{E}\left[\frac{1}{2}\int_0^T |u(s)|_U^2 ds + h(X^{\varepsilon,\delta,u})\right].$$

Large deviations in terms of Laplace principle : Weak convergence method.

- Here U is the control space, and the infimum is over all controls $u \in U$ with finite $L^2([0, T]; |\bullet|)$–norm.
- The controlled slow motion $X^{\varepsilon,\delta,u}$ that appears on the right hand side of the Laplace principle comes from a controlled slow–fast system $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$ of reaction–diffusion equations corresponding to (1).

Large deviations in terms of Laplace principle : Associated Control System.

- $$\begin{cases} \dfrac{\partial X^{\varepsilon,\delta,u}}{\partial t}(t,x) = \mathcal{A}_1 X^{\varepsilon,\delta,u}(t,x) + b_1(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x)) \\ \qquad\qquad +\sigma_1(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))(Q_1 u(t))(x) \\ \qquad\qquad +\sqrt{\varepsilon}\sigma_1(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))\dfrac{\partial W^{Q_1}}{\partial t}(t,x) \ , \\ \dfrac{\partial Y^{\varepsilon,\delta,u}}{\partial t}(t,x) \\ \qquad\qquad = \dfrac{1}{\delta^2}\left[\mathcal{A}_2 Y^{\varepsilon,\delta,u}(t,x) + b_2(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))\right] \\ \qquad\qquad +\dfrac{1}{\delta\sqrt{\varepsilon}}\sigma_2(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))(Q_2 u(t))(x) \\ \qquad\qquad +\dfrac{1}{\delta}\sigma_2(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))\dfrac{\partial W^{Q_2}}{\partial t}(t,x) \ , \\ X^{\varepsilon,\delta,u}(0,x) = X_0(x) \ , \ Y^{\varepsilon,\delta,u}(0,x) = Y_0(x) \ , \ x \in D \ , \\ \mathcal{N}_1 X^{\varepsilon,\delta,u}(t,x) = \mathcal{N}_2 Y^{\varepsilon,\delta,u}(t,x) = 0 \ , \ t \geq 0 \ , \ x \in \partial D \ . \end{cases} \tag{2}$$

Large deviations in terms of Laplace principle : Associated Control System.

- Need to analyze the limit as $\varepsilon \to 0$ (and thus $\delta \to 0$) of the controlled fast–slow system $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$ in (2).
- Recall that for any bounded continuous function $h : C([0,T]; L^2(D)) \to \mathbb{R}$, that

$$
\begin{aligned}
-\varepsilon \ln \mathbf{E}\left[\exp\left(-\frac{1}{\varepsilon} h(X^{\varepsilon,\delta})\right)\right] \\
= \inf_{u \in L^2([0,T];U)} \mathbf{E}\left[\frac{1}{2}\int_0^T |u(s)|_U^2 ds + h(X^{\varepsilon,\delta,u})\right].
\end{aligned}
$$

- u is in terms of a feedback control !

Set–up and assmptions.

- $H = L^2(D)$;
- $W(t) = \bigotimes_{k=1}^{\infty} \beta_k(t)$;
- $Q_i : \mathbb{R}^\infty \to H$ where $i = 1, 2$ add color to the noise and also decide if the noise are independent;
- $W^{Q_i}(t,x) = Q_i W(t)(x)$.
- $U = \{x \in \mathbb{R}^\infty, \sum_{i=1}^{\infty} x_i^2 < \infty\}$ with norm $|x|_U^2 = \sum_{i=1}^{\infty} x_i^2$ is the space in which the control lives in.

Limit of the controlled fast–slow process.

- We only know that the control $u \in L^2([0, T]; U)$;
- This makes the passage to the limit in the controlled fast–slow system $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$ as $\varepsilon \to 0$ (and therefore $\delta \to 0$) difficult.
- Need more *Hypotheses*.

Hypothesis 1.

- *Hypothesis 1.* For $i = 1, 2$, there exist complete orthonormal systems $\{e_{i,k}\}_{k\in\mathbb{N}}$ in H, and sequences of non–negative real numbers $\{\alpha_{i,k}\}_{k\in\mathbb{N}}$, such that

$$A_i e_{i,k} = -\alpha_{i,k} e_{i,k} \ , \ \ k \geq 1 \ .$$

The covariance operators $Q_i : U \to H$, $i = 1, 2$ are diagonalized by the same orthonormal basis $\{e_{i,k}\}_{k\in\mathbb{N}}$ in the following sense. For $i = 1, 2$, there exists an orthonormal set $\{f_{i,k}\}_{k\in\mathbb{N}}$. The set of $\{f_{i,k}\}_{k\in\mathbb{N}}$ is not necessarily complete. There exist sequences of non–negative real numbers $\{\lambda_{i,k}\}_{\substack{i=1,2\\ k\in\mathbb{N}}}$ satisfying

$$Q_i f_{i,k} = \lambda_{i,k} e_{i,k} .$$

Notice that if $\text{span}\{f_{1,k}\}_{k\in\mathbb{N}} \perp \text{span}\{f_{2,k}\}_{k\in\mathbb{N}}$, then the driving noises of the fast and the slow motion are independent.

Hypothesis 1.

- *Hypothesis 1 (cont'd).* If $d = 1$, then we have

$$\kappa_i := \sup_{k \in \mathbb{N}} \lambda_{i,k} |e_{i,k}|_0 < \infty \ , \ \zeta_i := \sum_{k=1}^{\infty} \alpha_{i,k}^{-\beta_i} |e_{i,k}|_0^2 < \infty$$

for some constant $\beta_i \in (0,1)$, and if $d \geq 2$, we have

$$\kappa_i := \sum_{k=1}^{\infty} \lambda_{i,k}^{\rho_i} |e_{i,k}|_0^2 < \infty \ , \ \zeta_i := \sum_{k=1}^{\infty} \alpha_{i,k}^{-\beta_i} |e_{i,k}|_0^2 < \infty$$

for some constants $\beta_i \in (0, +\infty)$ and $\rho_i \in (2, +\infty)$ such that

$$\frac{\beta_i(\rho_i - 2)}{\rho_i} < 1 \ .$$

Moreover

$$\inf_{\substack{k \in \mathbb{N} \\ i=1,2}} \alpha_{i,k} =: \lambda > 0 \ .$$

Hypothesis 2.

- *Hypothesis 2.* 1. The mappings $b_i : D \times \mathbb{R}^2 \to \mathbb{R}$ and $\sigma_i : D \times \mathbb{R}^2 \to \mathbb{R}$ are measurable, both for $i = 1$ and for $i = 2$, and $\sum_{i=1,2} (L^X_{b_i} + L^X_{\sigma_i} + L^Y_{b_i} + L^Y_{\sigma_i}) \le M$ for some $M > 0$.
  Moreover,

$$\sup_{x \in D} |b_2(x, 0, 0)| < \infty \ , \ \sup_{x \in D} |\sigma_2(x, 0, 0)| < \infty \ .$$

  2. Recalling λ, the constant introduced in (18), we have that

$$L^Y_{b_2} < \lambda.$$

Hypothesis 2.

- *Hypothesis 2 (cont'd).* 3. σ_2 grows linearly in X, but is bounded in Y. There exists $c > 0$,

$$\sup_{x \in D} \sup_{Y \in \mathbb{R}} |\sigma_2(x, X, Y)| \leq c(1 + |X|).$$

4. The Lipschitz constants $L^Y_{b_2}$ and $L^Y_{\sigma_2}$ are so chosen that

$$\frac{L^Y_{b_2}}{\lambda} + \sqrt{K_2 (L^Y_{\sigma_2})^2 \int_0^\infty s^{-\beta_2 \frac{\rho_2 - 2}{\rho_2}} e^{-\lambda \frac{\rho_2 + 2}{\rho_2} s} ds} =: \mathfrak{L}^Y_{b_2, \sigma_2} < 1$$

where

$$K_2 = \left(\frac{\beta_2}{e} \right)^{\beta_2 \frac{\rho_2 - 2}{\rho_2}} \zeta_2^{\frac{\rho_2 - 2}{\rho_2}} \kappa_2^{\frac{2}{\rho_2}}$$

and $\lambda, \beta_2, \rho_2, \zeta_2, \kappa_2$ are all from Hypothesis 1.

Hypothesis 3.

- *Hypothesis 3.* b_1 and σ_1 grow at most linearly in X and sublinearly in Y. To be precise, there exists $0 \leq \zeta < 1 - \frac{\beta_1(\rho_1 - 2)}{\rho_1}$ and a constant $C > 0$ such that

$$\sup_{x \in D} \left(|b_1(x, X, Y)| + |\sigma_1(x, X, Y)| \right) \leq C \left(1 + |X| + |Y|^{\zeta} \right) .$$

Limit of the controlled fast–slow process.

- Controlled pair of fast–slow process $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$.
- The right way to describe the limit is a pair (ψ, P), which we call a viable pair.
- $\psi \in C([0,T]; H)$ is the limit dynamics, and $\mathrm{P} \in \mathscr{P}(U \times \mathcal{Y} \times [0,T])$ is a certain invariant measure.
- This is an averaging procedure.
- Constructions of the same type have been carried out thoroughly in the case of finite dimensions, but for the first time in infinite dimensions in our work.

Limit of the controlled fast–slow process : Tightness.

- We introduce the family of random occupation measures

$$\mathrm{P}^{\varepsilon,\Delta}(dudYdt) = \frac{1}{\Delta}\int_t^{t+\Delta} \mathbf{1}_{du}(u(s))\mathbf{1}_{dY}\left(Y^{\varepsilon,\delta,u}(s)\right) dsdt \ .$$

- After very technical proof we can show that the pair $(X^{\varepsilon,\delta,u}, \mathrm{P}^{\varepsilon,\Delta})$ is tight in the space $C([0,T];H) \times \mathscr{P}(U \times \mathcal{Y} \times [0,T])$.
- We take weak topology on U and norm topology on $\mathcal{Y}$, and we make use of a–priori bounds and tightness functions.

Limit of the controlled fast–slow process : Limiting slow dynamics.

- Tightness guarantees that $(X^{\varepsilon,\delta,u}, \mathrm{P}^{\varepsilon,\Delta}) \to (\bar{X}, \mathrm{P})$ as $\varepsilon \to 0$ (and therefore $\delta \to 0$) for some limiting $(\bar{X}, \mathrm{P})$.
- Recall that in mild form we can write

$$X^{\varepsilon,\delta,u}(t) = S_1(t)X_0 + \int_0^t S_1(t-s)B_1(X^{\varepsilon,\delta,u}(s), Y^{\varepsilon,\delta,u}(s))ds + \int_0^t S_1(t-s)\Sigma_1(X^{\varepsilon,\delta,u}(s), Y^{\varepsilon,\delta,u}(s))Q_1 u(s)ds + \sqrt{\varepsilon}\int_0^t S_1(t-s)\Sigma_1(X^{\varepsilon,\delta,u}(s), Y^{\varepsilon,\delta,u}(s))dW^{Q_1}(s)\ .$$

- Limiting slow dynamics

$$\psi(t) = S_1(t)X_0 + \int_{U\times\mathcal{Y}\times[0,t]} S_1(t-s)\xi(\psi(s), Y, u)\mathrm{P}(dudYds)\ ,$$

where

$$\xi(X, Y, u) = \Sigma_1(X, Y)Q_1 u + B_1(X, Y)\ .$$

Limit of the controlled fast–slow process : Limiting occupation measure.

- In regards to the limit for the controlled fast process $Y^{\varepsilon,\delta,u}$, the limiting measure P characterizes simultaneously the structure of the invariant measure of $Y^{\varepsilon,\delta,u}$ and the control function u.
- In general, these two objects are intertwined and coupled together into the measure P, so that the averaging with respect to the measure P is different and hard to perform as in the classical averaging principle.

Limit of the controlled fast–slow process : Limiting occupation measure.

- $\mathrm{P}(dudYdt) = \eta_t(du|Y)\mu^{\psi_t}(dY|u)dt$, and we cannot guarantee that μ is invariant measure to some process, as this process is a controlled process and we have no regularity properties of the optimal controls known other than being square integrable.
- In finite dimensional case with periodic coefficients, the problem can be resolved using the characterization of optimal controls through solutions to Hamilton–Jacobi–Bellman equations.
- Such a characterization is not rigorously known in infinite dimensions and even if that becomes the case, one would need to establish sufficient properties of such equations that would then imply that the resulting controlled process has a well defined invariant measure that is regular enough.

Limit of the controlled fast–slow process : Limiting occupation measure.

- Is it possible to have $\mathrm{P}(dudYdt) = \eta_t(du|Y)\mu^{\psi_t}(dY)dt$?
- Recall that the fast motion can be written formally as

$$\begin{aligned} dY^{\varepsilon,\delta,u}(t) = & \frac{1}{\delta^2}\left[A_2 Y^{\varepsilon,\delta,u}(t) + B_2(X^{\varepsilon,\delta,u}(t), Y^{\varepsilon,\delta,u}(t))\right] dt + \\ & + \frac{1}{\delta^2}\frac{\delta}{\sqrt{\varepsilon}}\Sigma_2(X^{\varepsilon,\delta,u}(t), Y^{\varepsilon,\delta,u}(t))Q_2 u(t)dt + \\ & + \frac{1}{\delta}\Sigma_2(X^{\varepsilon,\delta,u}(t), Y^{\varepsilon,\delta,u}(t))dW^{Q_2} \ , \\ Y^{\varepsilon,\delta,u}(0) = & \ Y_0 \in H \ . \end{aligned}$$

- Send $\dfrac{\delta}{\sqrt{\varepsilon}} \to 0$ as $\varepsilon \to 0$ (and therefore $\delta \to 0$).

Hypothesis 5.

- *Hypothesis 5.* We assume that $\varepsilon \downarrow 0$, $\delta = \delta(\varepsilon) \downarrow 0$ and $\Delta = \Delta(\delta, \varepsilon) \downarrow 0$, such that

$$\lim_{\varepsilon \downarrow 0} \frac{\delta}{\sqrt{\varepsilon}} = 0, \text{ and } \lim_{\varepsilon \downarrow 0} \frac{\delta}{\Delta \sqrt{\varepsilon}} = 0 \ .$$

Limit of the controlled fast–slow process : Limiting occupation measure.

- From Hypothesis 5 we know that

$$\lim_{\varepsilon \downarrow 0} \frac{\delta}{\sqrt{\varepsilon}} = 0, \text{ and } \lim_{\varepsilon \downarrow 0} \frac{\delta}{\Delta\sqrt{\varepsilon}} = 0 \ .$$

- Fast motion without control but driven by the controlled slow process

$$\begin{aligned} d\mathrm{Y}^{\varepsilon,\delta}(t) &= \frac{1}{\delta^2}\left[A_2\mathrm{Y}^{\varepsilon,\delta}(t) + B_2(X^{\varepsilon,\delta,u}(t), \mathrm{Y}^{\varepsilon,\delta}(t))\right] dt + \\ &\quad + \frac{1}{\delta}\Sigma_2(X^{\varepsilon,\delta,u}(t), \mathrm{Y}^{\varepsilon,\delta}(t)) dW^{Q_2} \ , \\ \mathrm{Y}^{\varepsilon,\delta}(0) &= Y_0 \in H \ . \end{aligned}$$

- One can show that

$$\mathbf{E}\frac{1}{\Delta}\int_0^T |Y^{\varepsilon,\delta,u}(t) - \mathrm{Y}^{\varepsilon,\delta}(t)|_H^2 dt \to 0 \ ,$$

as $\varepsilon \to 0$ (and therefore $\delta \to 0$).

Limit of the controlled fast–slow process : Limiting occupation measure.

- Slogan : Ergodic properties are stable with respect to mildly regular small perturbations.
- Knowing closeness of two processes in mean square sense is already enough to draw conclusions about the equivalency of their invariant measures.
- $$\mathrm{P}(dudYdt) = \eta_t(du|Y)\mu^{\psi_t}(dY)dt \ .$$

Averaging of the controlled fast–slow process.

- Definition

  A pair $(\psi, \mathrm{P}) \in C([0,T]; L^2(D)) \times \mathscr{P}(U \times \mathcal{Y} \times [0,T])$ will be called viable with respect to $(\xi, \mathcal{L})$, or simply viable if there is no confusion, if the following are satisfied. The function $\psi(t)$ is absolutely continuous as a function in the space $C([0,T]; H)$, P is square integrable in the sense that

$$\int_{U\times\mathcal{Y}\times[0,T]} \left(|u|_U^2 + |Y|_H^2\right) \mathrm{P}(dudYds) < \infty$$

  and the following hold for all $t \in [0,T]$:

$$\psi(t) = S_1(t)X_0 + \int_{U\times\mathcal{Y}\times[0,t]} S_1(t-s)\xi(\psi(s), Y, u)\mathrm{P}(dudYds) \ ,$$

Averaging of the controlled fast–slow process.

- Definition
  the measure P is such that

$$\mathrm{P} \in \mathbb{P} = \left\{ \begin{array}{c} P \in \mathscr{P}(U \times \mathcal{Y} \times [0,T]) : \mathrm{P}(dudYdt) = \eta(du|Y,t)\mu(dY|t)dt, \\ \mu(dY|t) = \mu^{\psi(t)}(dY) \text{ for } t \in [0,T] \end{array} \right\}$$

  where μ^X is the invariant measure for the uncontrolled fast process with frozen slow component X, and

$$\mathrm{P}(U \times \mathcal{Y} \times [0,t]) = t \ .$$

Averaging of the controlled fast–slow process.

- Theorem

  Let $u \in \mathcal{P}_2^N(U)$, $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$ be the mild solution to controlled fast process and $T < \infty$. Let also $\mathrm{P}^{\varepsilon,\Delta}(dudYdt)$ be the occupation measure. Assume Hypotheses 1, 2, 3 and 5, $X_0 \in H_1^\theta$ with $\theta > 0$ sufficiently small and $Y_0 \in H$. Then, the family of processes $X^{\varepsilon,\delta,u}$ is tight in $C([0,T];H)$ and the family of measures $\mathrm{P}^{\varepsilon,\Delta}$ is tight in $\mathscr{P}(U \times \mathcal{Y} \times [0,T])$, where $U \times \mathcal{Y} \times [0,T]$ is endowed with the weak topology on U, the norm topology on $\mathcal{Y}$ and the standard topology on $[0,T]$. Hence, given any subsequence of $\{(X^{\varepsilon,\delta,u}, \mathrm{P}^{\varepsilon,\Delta}), \varepsilon, \delta, \Delta > 0\}$, there exists a subsubsequence that converges in distribution with limit $(\bar{X}, \mathrm{P})$. With probability 1, the accumulation point $(\bar{X}, \mathrm{P})$ is a viable pair with respect to $(\xi, \mathcal{L})$.

Large Deviations.

- Back to our large deviations problem : For every bounded and continuous function $h : C([0,T]; H) \to \mathbb{R}$ we want

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbf{E}_{X_0, Y_0} \left[\exp\left(-\frac{1}{\varepsilon} h(X^{\varepsilon,\delta}) \right) \right] = \inf_{\phi \in C([0,T];H)} [S_{X_0}(\phi) + h(\phi)] \, .$$

- Representation formula

$$-\varepsilon \ln \mathbf{E} \left[\exp\left(-\frac{1}{\varepsilon} h(X^{\varepsilon,\delta}) \right) \right] \\ = \inf_{u \in L^2([0,T];U)} \mathbf{E} \left[\frac{1}{2} \int_0^T |u(s)|_U^2 ds + h(X^{\varepsilon,\delta,u}) \right] \, .$$

Large Deviations.

- Theorem
  Let $(X^{\varepsilon,\delta}, Y^{\varepsilon,\delta})$ be the mild solution to fast–slow SRDE and let $T<\infty$. Assume Hypothesis 1, 2, 4, and 5 and let $Y_0 \in H$ and $X_0 \in H_1^{\theta}$. Define

$$S(\phi) = S_{X_0}(\phi) = \inf_{(\phi,P)\in\mathcal{V}_{(\xi,\mathcal{L})}} \left[\frac{1}{2}\int_{U\times\mathcal{Y}\times[0,T]} |u|_U^2 \mathrm{P}(dudYdt)\right] ,$$

with the convention that the infimum over the empty set is ∞.

Large Deviations.

- Theorem

  Then, there exists $\bar{\theta} > 0$ *such that for all* $\theta \in (0, \bar{\theta}]$ *and* $X_0 \in H_1^\theta$*, and for every bounded and continuous function* $h : C([0,T]; H) \to \mathbb{R}$ *we have*

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbf{E}_{X_0, Y_0} \left[\exp \left(-\frac{1}{\varepsilon} h(X^{\varepsilon,\delta}) \right) \right] = \inf_{\phi \in C([0,T];H)} [S_{X_0}(\phi) + h(\phi)] \, .$$

  In particular, $\{X^{\varepsilon,\delta}\}$ *satisfies the large deviations principle in* $C([0,T]; H)$ *with action functional* $S_{X_0}(\cdot)$*, uniformly for* X_0 *in compact subsets of* H_1^θ*.*

Large Deviations : Lower bound.

- By Fatou's Lemma we have

$$
\begin{aligned}
&\liminf_{\varepsilon\downarrow 0}\left(-\varepsilon \ln \mathbf{E}_{X_0}\left[\exp\left(-\frac{h(X^{\varepsilon,\delta})}{\varepsilon}\right)\right]\right)\\
\geq\ &\liminf_{\varepsilon\downarrow 0}\left(\mathbf{E}_{X_0}\left[\frac{1}{2}\int_0^T |u^\varepsilon(t)|_U^2 dt + h(X^{\varepsilon,\delta})\right] - \varepsilon\right)\\
\geq\ &\liminf_{\varepsilon\downarrow 0}\left(\mathbf{E}_{X_0}\left[\frac{1}{2}\int_0^T \frac{1}{\Delta}\int_t^{t+\Delta} |u^\varepsilon(s)|_U^2 ds dt + h(X^{\varepsilon,\delta})\right]\right)\\
=\ &\liminf_{\varepsilon\downarrow 0}\left(\mathbf{E}_{X_0}\left[\frac{1}{2}\int_{U\times\mathcal{Y}\times[0,T]} |u|_U^2 \mathrm{P}^{\varepsilon,\Delta}(dudYdt) + h(X^{\varepsilon,\delta})\right]\right)\\
\geq\ &\inf_{(\phi,\mathrm{P})\in\mathcal{V}_{(\xi,\mathcal{L})}}\left[\frac{1}{2}\int_{U\times\mathcal{Y}\times[0,T]} |u|_U^2 \mathrm{P}(dudYdt) + h(\phi)\right],
\end{aligned}
$$

which concludes the proof of the Laplace principle lower bound.

Large Deviations : Upper bound.

- In order to prove the Laplace principle upper bound, we have to construct a nearly optimal control that achieves the upper bound.
- The nearly optimal control will be in feedback form with respect to the Y variable,
- Due to infinite dimensionality we can only work it out in some special cases.

Hypothesis 4.

- *Hypothesis 4.* b_1 is as in Hypothesis 3 and if $d = 1$, then there are positive constants $0 < c_0 \leq c_1 < \infty$ such that $0 < c_0 \leq \sigma_1^2(x, X, Y) \leq c_1$, or if $d > 1$ then $\sigma_1(x, X, Y) = \sigma_1(x, X)$ is independent of Y and can grow at most linearly in X uniformly in $x \in D$.

Thank you for your attention !