Hypoelliptic multiscale Langevin diffusions and Slow–fast stochastic reaction–diffusion equations.

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Multiscale Langevin equation.

- $$\tau \ddot{q}_t^\varepsilon = \frac{\varepsilon}{\delta} b\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right) + c\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right) - \lambda\left(q_t^\varepsilon\right)\dot{q}_t^\varepsilon + \sqrt{\varepsilon}\sigma\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right)\dot{W}_t . \tag{1}$$
- Parameters $0 < \varepsilon, \delta \ll 1$ and $\delta = \delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.
- Parameter ε represents the strength of the noise, whereas δ is the parameter that separates the scales.
- We want to send the small parameters ε and δ to 0 in a way that
$$\frac{\varepsilon}{\delta} \to \infty$$
(homogenization case)

Systems of slow–fast stochastic reaction–diffusion equations.

- $$\begin{cases} \dfrac{\partial X^{\varepsilon,\delta}}{\partial t}(t,x) = \mathcal{A}_1 X^{\varepsilon,\delta}(t,x) + b_1(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x)) \\ \qquad\qquad + \sqrt{\varepsilon}\sigma_1(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x)) \dfrac{\partial W^{Q_1}}{\partial t}(t,x) \ , \\ \dfrac{\partial Y^{\varepsilon,\delta}}{\partial t}(t,x) = \dfrac{1}{\delta^2}\left[\mathcal{A}_2 Y^{\varepsilon,\delta}(t,x) + b_2(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x))\right] \\ \qquad\qquad + \dfrac{1}{\delta}\sigma_2(x, X^{\varepsilon,\delta}(t,x), Y^{\varepsilon,\delta}(t,x)) \dfrac{\partial W^{Q_2}}{\partial t}(t,x) \ , \\ X^{\varepsilon,\delta}(0,x) = X_0(x) \ , \ \ Y^{\varepsilon,\delta}(0,x) = Y_0(x) \ , \ \ x \in D \ , \\ \mathcal{N}_1 X^{\varepsilon,\delta}(t,x) = \mathcal{N}_2 Y^{\varepsilon,\delta}(t,x) = 0 \ , \ \ t \geq 0 \ , \ \ x \in \partial D \ . \end{cases} \tag{2}$$

Systems of slow–fast stochastic reaction–diffusion equations.

- Here $\varepsilon > 0$ is a small parameter and $\delta = \delta(\varepsilon) > 0$ is such that $\delta \to 0$ as $\varepsilon \downarrow 0$.
- The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are two strictly elliptic operators and $b_i(x, X, Y)$, $i = 1, 2$ are the nonlinear terms.
- The noise processes W^{Q_1} and W^{Q_2} are two cylindrical Wiener processes with covariance matrices Q_1 and Q_2, and $\sigma_i(x, X, Y)$, $i = 1, 2$ give the corresponding multiplicative noises.
- The initial values X_0 and Y_0 are assumed to be in $L^2(D)$.
- The boundary conditions are given by operators $\mathcal{N}_i$, $i = 1, 2$ which may correspond to either Dirichlet or Neumann conditions.

Background and Motivation.

- Chemical Physics and Biology.
- The dynamical behavior of proteins such as their folding and binding kinetics.
- The potential surface of a protein might have a hierarchical structure with potential minima within potential minima. The presence of multiple energy scales associated with the building blocks of proteins implies that the underlying energy landscapes of certain biomolecules can be rugged (i.e., consist of many minima separated by barriers of varying heights).
- As a consequence, the roughness of the energy landscapes that describe proteins has numerous effects on their kinetic properties as well as on their behavior at equilibrium.

Background and Motivation.

- Fast–slow stochastic partial differential equations of reaction–diffusion type have many applications in chemistry and biology.
- In the classical chemical kinetics, the evolution of concentrations of various components in a reaction is described by ordinary differential equations. Such a description turns out to be unsatisfactory in a number of applications, especially in biology.

Background and Motivation.

- There are several ways to construct a more adequate mathematical model. If the reaction is fast enough, one should take into account that the concentration is not constant in space in the volume where the reaction takes place. Then the change of concentration due to the spatial transport, as a role the diffusion, should be taken into consideration and the system of ordinary differential equations should be replaced by a system of partial differential equations of reaction–diffusion type.

Background and Motivation.

- In some cases, one should also take into account random change in time of the rates of reaction. Then the ordinary differential equation is replaced by a stochastic differential equation. If the rates change randomly not just in time but also in space, the evolution of concentrations can be described by a system of stochastic partial differential equations.
- On the other hand, the rates of chemical reactions in the system and the diffusion coefficients may have, and as a rule have, different orders. Some of them are much smaller than others and this leads to mathematical models based on slow–fast stochastic reaction–diffusion equations.

Large Deviations : the position process in the hypo–elliptic Langevin diffusion.

- We want to have large deviation results for the position process q^{ε} in the hypo–elliptic Langevin diffusion in terms of Laplace principle.
- Let $\{q^{\varepsilon}, \varepsilon > 0\}$ be a family of random variables taking values on a Polish space $\mathcal{S}$ and let I be a rate function on $\mathcal{S}$.
- For every bounded and continuous function $h : \mathcal{S} \to \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbf{E} \left[\exp \left(-\frac{h(q^{\varepsilon})}{\varepsilon} \right) \right] = \inf_{x \in \mathcal{S}} \left[I(x) + h(x) \right].$$

Large Deviations : slow motion in the system of slow–fast stochastic reaction–diffusion equations.

- We want to have large deviation results for the slow motion $X^{\varepsilon,\delta}$ in terms of Laplace principle.
- For every bounded and continuous function $h : C([0, T]; H) \to \mathbb{R}$ we have

$$\lim_{\varepsilon\downarrow 0} -\varepsilon \ln \mathbf{E}_{X_0,Y_0}\left[\exp\left(-\frac{1}{\varepsilon}h(X^{\varepsilon,\delta})\right)\right] = \inf_{\phi\in C([0,T];H)}[S_{X_0}(\phi)+h(\phi)] .$$

  (Laplace principle.)
- Calculate the action functional (rate function) $S_{X_0}(\phi)$.

Weak convergence framework of large deviations.

- One of the most effective methods in analyzing large deviation effects is the weak convergence method.
- Roughly speaking, by a variational representation of exponential functionals of Wiener processes, one can represent the exponential functional of the process that appears in the Laplace principle (which is equivalent to large deviations principle) as a variational infimum over a family of *controlled* processes.
- This has been applied by us in both finite and infinite dimensions.

Weak convergence : the position process in the hypo–elliptic Langevin diffusion.

- Boué–Dupuis identity [4] to hypo–elliptic Langevin diffusion :

$$-\varepsilon \ln \mathbf{E}_{q_0}\left[\exp\left(-\frac{h(q^\varepsilon_\bullet)}{\varepsilon}\right)\right] = \inf_{u\in\mathcal{A}} \mathbf{E}_{q_0}\left[\frac{1}{2}\int_0^T |u_s|^2 ds + h(\overline{q}^\varepsilon_\bullet)\right] .$$

- The control set

$$\mathcal{A} = \Big\{u = \{u_s \in \mathbb{R}^d : 0 \le s \le T\} \text{ progressively } \mathcal{F}_s\text{–measurable and } \mathbf{E}\int_0^T |u_s|^2 ds < \infty\Big\} .$$

- $\overline{q}^\varepsilon_\bullet$: controlled hypoelliptic Langevin diffusion.

4. Boué, M., Dupuis, P., A variational representation for certain functionals of Brownian motion, *Annals of Probability*, **26**, 4, (1998), pp. 1641–1659.

Weak convergence : the position process in the hypo–elliptic Langevin diffusion.

- We found a parametrization : $\tau = \tau(\varepsilon, \delta) = m\dfrac{\delta^2}{\varepsilon}$, $m > 0$ and we have

$$m\frac{\delta^2}{\varepsilon}\ddot{q}_t^\varepsilon = \frac{\varepsilon}{\delta}b\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right) + c\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right) - \lambda\left(q_t^\varepsilon\right)\dot{q}_t^\varepsilon + \sqrt{\varepsilon}\sigma\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right)\dot{W}_t \, . \tag{3}$$

- It is only in this parametrization that we can derive the large deviation principle for $\{q^\varepsilon, \varepsilon > 0\}$.

Weak convergence : the position process in the hypo–elliptic Langevin diffusion.

- $$\begin{cases} \dot{\overline{q}}_t^\varepsilon &= \dfrac{1}{\sqrt{m}}\dfrac{\varepsilon}{\delta}\overline{p}_t^\varepsilon \ , \\ \dot{\overline{p}}_t^\varepsilon &= \dfrac{1}{\sqrt{m}}\dfrac{1}{\delta}\left[\dfrac{\varepsilon}{\delta}b\left(\overline{q}_t^\varepsilon, \dfrac{\overline{q}_t^\varepsilon}{\delta}\right) + c\left(\overline{q}_t^\varepsilon, \dfrac{\overline{q}_t^\varepsilon}{\delta}\right)\right] - \dfrac{\lambda(q_t^\varepsilon)}{m}\dfrac{\varepsilon}{\delta^2}\overline{p}_t^\varepsilon \\ & \qquad + \dfrac{1}{\delta}\dfrac{\sigma\left(q_t^\varepsilon, \dfrac{q_t^\varepsilon}{\delta}\right)}{\sqrt{m}}u_t + \dfrac{\sqrt{\varepsilon}}{\delta}\dfrac{\sigma\left(q_t^\varepsilon, \dfrac{q_t^\varepsilon}{\delta}\right)}{\sqrt{m}}\dot{W}_t \ . \end{cases} \tag{4}$$
- Initial conditions $\overline{q}_0^\varepsilon = q_o \in \mathbb{R}^d$, $\overline{p}_0^\varepsilon = p_o \in \mathbb{R}^d$.

Weak convergence : slow motion in the system of slow–fast stochastic reaction–diffusion equations.

- Boué–Dupuis identity in infinite dimensions [5] : For any $h : C([0,T]; L^2(D)) \to \mathbb{R}$, that

$$-\varepsilon \ln \mathbf{E}\left[\exp\left(-\frac{1}{\varepsilon}h(X^{\varepsilon,\delta})\right)\right] = \inf_{u\in L^2([0,T];U)} \mathbf{E}\left[\frac{1}{2}\int_0^T |u(s)|_U^2 ds + h(X^{\varepsilon,\delta,u})\right] .$$

5. Budhiraja, A., Dupuis, P., A variational representation for positive functionals of infinite dimensional Brownian motions, *Probab. Math. Statist.* **20**, 2000, pp. 39–61.

Weak convergence : slow motion in the system of slow–fast stochastic reaction–diffusion equations.

- Here U is the control space, and the infimum is over all controls $u \in U$ with finite $L^2([0,T]; |\bullet|)$–norm.
- The controlled slow motion $X^{\varepsilon,\delta,u}$ that appears on the right hand side of the Laplace principle comes from a controlled slow–fast system $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$ of reaction–diffusion equations corresponding to (2).

Weak convergence : slow motion in the system of slow–fast stochastic reaction–diffusion equations.

- $$\begin{cases} \dfrac{\partial X^{\varepsilon,\delta,u}}{\partial t}(t,x) = \mathcal{A}_1 X^{\varepsilon,\delta,u}(t,x) + b_1(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x)) \\ \qquad\qquad +\sigma_1(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))(Q_1 u(t))(x) \\ \qquad\qquad +\sqrt{\varepsilon}\sigma_1(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))\dfrac{\partial W^{Q_1}}{\partial t}(t,x) \ , \\ \dfrac{\partial Y^{\varepsilon,\delta,u}}{\partial t}(t,x) \\ \qquad\qquad = \dfrac{1}{\delta^2}\left[\mathcal{A}_2 Y^{\varepsilon,\delta,u}(t,x) + b_2(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))\right] \\ \qquad\qquad +\dfrac{1}{\delta\sqrt{\varepsilon}}\sigma_2(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))(Q_2 u(t))(x) \\ \qquad\qquad +\dfrac{1}{\delta}\sigma_2(x, X^{\varepsilon,\delta,u}(t,x), Y^{\varepsilon,\delta,u}(t,x))\dfrac{\partial W^{Q_2}}{\partial t}(t,x) \ , \\ X^{\varepsilon,\delta,u}(0,x) = X_0(x) \ , \ \ Y^{\varepsilon,\delta,u}(0,x) = Y_0(x) \ , \ \ x \in D \ , \\ \mathcal{N}_1 X^{\varepsilon,\delta,u}(t,x) = \mathcal{N}_2 Y^{\varepsilon,\delta,u}(t,x) = 0 \ , \ \ t \geq 0 \ , \ \ x \in \partial D \ . \end{cases} \tag{5}$$

Weak convergence : Feedback Control.

- Compare the two expressions

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbf{E}\left[\exp\left(-\frac{h(q^\varepsilon)}{\varepsilon}\right)\right] = \inf_{x \in \mathcal{S}} \left[I(x) + h(x)\right],$$

and

$$-\varepsilon \ln \mathbf{E}_{q_0}\left[\exp\left(-\frac{h(q_\bullet^\varepsilon)}{\varepsilon}\right)\right] = \inf_{u \in \mathcal{A}} \mathbf{E}_{q_0}\left[\frac{1}{2}\int_0^T |u_s|^2 ds + h(\overline{q}_\bullet^\varepsilon)\right].$$

- Goal : To take the limit as $\varepsilon \downarrow 0$ of the controlled process $\{\overline{q}^\varepsilon : \varepsilon > 0\}$ as $\varepsilon \downarrow 0$.
- Fast–slow dynamics, averaging, homogenization...

Weak convergence : Feedback Control.

- Need to analyze the limit as $\varepsilon \to 0$ (and thus $\delta \to 0$) of the controlled fast–slow system $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$ in (5).
- Recall that for any bounded continuous function $h : C([0,T]; L^2(D)) \to \mathbb{R}$, that

$$
\begin{aligned}
-\varepsilon \ln \mathbf{E}\left[\exp\left(-\frac{1}{\varepsilon}h(X^{\varepsilon,\delta})\right)\right] \\
= \inf_{u \in L^2([0,T];U)} \mathbf{E}\left[\frac{1}{2}\int_0^T |u(s)|_U^2 ds + h(X^{\varepsilon,\delta,u})\right].
\end{aligned}
$$

- u is in terms of a feedback control !

Large deviations for second order Langevin equation.

- **Condition 1.** *The functions* $b(q,r), c(q,r), \sigma(q,r)$ *are (i) periodic with period* 1 *in the second variable in each direction, and (ii)* $C^1(\mathbb{R}^d)$ *in* r *and* $C^2(\mathbb{R}^d)$ *in* q *with all partial derivatives continuous and globally bounded in* q *and* r*.*
- Control space $\mathcal{Z} = \mathbb{R}^d$; Fast variable space $\mathcal{Y} = \mathbb{R}^d \times \mathbb{T}^d$.
- Fast variable is actually $\left(\overline{p}_s^\varepsilon, \dfrac{\overline{q}_s^\varepsilon}{\delta}\right)$.

Large deviations for second order Langevin equation.

- Define an operator

$$\begin{aligned}\mathcal{L}_q^m \Phi(p,r) &= \frac{1}{\sqrt{m}}\left[p\cdot\nabla_r\Phi(p,r) + b(q,r)\cdot\nabla_p\Phi(p,r)\right] \\ &\quad + \frac{1}{m}\left[-\lambda(q)p\cdot\nabla_p\Phi(p,r) + \frac{1}{2}\alpha(q,r) : \nabla_p^2\Phi(p,r)\right]\end{aligned}$$

  where $\alpha(q,r) = \sigma(q,r)\sigma^T(q,r)$.
- For each fixed q, the operator $\mathcal{L}_q^m$ defines a hypoelliptic diffusion process on $(p,r) \in \mathcal{Y} = \mathbb{R}^d \times \mathbb{T}^d$.
- Let $\mu(dpdr|q)$ be the unique invariant measure for this process.
- Notice that $\mathcal{L}_q^m$ is effectively the operator corresponding to the fast motion.

Large deviations for second order Langevin equation.

- **Condition 2.** (Centering condition) *We assume that for every* $q \in \mathbb{R}^d$ *we have*

$$\int_{\mathcal{Y}} b(q,r)\mu(dpdr|q) = 0 \ .$$

Large deviations for second order Langevin equation.

- Preliminary cell problem

$$\mathcal{L}_q^m \Phi(p,r) = -\frac{1}{\sqrt{m}} p \ , \qquad \int_{\mathcal{Y}} \Phi(p,r)\mu(drdp|q) = 0 \ ,$$

has a unique, smooth solution that does not grow too fast at infinity.

- $\Phi(p,r) = (\Phi_1(p,r), ..., \Phi_d(p,r))$.

Large deviations for second order Langevin equation.

- Theorem

  Let $\{q^\varepsilon, \varepsilon > 0\}$ *be the unique solution to* (3)*. Under Conditions* 1 *and* 2, $\{q^\varepsilon, \varepsilon > 0\}$ *satisfies the large deviations principle with rate function*

$$S_m(\phi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T (\dot{\phi}_s - r_m(\phi_s))^T Q_m^{-1}(\phi_s)(\dot{\phi}_s - r_m(\phi_s))ds \\ \qquad \text{if } \phi \in \mathcal{AC}([0,T];\mathbb{R}^d) \ , \ \phi_0 = q_o \ ; \\ +\infty \quad , \ \text{otherwise} \ . \end{cases}$$

  where

$$r_m(q) = \frac{1}{\sqrt{m}} \int_{\mathcal{Y}} \nabla_p \Phi(p,r) c(q,r) \mu(dpdr|q) \ ,$$

$$Q_m(q) = \frac{1}{m} \int_{\mathcal{Y}} \nabla_p \Phi(p,r) \alpha(q,r) (\nabla_p \Phi(p,r))^T \mu(dpdr|q) \ .$$

Small mass limit : approximation by first order Langevin equation.

- Recall the multiscale Langevin equation

$$m\frac{\delta^2}{\varepsilon}\ddot{q}_t^\varepsilon = \frac{\varepsilon}{\delta} b\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right) + c\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right) - \lambda(q_t^\varepsilon)\dot{q}_t^\varepsilon + \sqrt{\varepsilon}\sigma\left(q_t^\varepsilon, \frac{q_t^\varepsilon}{\delta}\right)\dot{W}_t \ .$$

- Small mass limit : let $m \to 0$.
- q_t^ε is approximated by $\widetilde{q}_t^\varepsilon$ [6] :

$$\begin{aligned}\dot{\widetilde{q}}_t^\varepsilon = & \frac{\varepsilon}{\delta}\frac{b\left(\widetilde{q}_t^\varepsilon, \frac{\widetilde{q}_t^\varepsilon}{\delta}\right)}{\lambda(\widetilde{q}_t^\varepsilon)} + \frac{c\left(\widetilde{q}_t^\varepsilon, \frac{\widetilde{q}_t^\varepsilon}{\delta}\right)}{\lambda(\widetilde{q}_t^\varepsilon)} \\ & -\varepsilon\frac{\nabla\lambda(\widetilde{q}_t^\varepsilon)}{2\lambda^3(\widetilde{q}_t^\varepsilon)}\alpha\left(\widetilde{q}_t^\varepsilon, \frac{\widetilde{q}_t^\varepsilon}{\delta}\right) + \sqrt{\varepsilon}\frac{\sigma\left(\widetilde{q}_t^\varepsilon, \frac{\widetilde{q}_t^\varepsilon}{\delta}\right)}{\lambda(\widetilde{q}_t^\varepsilon)}\dot{W}_t \ .\end{aligned} \tag{6}$$

- $\widetilde{q}_t^\varepsilon$ the solution of first order Langevin equation.

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6. Freidlin, M., Hu, W., Smoluchowski–Kramers approximation in the case of variable friction, *Journal of Mathematical Sciences*, **79**, 1 (2011), pp. 184–207.

Large deviations for first order Langevin equation.

- Large deviations for first order Langevin equation $\widetilde{q}_t^\varepsilon$ is well established.
- Let $\mu_0(dr|q)$ be the unique invariant measure corresponding to the operator

$$\mathcal{L}_q^0 = \frac{1}{\lambda(q)} b(q,r) \cdot \nabla_r + \frac{1}{2\lambda(q)} \alpha(q,r) : \nabla_r^2$$

  equipped with periodic boundary conditions in r (q is being treated as a parameter here) on $\bar{\mathcal{Y}} = \mathbb{T}^d$
- Centering condition :

$$\int_{\bar{\mathcal{Y}}} b(q,r)\mu_0(dr|q) = 0 \ .$$

Large deviations for first order Langevin equation.

- Cell problem

$$\mathcal{L}^0_q \chi_\ell(q,r) = -\frac{1}{\lambda(q)} b_\ell(q,r) \ , \ \int_{\bar{\mathcal{Y}}} \chi_\ell(q,r)\mu_0(dr|q) = 0 \ ,$$

$\ell = 1, 2, ..., d$, has a unique bounded and sufficiently smooth solution $\chi = (\chi_1, ..., \chi_d)$.

Large deviations for first order Langevin equation.

- Theorem

  Let $\{\widetilde{q}^{\varepsilon}, \varepsilon > 0\}$ be the unique solution to the first order Langevin equation. Under Conditions 1 and 2, $\{\widetilde{q}^{\varepsilon}, \varepsilon > 0\}$ satisfies a large deviations principle with rate function

$$S_0(\phi) = \begin{cases} \frac{1}{2}\int_0^T (\dot{\phi}_s - r_0(\phi_s))^T Q_0^{-1}(\phi_s)(\dot{\phi}_s - r_0(\phi_s))ds \ , \\ \qquad\qquad \textit{if } \phi \in \mathcal{AC}([0,T];\mathbb{R}^d) \ , \phi_0 = q_o \ ; \\ +\infty \ , \ \textit{otherwise.} \end{cases}$$

where

$$r_0(q) = \frac{1}{\lambda(q)} \int_{\bar{\mathcal{Y}}} \left((I + \frac{\partial \chi}{\partial r}(q,r)) \right) c(q,r)\mu_0(dr|q)$$

and

$$Q_0(q) = \frac{1}{\lambda^2(q)} \int_{\bar{\mathcal{Y}}} \left(I + \frac{\partial \chi}{\partial r}(q,r) \right) \alpha(q,r) \left(I + \frac{\partial \chi}{\partial r}(q,r) \right)^T \mu_0(dr|q).$$

Approximation of the rate function.

- When do we have

$$\lim_{m \to 0} S_m(\phi) = S_0(\phi) \ ?$$

- This is very hard in general.
- We can work out the case when $\sigma(q, r) = \sqrt{2\beta\lambda(q)}I$, $\beta > 0$ (fluctuation–dissipation balance).
- Villani's *Hypercoercivity*.

Limit of the controlled fast–slow SRDEs.

- Controlled pair of fast–slow process $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$.
- The right way to describe the limit is a pair (ψ, P), which we call a viable pair.
- $\psi \in C([0,T];H)$ is the limit dynamics, and $\mathrm{P} \in \mathscr{P}(U \times \mathcal{Y} \times [0,T])$ is a certain invariant measure.
- This is an averaging procedure.
- Constructions of the same type have been carried out thoroughly in the case of finite dimensions, but for the first time in infinite dimensions in our work.

Limit of the controlled fast–slow process : Tightness.

- We introduce the family of random occupation measures

$$\mathrm{P}^{\varepsilon,\Delta}(dudYdt) = \frac{1}{\Delta}\int_t^{t+\Delta} \mathbf{1}_{du}(u(s))\mathbf{1}_{dY}\left(Y^{\varepsilon,\delta,u}(s)\right) dsdt \ .$$

- After very technical proof we can show that the pair $(X^{\varepsilon,\delta,u}, \mathrm{P}^{\varepsilon,\Delta})$ is tight in the space $C([0,T];H) \times \mathscr{P}(U \times \mathcal{Y} \times [0,T])$.
- We take weak topology on U and norm topology on $\mathcal{Y}$, and we make use of a–priori bounds and tightness functions.

Limit of the controlled fast–slow process : Limiting slow dynamics.

- Tightness guarantees that $(X^{\varepsilon,\delta,u}, \mathrm{P}^{\varepsilon,\Delta}) \to (\bar{X}, \mathrm{P})$ as $\varepsilon \to 0$ (and therefore $\delta \to 0$) for some limiting $(\bar{X}, \mathrm{P})$.
- Recall that in mild form we can write

$$X^{\varepsilon,\delta,u}(t) = S_1(t)X_0 + \int_0^t S_1(t-s)B_1(X^{\varepsilon,\delta,u}(s), Y^{\varepsilon,\delta,u}(s))ds + \int_0^t S_1(t-s)\Sigma_1(X^{\varepsilon,\delta,u}(s), Y^{\varepsilon,\delta,u}(s))Q_1 u(s)ds + \sqrt{\varepsilon}\int_0^t S_1(t-s)\Sigma_1(X^{\varepsilon,\delta,u}(s), Y^{\varepsilon,\delta,u}(s))dW^{Q_1}(s) \, .$$

- Limiting slow dynamics

$$\psi(t) = S_1(t)X_0 + \int_{U\times\mathcal{Y}\times[0,t]} S_1(t-s)\xi(\psi(s), Y, u)\mathrm{P}(dudYds) \, ,$$

where

$$\xi(X, Y, u) = \Sigma_1(X, Y)Q_1 u + B_1(X, Y) \, .$$

Limit of the controlled fast–slow process : Limiting occupation measure.

- In regards to the limit for the controlled fast process $Y^{\varepsilon,\delta,u}$, the limiting measure P characterizes simultaneously the structure of the invariant measure of $Y^{\varepsilon,\delta,u}$ and the control function u.
- In general, these two objects are intertwined and coupled together into the measure P, so that the averaging with respect to the measure P is different and hard to perform as in the classical averaging principle.

Limit of the controlled fast–slow process : Limiting occupation measure.

- $\mathrm{P}(dudYdt) = \eta_t(du|Y)\mu^{\psi_t}(dY|u)dt$, and we cannot guarantee that μ is invariant measure to some process, as this process is a controlled process and we have no regularity properties of the optimal controls known other than being square integrable.
- In finite dimensional case with periodic coefficients, the problem can be resolved using the characterization of optimal controls through solutions to Hamilton–Jacobi–Bellman equations.
- Such a characterization is not rigorously known in infinite dimensions and even if that becomes the case, one would need to establish sufficient properties of such equations that would then imply that the resulting controlled process has a well defined invariant measure that is regular enough.

Limit of the controlled fast–slow process : Limiting occupation measure.

- Is it possible to have $\mathrm{P}(dudYdt) = \eta_t(du|Y)\mu^{\psi_t}(dY)dt$?
- Recall that the fast motion can be written formally as

$$
\begin{aligned}
dY^{\varepsilon,\delta,u}(t) = \frac{1}{\delta^2}\Big[A_2 Y^{\varepsilon,\delta,u}(t) + B_2(X^{\varepsilon,\delta,u}(t), Y^{\varepsilon,\delta,u}(t))\Big]\, dt + \\
+ \frac{1}{\delta^2}\frac{\delta}{\sqrt{\varepsilon}}\Sigma_2(X^{\varepsilon,\delta,u}(t), Y^{\varepsilon,\delta,u}(t))Q_2 u(t)dt + \\
+ \frac{1}{\delta}\Sigma_2(X^{\varepsilon,\delta,u}(t), Y^{\varepsilon,\delta,u}(t))dW^{Q_2} \ , \\
Y^{\varepsilon,\delta,u}(0) = Y_0 \in H \ .
\end{aligned}
$$

- Send $\dfrac{\delta}{\sqrt{\varepsilon}} \to 0$ as $\varepsilon \to 0$ (and therefore $\delta \to 0$).

Hypothesis.

- We assume that $\varepsilon \downarrow 0$, $\delta = \delta(\varepsilon) \downarrow 0$ and $\Delta = \Delta(\delta, \varepsilon) \downarrow 0$, such that
$$\lim_{\varepsilon \downarrow 0} \frac{\delta}{\sqrt{\varepsilon}} = 0, \text{ and } \lim_{\varepsilon \downarrow 0} \frac{\delta}{\Delta\sqrt{\varepsilon}} = 0 \ .$$

Limit of the controlled fast–slow process : Limiting occupation measure.

- From Hypothesis we know that

$$\lim_{\varepsilon\downarrow 0}\frac{\delta}{\sqrt{\varepsilon}}=0,\ \text{and}\ \lim_{\varepsilon\downarrow 0}\frac{\delta}{\Delta\sqrt{\varepsilon}}=0\ .$$

- Fast motion without control but driven by the controlled slow process

$$\begin{aligned} d\mathrm{Y}^{\varepsilon,\delta}(t) &= \frac{1}{\delta^2}\left[A_2\mathrm{Y}^{\varepsilon,\delta}(t)+B_2(X^{\varepsilon,\delta,u}(t),\mathrm{Y}^{\varepsilon,\delta}(t))\right]dt+ \\ &\quad +\frac{1}{\delta}\Sigma_2(X^{\varepsilon,\delta,u}(t),\mathrm{Y}^{\varepsilon,\delta}(t))dW^{Q_2}\ , \\ \mathrm{Y}^{\varepsilon,\delta}(0) &= Y_0\in H\ . \end{aligned}$$

- One can show that

$$\mathbf{E}\frac{1}{\Delta}\int_0^T|Y^{\varepsilon,\delta,u}(t)-\mathrm{Y}^{\varepsilon,\delta}(t)|_H^2dt\to 0\ ,$$

as $\varepsilon\to 0$ (and therefore $\delta\to 0$).

Limit of the controlled fast–slow process : Limiting occupation measure.

- Slogan : Ergodic properties are stable with respect to mildly regular small perturbations.
- Knowing closeness of two processes in mean square sense is already enough to draw conclusions about the equivalency of their invariant measures.
- $$\mathrm{P}(dudYdt) = \eta_t(du|Y)\mu^{\psi_t}(dY)dt \ .$$

Averaging of the controlled fast–slow process.

- Definition

  A pair $(\psi, \mathrm{P}) \in C([0, T]; L^2(D)) \times \mathscr{P}(U \times \mathcal{Y} \times [0, T])$ will be called viable with respect to $(\xi, \mathcal{L})$, or simply viable if there is no confusion, if the following are satisfied. The function $\psi(t)$ is absolutely continuous as a function in the space $C([0, T]; H)$, P is square integrable in the sense that

$$\int_{U\times\mathcal{Y}\times[0,T]} \left(|u|_U^2 + |Y|_H^2\right) \mathrm{P}(dudYds) < \infty$$

  and the following hold for all $t \in [0, T]$:

$$\psi(t) = S_1(t)X_0 + \int_{U\times\mathcal{Y}\times[0,t]} S_1(t-s)\xi(\psi(s), Y, u)\mathrm{P}(dudYds) \ ,$$

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- Definition
  the measure P is such that

$$\mathrm{P} \in \mathbb{P} =$$

$$\left\{ \begin{array}{c} P \in \mathscr{P}(U \times \mathcal{Y} \times [0, T]) : \mathrm{P}(dudYdt) = \eta(du|Y, t)\mu(dY|t)dt, \\ \mu(dY|t) = \mu^{\psi(t)}(dY) \text{ for } t \in [0, T] \end{array} \right\}$$

  where μ^X is the invariant measure for the uncontrolled fast process with frozen slow component X, and

$$\mathrm{P}(U \times \mathcal{Y} \times [0, t]) = t \ .$$

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- Theorem

  Let $u \in \mathcal{P}_2^N(U)$, $(X^{\varepsilon,\delta,u}, Y^{\varepsilon,\delta,u})$ be the mild solution to controlled fast process and $T < \infty$. Let also $\mathrm{P}^{\varepsilon,\Delta}(dudYdt)$ be the occupation measure. Assume Hypotheses 1, 2, 3 and 5, $X_0 \in H_1^\theta$ with $\theta > 0$ sufficiently small and $Y_0 \in H$. Then, the family of processes $X^{\varepsilon,\delta,u}$ is tight in $C([0,T];H)$ and the family of measures $\mathrm{P}^{\varepsilon,\Delta}$ is tight in $\mathscr{P}(U \times \mathcal{Y} \times [0,T])$, where $U \times \mathcal{Y} \times [0,T]$ is endowed with the weak topology on U, the norm topology on $\mathcal{Y}$ and the standard topology on $[0,T]$. Hence, given any subsequence of $\{(X^{\varepsilon,\delta,u}, \mathrm{P}^{\varepsilon,\Delta}), \varepsilon, \delta, \Delta > 0\}$, there exists a subsubsequence that converges in distribution with limit $(\bar{X}, \mathrm{P})$. With probability 1, the accumulation point $(\bar{X}, \mathrm{P})$ is a viable pair with respect to $(\xi, \mathcal{L})$.

Large Deviations.

- Back to our large deviations problem : For every bounded and continuous function $h : C([0,T];H) \to \mathbb{R}$ we want

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbf{E}_{X_0,Y_0}\left[\exp\left(-\frac{1}{\varepsilon}h(X^{\varepsilon,\delta})\right)\right] = \inf_{\phi \in C([0,T];H)} [S_{X_0}(\phi)+h(\phi)] \, .$$

- Representation formula

$$-\varepsilon \ln \mathbf{E}\left[\exp\left(-\frac{1}{\varepsilon}h(X^{\varepsilon,\delta})\right)\right] = \inf_{u \in L^2([0,T];U)} \mathbf{E}\left[\frac{1}{2}\int_0^T |u(s)|_U^2 ds + h(X^{\varepsilon,\delta,u})\right] \, .$$

Large Deviations.

- Theorem
  Let $(X^{\varepsilon,\delta}, Y^{\varepsilon,\delta})$ be the mild solution to fast–slow SRDE and let $T < \infty$. Under some assumptions to be ignored here, let $Y_0 \in H$ and $X_0 \in H_1^{\theta}$. Define

$$S(\phi) = S_{X_0}(\phi) = \inf_{(\phi,P)\in\mathcal{V}_{(\xi,\mathcal{L})}} \left[\frac{1}{2} \int_{U\times\mathcal{Y}\times[0,T]} |u|_U^2 \mathrm{P}(dudYdt) \right] ,$$

  with the convention that the infimum over the empty set is ∞.

Large Deviations.

- Theorem

  Then, there exists $\bar{\theta} > 0$ *such that for all* $\theta \in (0, \bar{\theta}]$ *and* $X_0 \in H_1^\theta$*, and for every bounded and continuous function* $h : C([0,T]; H) \to \mathbb{R}$ *we have*

$$\lim_{\varepsilon \downarrow 0} -\varepsilon \ln \mathbf{E}_{X_0, Y_0} \left[\exp\left(-\frac{1}{\varepsilon} h(X^{\varepsilon,\delta}) \right) \right] = \inf_{\phi \in C([0,T];H)} [S_{X_0}(\phi) + h(\phi)] \, .$$

  In particular, $\{X^{\varepsilon,\delta}\}$ *satisfies the large deviations principle in* $C([0,T]; H)$ *with action functional* $S_{X_0}(\cdot)$*, uniformly for* X_0 *in compact subsets of* H_1^θ*.*

Large Deviations : Lower bound.

- By Fatou's Lemma we have

$$
\begin{aligned}
& \liminf_{\varepsilon \downarrow 0} \left(-\varepsilon \ln \mathbf{E}_{X_0} \left[\exp\left(-\frac{h(X^{\varepsilon,\delta})}{\varepsilon} \right) \right] \right) \\
\geq \; & \liminf_{\varepsilon \downarrow 0} \left(\mathbf{E}_{X_0} \left[\frac{1}{2} \int_0^T |u^\varepsilon(t)|_U^2 dt + h(X^{\varepsilon,\delta}) \right] - \varepsilon \right) \\
\geq \; & \liminf_{\varepsilon \downarrow 0} \left(\mathbf{E}_{X_0} \left[\frac{1}{2} \int_0^T \frac{1}{\Delta} \int_t^{t+\Delta} |u^\varepsilon(s)|_U^2 ds dt + h(X^{\varepsilon,\delta}) \right] \right) \\
= \; & \liminf_{\varepsilon \downarrow 0} \left(\mathbf{E}_{X_0} \left[\frac{1}{2} \int_{U \times \mathcal{Y} \times [0,T]} |u|_U^2 \mathrm{P}^{\varepsilon,\Delta}(du dY dt) + h(X^{\varepsilon,\delta}) \right] \right) \\
\geq \; & \inf_{(\phi,\mathrm{P}) \in \mathcal{V}_{(\xi,\mathcal{L})}} \left[\frac{1}{2} \int_{U \times \mathcal{Y} \times [0,T]} |u|_U^2 \mathrm{P}(du dY dt) + h(\phi) \right],
\end{aligned}
$$

which concludes the proof of the Laplace principle lower bound.

Large Deviations : Upper bound.

- In order to prove the Laplace principle upper bound, we have to construct a nearly optimal control that achieves the upper bound.
- The nearly optimal control will be in feedback form with respect to the Y variable,
- Due to infinite dimensionality we can only work it out in some special cases.

Special cases in which we can obtain the large deviations upper bound.

- If $d = 1$, then there are positive constants $0 < c_0 \leq c_1 < \infty$ such that $0 < c_0 \leq \sigma_1^2(x, X, Y) \leq c_1$, or if $d > 1$ then $\sigma_1(x, X, Y) = \sigma_1(x, X)$ is independent of Y and can grow at most linearly in X uniformly in $x \in D$.

Thank you for your attention !