Small mass asymptotic for the motion with variable and vanishing friction.

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Langevin equation.

- The Langevin equation for a particle in a fluid is the Newton's equation of a form

$$\mu\ddot{\mathbf{q}}_t^\mu = \mathbf{b}(\mathbf{q}_t^\mu) - \lambda\dot{\mathbf{q}}_t^\mu + \sigma(\mathbf{q}_t^\mu)\dot{\mathbf{W}}_t \ , \ \mathbf{q}_0^\mu = \mathbf{q} \in \mathbb{R}^d \ , \dot{\mathbf{q}}_0^\mu = \mathbf{p} \in \mathbb{R}^d \ .$$

- $\mathbf{q}_t^\mu$ the position of the particle ; μ is the small mass ; $\lambda > 0$ is a constant friction ; $\mathbf{b}(\bullet)$ is the drift ; $\sigma(\bullet)$ is a diffusion matrix ; $\mathbf{W}_t$ is a multidimensional Wiener process.

Small mass asymptotic (Smoluchowski-Kramers approximation).

- Langevin equation

$$\mu\ddot{\mathbf{q}}_t^\mu = \mathbf{b}(\mathbf{q}_t^\mu) - \lambda\dot{\mathbf{q}}_t^\mu + \sigma(\mathbf{q}_t^\mu)\dot{\mathbf{W}}_t \ , \ \mathbf{q}_0^\mu = \mathbf{q} \in \mathbb{R}^d \ , \dot{\mathbf{q}}_0^\mu = \mathbf{p} \in \mathbb{R}^d \ .$$

- Let $\mu = 0$ we get

$$\dot{\mathbf{q}}_t = \frac{1}{\lambda}\mathbf{b}(\mathbf{q}_t) + \frac{1}{\lambda}\sigma(\mathbf{q}_t)\dot{\mathbf{W}}_t \ , \ \mathbf{q}_0 = \mathbf{q}_0^\mu = \mathbf{q} \in \mathbb{R}^d \ .$$

- For any $\kappa > 0$ we have

$$\lim_{\mu\downarrow 0} \mathbf{P}\left(\max_{0\leq t\leq T} |\mathbf{q}_t^\mu - \mathbf{q}_t|_{\mathbb{R}^d} > \kappa\right) = 0 \ .$$

- The above approximation is called *Smoluchowski-Kramers approximation*.

Variable friction $\lambda = \lambda(\mathbf{q})$?

- Variable friction : $\lambda = \lambda(\mathbf{q})$ is a function of the position.
- First suppose that $0 < \lambda_0 \leq \lambda(\mathbf{q}) \leq \Lambda < \infty$.
- Langevin equation with variable friction

$$\mu \ddot{\mathbf{q}}_t^\mu = \mathbf{b}(\mathbf{q}_t^\mu) - \lambda(\mathbf{q}_t)\dot{\mathbf{q}}_t^\mu + \sigma(\mathbf{q}_t^\mu)\dot{\mathbf{W}}_t \ , \ \mathbf{q}_0^\mu = \mathbf{q} \in \mathbb{R}^d \ , \dot{\mathbf{q}}_0^\mu = \mathbf{p} \in \mathbb{R}^d \ .$$

- Let $\mu = 0$ we get

$$\dot{\mathbf{q}}_t = \frac{\mathbf{b}(\mathbf{q}_t)}{\lambda(\mathbf{q}_t)} + \frac{\sigma(\mathbf{q}_t)}{\lambda(\mathbf{q}_t)}\dot{\mathbf{W}}_t \ .$$

- Is it again true that for any $\kappa > 0$ we have

$$\lim_{\mu \downarrow 0} \mathbf{P}\left(\max_{0 \leq t \leq T} |\mathbf{q}_t^\mu - \mathbf{q}_t|_{\mathbb{R}^d} > \kappa\right) = 0 \ ?$$

Variable friction $\lambda = \lambda(\mathbf{q})$?

- The answer is NO in general.
- One has to use a further regularization.
- *Remark.* If friction is variable, the limit still can exist without regularization. It is a bit different from the constant friction case, but it coincides with the regularized result. See [Hottovy et al., *J Stat Phys* (2012) **146**, 762–773].

Approximation of the Wiener process.

- Approximation of the Wiener process

$$\mathbf{W}_t^\delta = \frac{1}{\delta}\int_0^\infty \mathbf{W}_s \rho\left(\frac{s-t}{\delta}\right) ds = \frac{1}{\delta}\int_0^\delta \mathbf{W}_{s+t}\rho\left(\frac{s}{\delta}\right) ds \ ,$$

  where $\rho(\bullet)$ is a smooth C^∞ function whose support is contained in the interval $[0,1]$ such that

$$\int_0^1 \rho(s)ds = 1 \ .$$

- $\dot{\mathbf{W}}_t^\delta$ is a small δ-correlated noise.

Regularized Smoluchowski-Kramers approximation.

- Langevin equation with an approximated Wiener process

$$\mu\ddot{\mathbf{q}}_t^{\mu,\delta}=\mathbf{b}(\mathbf{q}_t^{\mu,\delta})-\lambda(\mathbf{q}_t^{\mu,\delta})\dot{\mathbf{q}}_t^{\mu,\delta}+\dot{\mathbf{W}}_t^{\delta}\ ,\ \mathbf{q}_0^{\mu,\delta}=\mathbf{q}\in\mathbb{R}^d\ ,\ \dot{\mathbf{q}}_0^{\mu,\delta}=\mathbf{p}\in\mathbb{R}^d.$$

- First let $\mu\downarrow 0$ we have

$$\dot{\widetilde{\mathbf{q}}}_t^{\delta}=\frac{\mathbf{b}(\widetilde{\mathbf{q}}_t^{\delta})}{\lambda(\widetilde{\mathbf{q}}_t^{\delta})}+\frac{1}{\lambda(\widetilde{\mathbf{q}}_t^{\delta})}\dot{\mathbf{W}}_t^{\delta}\ ,\ \widetilde{\mathbf{q}}_0^{\delta}=\mathbf{q}\in\mathbb{R}^d\ .$$

- Regularized Smoluchowski-Kramers approximation

$$\lim_{\mu\downarrow 0}\mathbf{P}\left(\max_{0\leq t\leq T}|\mathbf{q}_t^{\mu,\delta}-\widetilde{\mathbf{q}}_t^{\delta}|_{\mathbb{R}^d}>\kappa\right)=0\ .$$

Regularized Smoluchowski-Kramers approximation : Second limit as $\delta \downarrow 0$.

- First let $\mu \downarrow 0$ we have

$$\dot{\widetilde{\mathbf{q}}}_t^\delta = \frac{\mathbf{b}(\widetilde{\mathbf{q}}_t^\delta)}{\lambda(\widetilde{\mathbf{q}}_t^\delta)} + \frac{1}{\lambda(\widetilde{\mathbf{q}}_t^\delta)}\dot{\mathbf{W}}_t^\delta \ , \ \widetilde{\mathbf{q}}_0^\delta = \mathbf{q} \in \mathbb{R}^d \ .$$

- Second limit : as $\delta \downarrow 0$ we let

$$\dot{\widehat{\mathbf{q}}}_t = \frac{\mathbf{b}(\widehat{\mathbf{q}}_t)}{\lambda(\widehat{\mathbf{q}}_t)} + \frac{1}{\lambda(\widehat{\mathbf{q}}_t)} \circ \dot{\mathbf{W}}_t \ , \ \widehat{\mathbf{q}}_0 = \mathbf{q} \in \mathbb{R}^d \ .$$

- Limit as $\delta \downarrow 0$:

$$\lim_{\delta \to 0} \mathbf{E} \max_{t \in [0,T]} |\widetilde{\mathbf{q}}_t^\delta - \widehat{\mathbf{q}}_t|_{\mathbb{R}^d} = 0 \ .$$

- Because of the Wong-Zakai theorem we have Stratonovich integral instead of Itô integral.

Application 1. Fast oscillating friction and drift : homogenization problem.

- Fast oscillating friction and drift

$$\mu \ddot{\mathbf{q}}_t^{\mu,\delta,\varepsilon} = \mathbf{b}\left(\frac{\mathbf{q}_t^{\mu,\delta,\varepsilon}}{\varepsilon}\right) - \lambda\left(\frac{\mathbf{q}_t^{\mu,\delta,\varepsilon}}{\varepsilon}\right)\dot{\mathbf{q}}_t^{\mu,\delta,\varepsilon} + \dot{\mathbf{W}}_t^{\delta} \ ,$$

$$\mathbf{q}_0^{\mu,\delta,\varepsilon} = \mathbf{q} \in \mathbb{R}^d \ , \ \dot{\mathbf{q}}_0^{\mu,\delta,\varepsilon} = \mathbf{p} \in \mathbb{R}^d \ .$$

- Using homogenization theory to characterize the limit as first $\mu \downarrow 0$ and then $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$.

Application 2. Friction with jump : gluing condition.

- Friction with a jump, 1-d case :

$$\mu q_t^{\mu,\delta,\varepsilon} = b(q_t^{\mu,\delta,\varepsilon}) - \lambda_\varepsilon(q_t^{\mu,\delta,\varepsilon})\dot{q}_t^{\mu,\delta,\varepsilon} + \dot{W}_t^\delta \ ,$$

$$q_0^{\mu,\delta,\varepsilon} = q \in \mathbb{R}^1 \ , \ \dot{q}_0^{\mu,\delta,\varepsilon} = p \in \mathbb{R}^1 \ .$$

- $\lim\limits_{\varepsilon \downarrow 0} \lambda_\varepsilon(q) = \lambda_1 > 0$ for $q < 0$.
- $\lim\limits_{\varepsilon \downarrow 0} \lambda_\varepsilon(q) = \lambda_2 > 0$ for $q \geq 0$.
- $\lambda_1 \neq \lambda_2$.
- Gluing condition $\dfrac{1}{\lambda_1} f'_-(O) = \dfrac{1}{\lambda_2} f'_+(O)$. (Such boundary conditions are typical for diffusion processes on graphs !)

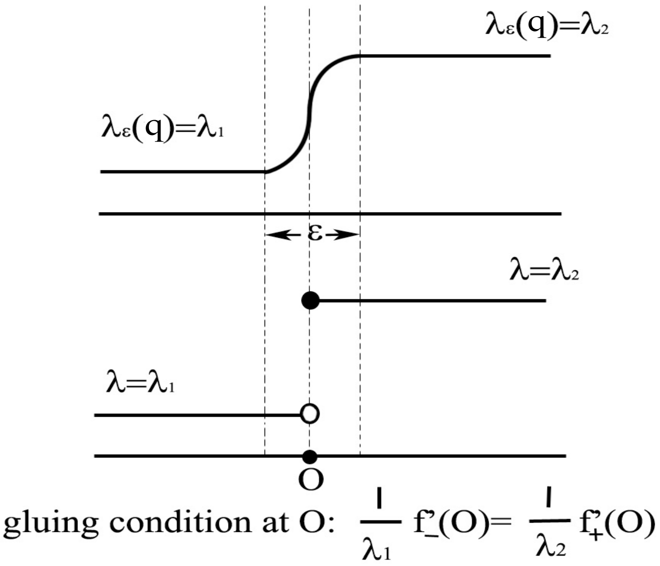


FIG.:

Vanishing friction.

- We assumed previously that $0 < \lambda_0 \leq \lambda(\mathbf{q}) \leq \Lambda < \infty$.
- What if in some regions we have $\lambda(\mathbf{q}) = 0$?

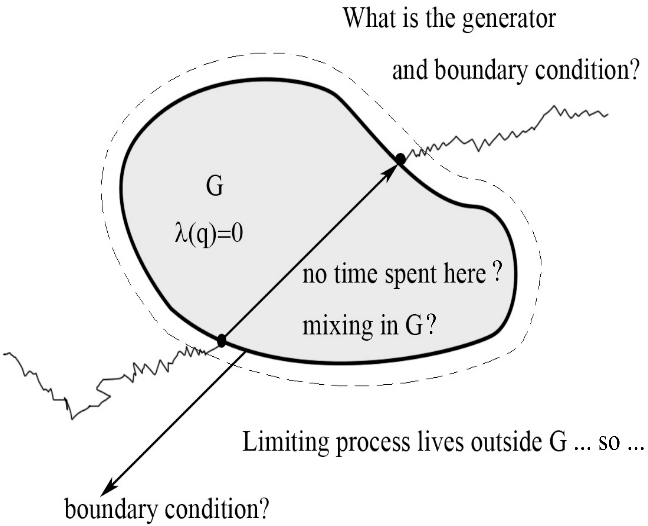

What is the generator
and boundary condition?
G
λ(q)=0
no time spent here ?
mixing in G ?
Limiting process lives outside G ... so ...
boundary condition?

FIG.:

Vanishing friction : Regularization.

- We apply a further regularization by adding a small ε in the friction :

$$\mu \ddot{\mathbf{q}}_t^{\mu,\delta,\varepsilon} = \mathbf{b}(\mathbf{q}_t^{\mu,\delta,\varepsilon}) - (\lambda(\mathbf{q}_t^{\mu,\delta,\varepsilon}) + \varepsilon)\dot{\mathbf{q}}_t^{\mu,\delta,\varepsilon} + \dot{\mathbf{W}}_t^{\delta} \ ,$$

$$\mathbf{q}_0^{\mu,\delta,\varepsilon} = \mathbf{q} \ , \ \dot{\mathbf{q}}_0^{\mu,\delta,\varepsilon} = \mathbf{p} \ .$$

- We assume that that $\lambda(\mathbf{q}) = 0$ for $\mathbf{q} \in [G] \subset \mathbb{R}^n$ and $\lambda(\mathbf{q}) > 0$ for $\mathbf{q} \in \mathbb{R}^n \backslash [G]$. Here G is a domain in $\mathbb{R}^n$ and $[G]$ its closure in the standard Euclidean metric.
- For simplicity of presentation we assume that $\sigma(\bullet) =$ identity.
- We study the limit as first $\mu \downarrow 0$ then $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$.

Vanishing friction : first $\mu \downarrow 0$ and then $\delta \downarrow 0$.

- As first $\mu \downarrow 0$ and then $\delta \downarrow 0$ previous results apply : the limiting process looks like

$$\dot{\mathbf{q}}_t^\varepsilon = \frac{1}{\lambda(\mathbf{q}_t^\varepsilon)+\varepsilon}\mathbf{b}(\mathbf{q}_t^\varepsilon) + \frac{1}{\lambda(\mathbf{q}_t^\varepsilon)+\varepsilon} \circ \dot{\mathbf{W}}_t \ , \ \mathbf{q}_0^\varepsilon = \mathbf{q} \ , \ \varepsilon > 0 \ ,$$

and we study the limit of $\mathbf{q}_t^\varepsilon$ as $\varepsilon \downarrow 0$.

- In Itô's form it is

$$\dot{\mathbf{q}}_t^\varepsilon = \frac{1}{\lambda(\mathbf{q}_t^\varepsilon)+\varepsilon}\mathbf{b}(\mathbf{q}_t^\varepsilon) - \frac{\nabla\lambda(\mathbf{q}_t^\varepsilon)}{2(\lambda(\mathbf{q}_t^\varepsilon)+\varepsilon)^3} + \frac{1}{\lambda(\mathbf{q}_t^\varepsilon)+\varepsilon}\dot{\mathbf{W}}_t \ , \ \mathbf{q}_0^\varepsilon = \mathbf{q} \ .$$

Vanishing friction : general theme.

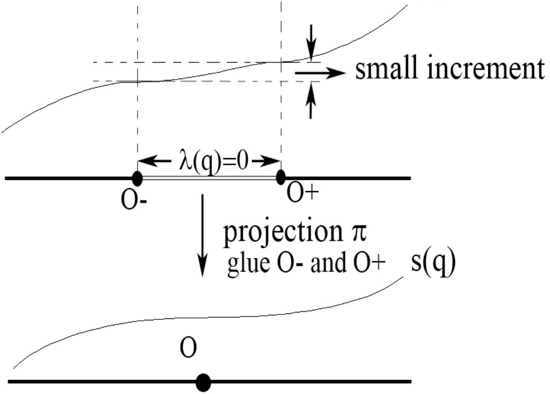
- Limiting process lives outside the domain G.
- Glue all points of $[G]$ and introduce a projection π.
- The projected space (image of π) is appropriate for a continuous version of the limiting process to live in.
- We have to show Markov property of the limiting process.
- We have to identify the generator of the limiting process (in Hille-Yosida sense).
- We have to specify the boundary condition (=domain of definition of the generator) at the image of $\pi([G])$.

Vanishing friction : 1-d case.

- $$\dot{q}_t^{\varepsilon} = \frac{b(q_t^{\varepsilon})}{\lambda(q_t^{\varepsilon}) + \varepsilon} - \frac{\lambda'(q_t^{\varepsilon})}{2(\lambda(q_t^{\varepsilon}) + \varepsilon)^3} + \frac{1}{\lambda(q_t^{\varepsilon}) + \varepsilon} \dot{W}_t \ , \ q_0^{\varepsilon} = q_0 \in \mathbb{R} \ .$$
- Feller's $D_{m^{\varepsilon}} D_{s^{\varepsilon}}$ process. m^{ε}-speed measure ; s^{ε}-scale function.
- $\lambda(q) = 0$ for $q \in [-1, 1]$. Outside $[-1, 1]$ we have $m^{\varepsilon}(\bullet) \to m(\bullet)$ and $s^{\varepsilon}(\bullet) \to s(\bullet)$ as $\varepsilon \downarrow 0$.
- Gluing condition :

$$D_m^- f(O) = D_m^+ f(O) \ .$$

original process generator $= D_{m^\varepsilon} D_{s^\varepsilon}$



limiting process is Markov

generator (in Hille-Yosida sense) is $D_m D_s$

gluing condition at O: $D_m^- f(O) = D_m^+ f(O)$

Vanishing friction : 2-d model problem.

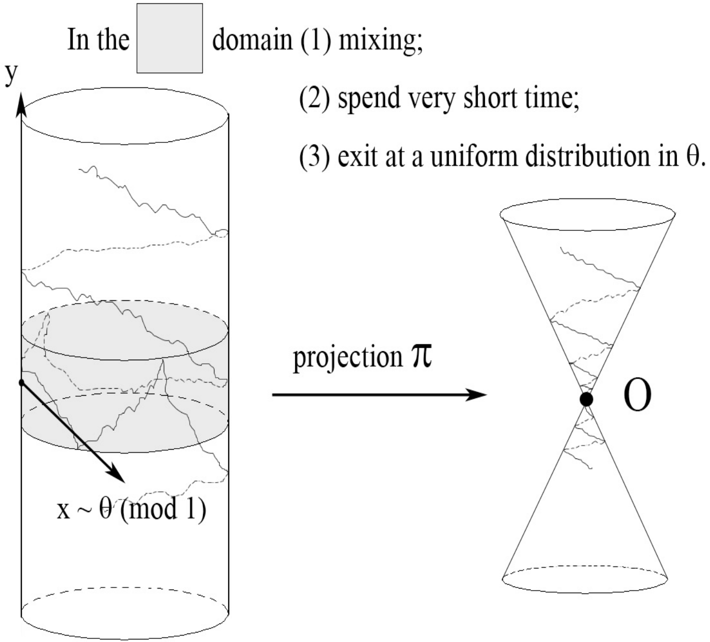
- Assume $\lambda(x,y) = \lambda(y)$ and $\mathbf{b}(\bullet) = \mathbf{0}$.
- $\lambda(y) = 0$ for $y \in [-1,1]$.
- The equation

$$\dot{\mathbf{q}}_t^\varepsilon = -\frac{\nabla\lambda(\mathbf{q}_t^\varepsilon)}{2(\lambda(\mathbf{q}_t^\varepsilon)+\varepsilon)^3} + \frac{1}{\lambda(\mathbf{q}_t^\varepsilon)+\varepsilon}\dot{\mathbf{W}}_t \ , \ \mathbf{q}_0^\varepsilon = \mathbf{q}_0 \in \mathbb{R}^2 \ , \ \varepsilon > 0 \ ,$$

  becomes

$$\begin{cases} \dot{x}_t^\varepsilon = \dfrac{1}{\lambda(y_t^\varepsilon)+\varepsilon}\dot{W}_t^1 \ , \ x_0^\varepsilon = x_0 \in \mathbb{R} \ , \\ \dot{y}_t^\varepsilon = -\dfrac{\lambda'(y_t^\varepsilon)}{2(\lambda(y_t^\varepsilon)+\varepsilon)^3} + \dfrac{1}{\lambda(y_t^\varepsilon)+\varepsilon}\dot{W}_t^2 \ , \ y_0^\varepsilon = y_0 \in \mathbb{R} \ . \end{cases}$$

- Let $x \sim \theta$ (mod1).



In the domain (1) mixing;
(2) spend very short time;
(3) exit at a uniform distribution in θ.
y
x ~ θ (mod 1)
projection π
O

Vanishing friction : 2-d model problem.

- Limiting process lives in a double cone.
- Existence in the sense of Hille-Yosida.
- Once it hits O it immediately leaves O.
- Gluing condition is a generalized Feller's boundary condition :

$$\int_0^{2\pi} \lim_{\theta' \to \theta, \widetilde{y} \to 0-} D_m f(\theta', \widetilde{y}) d\theta = \int_0^{2\pi} \lim_{\theta' \to \theta, \widetilde{y} \to 0+} D_m f(\theta', \widetilde{y}) d\theta \ .$$

(It is similar to that of a Walsh BM !)

Vanishing friction : general case.

- We still do not know.
- First difficulty is to establish *existence* of the limiting process with a specified boundary condition.

The end

Thank you for your attention !