Second order elliptic equations with a small parameter.

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Small parameter : general framework.

- We are interested in boundary problems for the operator $L_\varepsilon = L_0 + \varepsilon L_1$ and initial-boundary problems for $\dfrac{\partial u^\varepsilon}{\partial t} = L_\varepsilon u^\varepsilon$, $t > 0$, $x \in \partial G$.
- Operators

$$L_k = b^{(k)}(x) \cdot \nabla + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}^{(k)}(x) \frac{\partial^2}{\partial x_i \partial x_j} \ , \ k = 0, 1 \ .$$

- Vectors $b^{(k)}(x) = (b_1^{(k)}(x), ..., b_d^{(k)}(x))$, $k = 0, 1$.
- Coefficients $a_{ij}^{(k)}$ and $b_j^{(k)}$ are $\mathbf{C}^{(2)}$.
- For fixed $\varepsilon > 0$ the operator L_ε is elliptic.

Small parameter : non-degenerate case.

- We take the Dirichlet problem as an example :

$$L_\varepsilon u^\varepsilon(x) = (L_0 + \varepsilon L_1)\, u^\varepsilon(x) = 0 \ , \ \ u^\varepsilon(x)|_{\partial G} = \psi(x) \ .$$

- If L_0 is elliptic, then $u^\varepsilon \to u^0$ as $\varepsilon \downarrow 0$ where $L_0 u^0(x) = 0$, $u^0(x)\big|_{\partial G} = \psi(x)$.
- What about degenerate L_0 ?

Levinson case.

- Levinson case (1950) :

$$L_0 = b^{(0)}(x) \cdot \nabla$$

  and

$$L_1 = b^{(1)}(x) \cdot \nabla + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}^{(1)}(x) \frac{\partial^2}{\partial x_i \partial x_j} \ .$$

- Levinson condition : trajectories of the dynamical system $\dot{X}_t = b^{(0)}(X_t)$ leave the domain G in finite time and cross the boundary ∂G in a regular way.
- **Theorem.** (Levinson, 1950) *We have* $\lim\limits_{\varepsilon \downarrow 0} u^{\varepsilon}(x) = u^0(x)$ *where* $u^0(x)$ *is the solution of* $L_0 u^0(x) = 0$, $x \in G$ *and* $u^0\big|_{\partial_1 G} = \psi(x)$. *Here* $\partial_1 G$ *is the part of the boundary* ∂G *where* X_t *hits and leaves* ∂G.

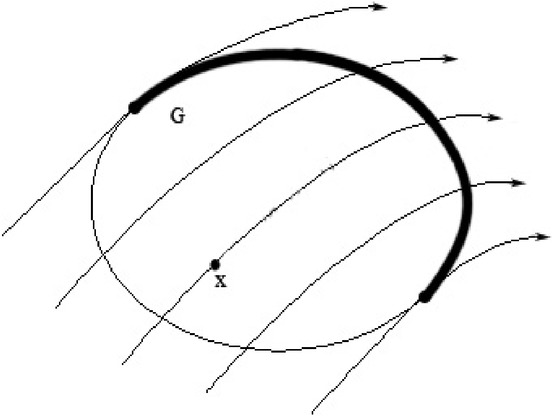

FIG.: Levinson case.

Levinson case from probabilistic point of view.

- Levinson's result can be easily explained from a probabilistic point of view.
- Let as before $\dot{X}_t = b^{(0)}(X_t)$.
- Consider a diffusion process with a small diffusion :

$$
\begin{array}{l}
dX_t^{\varepsilon} = \sqrt{\varepsilon}\sigma^{(1)}(X_t^{\varepsilon})dW_t + b^{(0)}(X_t^{\varepsilon})dt \ , \\
X_0^{\varepsilon} = x \in \mathbb{R}^d \ .
\end{array}
$$

- $\sigma^{(1)}(x)(\sigma^{(1)}(x))^* = a^{(1)}(x)$.
- As $\varepsilon \downarrow 0$ the process X_t^{ε} converges to X_t in a certain sense.
- Thus $u^{\varepsilon}(x) = \mathbf{E}_x\psi(X_{\tau}^{\varepsilon}) \to \mathbf{E}_x\psi(X_t) = u^0(x)$ as $\varepsilon \downarrow 0$.
- τ is the first hitting time of X_t^{ε} to ∂G.

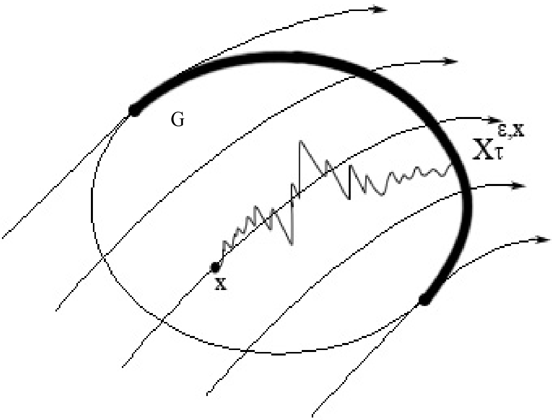

FIG.: Levinson case.

Summary.

- Summary : Convergence of the solution of corresponding PDE $\Longleftrightarrow$ (Weak) convergence of corresponding diffusion process.

Degenerate problems : Neumann case.

- Neumann problem
$$\left(\frac{1}{\varepsilon}L_0 + L_1\right) u^{\varepsilon}(x) = f(x) \ , \ \left.\frac{\partial u^{\varepsilon}(x)}{\partial \gamma^{\varepsilon}(x)}\right|_{\partial G} = 0 \ .$$
- We work with self-adjoint situation
$$L_k u(x) = \frac{1}{2}\nabla \cdot (a^{(k)}(x)\nabla u(x)) \ .$$
- Solvability and uniqueness condition : for Hölder continuous $\int_G f(x)dx = 0$ and for $x_O \in G \cup \partial G$ we have $u(x_O) = 0$.
- $a^{(0)}(x)$ is degenerate and non-negative definite ; $a^{(1)}(x)$ is positive definite. Suppose $a^{(0)}(x) = \sigma^{(0)}(x)(\sigma^{(0)}(x))^*$ and $a^{(1)}(x) = \sigma^{(1)}(x)(\sigma^{(1)}(x))^*$. The coefficients of $a^{(1)}(x)$ are in $\mathbf{C}^{(2)}$ and the coefficients of $a^{(0)}(x)$ are in at least $\mathbf{C}^{(1)}$.
- We specify the degeneration by looking at a first integral $H(x) : a^{(0)}(x)\nabla H(x) = 0$.
- We single out only one first integral by making a restriction $\mathbf{e} \cdot (a^{(0)}(x)\mathbf{e}) \geq \underline{a}(x)|\mathbf{e}|^2_{\mathbb{R}^d}$ for each $\mathbf{e}$ such that $\mathbf{e} \cdot \nabla H(x) = 0$.

Averaging principle.

- First we assume that $a^{(0)}(x)$ has constant rank $d-1$ and its coefficients are in $\mathbf{C}^{(2)}$.
- Corresponding process

$$\begin{aligned} & dX_t^\varepsilon \\ & = \frac{1}{\varepsilon} b^{(0)}(X_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}}\sigma^{(0)}(X_t^\varepsilon)dW_t^0 \text{ (fast motion)} \\ & + b^{(1)}(X_t^\varepsilon)dt + \sigma^{(1)}(X_t^\varepsilon)dW_t^1 \ . \end{aligned}$$

  with reflection w.r.t. inward co-normal $\gamma^\varepsilon(x)$ at ∂G.
- $b^{(0)}$ and $b^{(1)}$ are calculated from $a^{(0)}$ and $a^{(1)}$.
- Fast motion is moving on level surface $\{H(x) = \text{const}\}$ and has Lebesgue measure as its invariant measure.
- Averaging principle (Khasminski, Freidlin-Wentzell, ...) : The limit of slow motion $H(X_t^\varepsilon)$ can be calculated by averaging with respect to the fast motion : $H(X_t^\varepsilon) \to Y_t$ weakly as $\varepsilon \downarrow 0$.

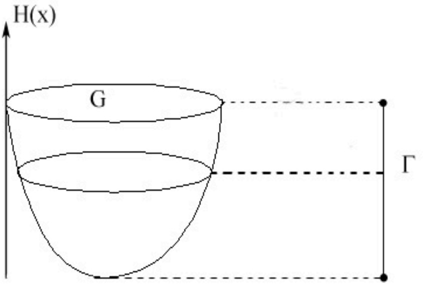

FIG.: Averaging Principle.

Averaging principle.

- $H(X_t^\varepsilon) \to Y_t$ weakly in $\mathbf{C}_{[0,T]}(\Gamma)$.
- Y_t is a 1-dimensional process.
- In the simplest case as in our example Y_t has a generator $\mathcal{L}f(H) = \frac{1}{2}M^{-1}(H)\frac{d}{dH}\left(M(H)\overline{a^{(1)}}(H)\frac{df}{dH}\right)$.
- Here $\overline{a^{(1)}}(h) = M^{-1}(h)\int_{C(h)} \frac{(a^{(1)}(x)\nabla H(x), \nabla H(x))}{|\nabla H(x)|_{\mathbb{R}^d}} d\sigma$ and $M(h) = \int_{C(h)} \frac{d\sigma}{|\nabla H(x)|_{\mathbb{R}^d}}$.

The implication of averaging principle on differential equations.

- Neumann problem :

$$\left(\frac{1}{\varepsilon}L_0 + L_1\right) u^\varepsilon(x) = f(x) \text{ for } x \in G \ , \ \left.\frac{\partial u^\varepsilon(x)}{\partial \gamma^\varepsilon(x)}\right|_{x\in\partial G} = 0 \ .$$

-

$$u^\varepsilon(x) = -\int_0^\infty \mathbf{E}_x f(X_t^\varepsilon)dt + \int_0^\infty \mathbf{E}_{x_O} f(X_t^\varepsilon)dt \ .$$

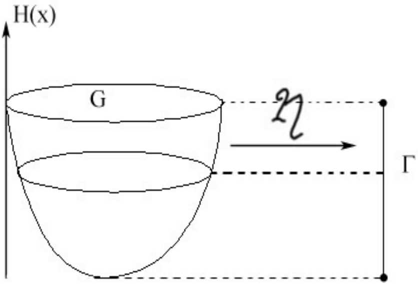

Fig.: Identification mapping $\mathfrak{Y}$.

The implication of averaging principle on differential equations.

- $$u^{\varepsilon}(x) = -\int_0^{\infty} \mathbf{E}_x f(X_t^{\varepsilon})dt + \int_0^{\infty} \mathbf{E}_{x_O} f(X_t^{\varepsilon})dt$$
  $$\to -\int_0^{\infty} \mathbf{E}_{\mathfrak{Y}(x)} \overline{f}(Y_t)dt + \int_0^{\infty} \mathbf{E}_{\mathfrak{Y}(x_O)} \overline{f}(Y_t)dt = v(\mathfrak{Y}(x)) \ .$$
- Here $\overline{f}(h) = \dfrac{1}{M(h)} \displaystyle\int_{C(h)} f(x) \dfrac{d\sigma}{|\nabla H(x)|_{\mathbb{R}^d}}$.
- $v(h)$ is the solution of ODE on Γ : $\mathcal{L}v(h) = -\overline{f}(y)$ and $v(\mathfrak{Y}(x_O)) = 0$.

A few remarks.

- The case when $H(x)$ has saddle point in G : gluing condition at the interior vertices of Γ. (Freidlin-Wentzell, PTRF, 2012)
- The case of attractor : In the sense of random perturbations of dynamical systems – large deviation principle. (Freidlin-Wentzell, 1969 ; Kifer, 1974)

Our problem.

- We are interested in the case that $a^{(0)}(x)$ has rank d in $\mathcal{E} \subset G$ and rank $d-1$ in $[G]\backslash\mathcal{E} = \cup_{k=1}^{r}[U_k]$.
- Global first integral $H(x) : H(x) = 0$ on $[\mathcal{E}]$ and $H(x) = H_k(x)$ on $[U_k]$.

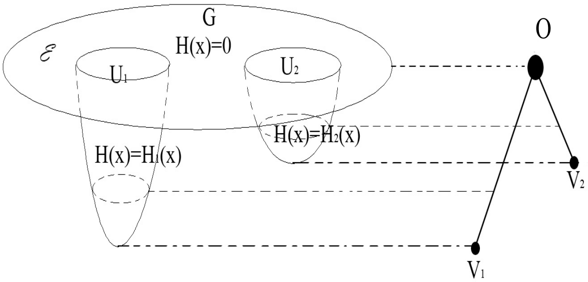

FIG.: Our problem.

Assumption on the degeneracy.

- On $\cup_{k=1}^{r}[U_k] : a^{(0)}(x)$ has rank $d-1$. Existence of first integrals H_k, $k=1,...,r$. Non-dengeneracy on $C_k(h)=\{x\in U_k : H_k(x)=h\}$, etc. similar assumptions as in the averaging principle.
- Denote $\gamma_k=\partial U_k$ and $\gamma=\cup_{k=1}^{r}\gamma_k$.
- $\text{const}_1\cdot\text{dist}^2(x,\gamma)\leq \mathbf{e}_d(x)\cdot(a^{(0)}(x)\mathbf{e}_d(x))\leq \text{const}_2\cdot\text{dist}^2(x,\gamma)$. ("quadratic degeneracy")
- Coefficients of $a^{(0)}(x)$ are in $\mathbf{C}^{(1)}$ on γ. We assume the decomposition $\sigma^{(0)}(x)(\sigma^{(0)}(x))^*=a^{(0)}(x)$.

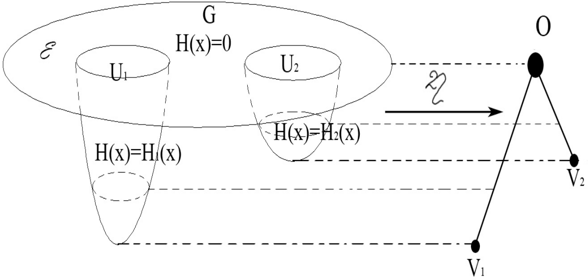

FIG.: Identification mapping $\mathfrak{Y}$.

Weak convergence of the process $Y_t^\varepsilon = \mathfrak{Y}(X_t^\varepsilon)$.

- Let $Y_t^\varepsilon = \mathfrak{Y}(X_t^\varepsilon)$.
- We can introduce the graph Γ with coordinate (k, H).
- Y_t^ε lives on Γ and is, in general, not Markov for fixed $\varepsilon > 0$.
- $Y_t^\varepsilon \to Y_t$ weakly as $\varepsilon \downarrow 0$ in $\mathbf{C}_{[0,T]}(\Gamma)$.
- Inside each I_k the process Y_t has a generator

$$\mathcal{L}_k f(k, H_k) = \frac{1}{2} M_k^{-1}(H_k) \frac{d}{dH_k} \left(M_k(H_k) \overline{a^{(1)}}(H_k) \frac{df}{dH_k} \right) .$$

- Here

$$\overline{a^{(1)}}(h) = M_k^{-1}(h) \int\limits_{C_k(h)} \frac{(a^{(1)}(x) \nabla H_k(x), \nabla H_k(x))}{|\nabla H_k(x)|_{\mathbb{R}^d}} d\sigma \ ,$$

and normalizing factor

$$M_k(h) = \int\limits_{C_k(h)} \frac{d\sigma}{|\nabla H_k(x)|_{\mathbb{R}^d}} \ .$$

Weak convergence of the process $Y_t^\varepsilon = \mathfrak{Y}(X_t^\varepsilon)$.

- What is more important : Y_t is a Markov process on Γ with generator A and domain of definition $D(A)$.
- Inside each I_k A agrees with $\mathcal{L}_k$. (standard averaging principle)
- The domain of definition $D(A)$ of the operator A consists of those functions f that are twice continuously differentiable inside each I_k having the limit $\lim\limits_{H_k \to 0} \dfrac{\partial f}{\partial H_k}(k, H_k)$. These functions satisfy the gluing condition at the vertex O :

$$0 = \text{Volume}(\mathcal{E}) \cdot Af(O) + \frac{1}{2}\sum_{k=1}^{r} p_k \cdot \lim_{H_k \to 0} \frac{\partial f}{\partial H_k}(k, H_k) \ .$$

  (gluing condition of "delay" type)
- Here Volume($\mathcal{E}$) is the d-dimensional volume of the domain $\mathcal{E}$ and

$$p_k = \int_{\gamma_k} \frac{(a^{(1)}(x)\nabla H_k(x), \nabla H_k(x))}{|\nabla H_k(x)|_{\mathbb{R}^d}} d\sigma \ .$$

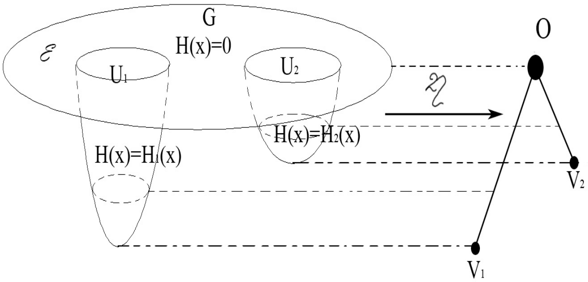

FIG.: gluing condition of "delay" type.

The answer to our problem.

- **Theorem.** (Freidlin-H, 2012, preprint) *Consider the Neumann problem*

$$\frac{1}{\varepsilon}L_\varepsilon u^\varepsilon(x) = \left(\frac{1}{\varepsilon}L_0 + L_1\right) u^\varepsilon(x) = f(x) \text{ for } x \in G \ ,$$

$$\left.\frac{\partial u^\varepsilon(x)}{\partial \gamma^\varepsilon(x)}\right|_{x \in \partial G} = 0$$

with a Hölder continuous function $f(x)$ *satisfying* $\int_G f(x)dx = 0$. *Let* $u^\varepsilon(x_O) = 0$ *for some* $x_O \in G$. *Then we have*

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(x) = v(\mathfrak{Y}(x))$$

where $v(y)$ *is a continuous function on* Γ *such that*

$$\mathcal{L}_k v(y) = -\bar{f}(y) \text{ for } y \in (I_k) \ , \ k = 1, ..., r \ .$$

The answer to our problem.

- **Theorem.** (continued) (Freidlin-H, 2012, preprint) *Here*

$$\bar{f}(y) = \frac{1}{\text{Volume}(\mathcal{E})} \int_{\mathcal{E}} f(x) dx$$

when $y = O$ *and*

$$\bar{f}(y) = \frac{1}{M_k(H_k)} \int_{C_k(H_k)} f(x) \frac{d\sigma}{|\nabla H_k(x)|_{\mathbb{R}^d}}$$

when $y = (k, H_k)$. *The function* $v(y)$ *satisfies the gluing condition*

$$0 = Volume(\mathcal{E}) \cdot Av(O) + \frac{1}{2} \sum_{k=1}^{r} p_k \cdot \lim_{H_k \to 0} \frac{\partial v}{\partial H_k}(k, H_k) \ .$$

and $v(\mathfrak{Y}(x_O)) = 0$. *Such a function* $v(y)$ *is unique.*

Why $Y_t^\varepsilon \to Y_t$ weakly in $\mathbf{C}_{[0,T]}(\Gamma)$? Heuristics.

- This proof of the convergence follows from the argument of Dolgopyat & Koralov, to appear in *Journal of AMS*. The basic idea can also be found in the classical monograph of Freidlin & Wentzell, Chapter 8.
- Situation is similar but actually simpler than the case of averaging for Hamiltonian flows on $\mathbb{T}^2$.
- For motion of X_t^ε inside each of the $[U_k]$'s, the result is just a consequence of averaging principle, as we just did before in our example.
- For the motion of X_t^ε in $\mathcal{E}$, the reason is that we have glued all points in $\mathcal{E}$ into one point O.

Why $Y_t^\varepsilon \to Y_t$ weakly in $\mathbf{C}_{[0,T]}(\Gamma)$? Heuristics.

- For fixed $\varepsilon > 0$ Lebesgue measure is invariant for the process X_t^ε in $[G]$.
- When we do the projection $\mathfrak{Y} : [G] \to \Gamma$, the limiting process Y_t, as a result, is expected to have an invariant measure μ on Γ, which is induced by Lebesgue measure of X_t^ε on $[G]$.
- In particular, $\mu(\{O\}) = \text{Volume}(\mathcal{E})$.
- We have

$$\int_\Gamma Au(k,h)d\mu = 0 \ .$$

- The above relation, when expanded, gives the gluing condition :

$$0 = \text{Volume}(\mathcal{E}) \cdot Av(O) + \frac{1}{2}\sum_{k=1}^{r} p_k \cdot \lim_{H_k \to 0} \frac{\partial v}{\partial H_k}(k, H_k) \ .$$

The proof : a short review of technicalities.

- The proof of convergence $\mathfrak{Y}(X_t^\varepsilon) \to Y_t$ in $\mathbf{C}_{[0,T]}(\Gamma)$ makes use of martingale problem techniques. I will omit the technicalities in this point here.
- We need to show that the process X_t^ε, as ε is small, quickly tend to its invariant measure.
- This is not immediately obvious since as ε is small the process X_t^ε is close to a degenerate one near γ. In other words, we expect the process X_t^ε to move slower and slower when it approaches γ.
- The key underlying reason that this is true is because of our assumption of "quadratic degeneracy" :

$$\text{const}_1 \cdot \text{dist}^2(x,\gamma) \leq \mathbf{e}_d(x) \cdot (a^{(0)}(x)\mathbf{e}_d(x)) \leq \text{const}_2 \cdot \text{dist}^2(x,\gamma) \ .$$

The proof : a short review of technicalities.

- We need the control of certain stopping times :

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in [\mathcal{E}]} \mathbf{E}_x \sigma = 0 \ ,$$

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \gamma} \mathbf{E}_x \tau = 0 \ ,$$

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \underline{\gamma}} \mathbf{E}_x \sigma = 0 \ .$$

- These are estimated via the construction of barrier functions.

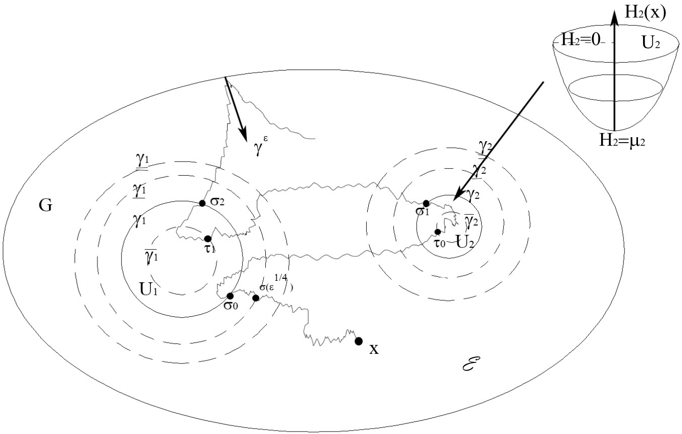

FIG.: Details in the proof.

The proof : a short review of technicalities.

- Geometric construction : extension of the first integral H_k to a neighborhood outside γ.
- "Global barrier" : first introduce a Riemannian coordinate $(\varphi_1^k, ..., \varphi_{d-1}^k, H_k)$ near γ.
- "barrier" function $u_k = u_k(H_k)$.
- Making use of some basic facts in Riemannian geometry it is possible to show that

$$\left(\frac{1}{\varepsilon}L_0 + L_1\right) u_k(x)$$
$$= \frac{1}{A(x)}\left[\frac{\partial}{\partial H_k}\left(\left(\frac{K_1(x)}{\varepsilon} + K_2(x)\right)\frac{du_k}{dH_k}(H_k)\right)\right.$$
$$\left. + K_3(x)\frac{du_k}{dH_k}(H_k)\right] .$$

The proof : a short review of technicalities.

- Making use of these barriers accompanied by some dangerous estimates our process X_t^{ε} is able to freely travel in and out of γ.
- This is the main technical part of the work yet I will omit it in this talk.

The End.

- Thank you for your attention !