On the long time behavior of a perturbed conservative system with degeneracy.

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Random perturbations of dynamical systems with group symmetry.

- Many Hamiltonian systems that arise in mechanics, mechanical engineering, as well as hydrodynamics are subject to group symmetry.
- As an example, in the study of the motion of an ideal incompressible fluid, V.I.Arnold had proposed a beautiful picture that describes the dynamics of ideal incompressible fluid as geodesic flows on the group of all diffeomorphisms of a certain domain.
- The studies of random perturbations of Hamiltonian systems, or general dynamical systems with symmetry, particular the long–time dynamics and problems about invariant measures of these systems are of interest.

Random perturbations of dynamical systems with group symmetry.

- Schematically, the general problem can be formulated as follows. We are given a dynamical system

$$\dot{x} = b(x)$$

  in an ambient space $x \in M$ (M can be a Riemanian manifold).
- Usually we assume $b(x)$ preserves the energy.
- Then we assume that for some group G the system we consider has some symmetry with respect to G.

Random perturbations of dynamical systems with group symmetry.

- The last sentence about symmetry of the system with respect to the group G is a bit vague and could be understood in many different ways.
- It can be understood in a strict way so that the group can act on the space M (in particular, it is such case when $G = M$) and the dynamics is invariant with respect to G–action.
- It can also be understood as a more "rough" symmetry, in the sense for example that the stable attractors have equivalent dynamical properties under G–action (for example, equivalence of "quasi–potential").

Random perturbations of dynamical systems with group symmetry.

- Our goal is to describe the effect of adding a small noise. That is, we study systems of type

$$\dot{\mathcal{X}}^\varepsilon = b(\mathcal{X}^\varepsilon) + \xi^\varepsilon$$

  where ξ^ε is a deterministic and/or stochastic perturbation depending on the small parameter(s) $\varepsilon = (\varepsilon_1, ..., \varepsilon_k)$.
- In general, an effective description of the long–time behavior of the perturbed system is the motion on the cone of invariant measures of the unperturbed system.

Model problem : AB–model.

- We consider here a model problem

$$\begin{cases} dx_t = -x_t y_t dt \ , \\ dy_t = x_t^2 dt \ . \end{cases}$$

- A phase picture is shown in the next Figure.
- We see that the whole line Oy_A contains stable equilibriums and the whole line Oy_B contains unstable equilibriums. This is different from the cases considered in the classical Freidlin-Wentzell theory.
- In this case we can understand the symmetry of our model in a more rough way : the stable and unstable equilibriums are symmetric with respect to shifts in the directions of Oy_A and Oy_B, respectively.
- Our model preserves the energy $E(x,y) = x^2 + y^2$. The driving vector field $b(x,y) = (-xy, x^2)$ is degenerate on $x = 0$.

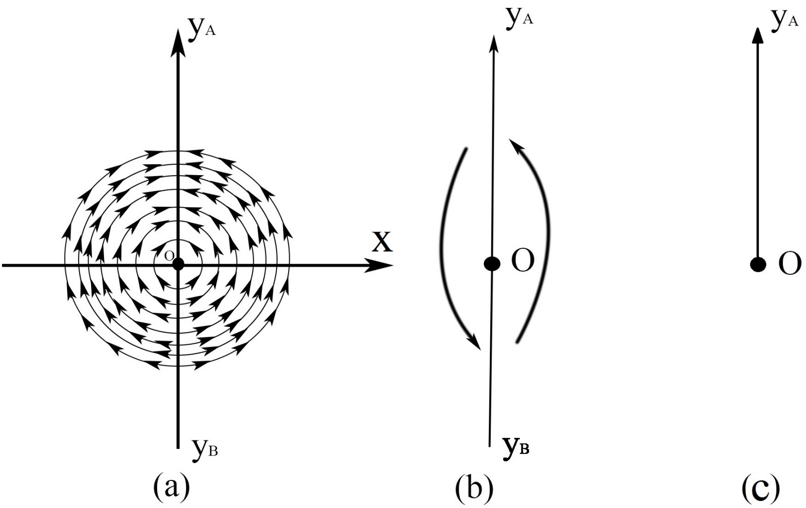

FIGURE: The *AB* model.

Randomly perturbed AB–model.

- We add friction and random perturbation to this system

$$\left\{\begin{array}{l} d\mathcal{X}_t^\varepsilon = -\mathcal{X}_t^\varepsilon \mathcal{Y}_t^\varepsilon dt - \varepsilon \mathcal{X}_t^\varepsilon dt + \sqrt{\varepsilon} dW_t^1 \ , \ \mathcal{X}_0^\varepsilon = x_0 \ , \\ d\mathcal{Y}_t^\varepsilon = (\mathcal{X}_t^\varepsilon)^2 dt - \varepsilon \mathcal{Y}_t^\varepsilon dt + \sqrt{\varepsilon} dW_t^2 \ , \ \mathcal{Y}_0^\varepsilon = y_0 \ . \end{array}\right.$$

- Here W_t^1 and W_t^2 are two independent standard 1–dimensional Brownian motions;
- The small parameter $\varepsilon > 0$ is the intensity of the friction, and the small parameter $\sqrt{\varepsilon} > 0$ is the intensity of the noise.

Randomly perturbed AB–model.

- Question : What is the long–time behavior of the system $(\mathcal{X}_t^\varepsilon, \mathcal{Y}_t^\varepsilon)$ as $t \to \infty$ and $\varepsilon \downarrow 0$?

Randomly perturbed AB–model : Background.

- Our model problem here differs from the set–up in the classical Freidlin–Wentzell theory in that the point–like asymptotically stable attractor is replaced by a manifold. We can view our limiting process as a “process–level attractor” of our system.

Randomly perturbed *AB*–model : Background.

- Finite dimensional models for the inviscid stochastic 2–d Navier–Stokes equations of the form

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega - \nu \Delta \omega = \sqrt{\nu}\xi(t,x)\,,\, u = \nabla^{\top} \Delta^{-1} \omega\,, \omega(x,0) = \omega_0(x)\,,$$

  in which $\xi(t,x)$ is a noise, and positive parameter $\nu \to 0$.
- Our consideration of the unperturbed system mimics the attractor for the 2–d Euler system, that has continuous sets of steady states.
- In fact, systems that arise in hydrodynamics, such as in the context of Euler's equation, typically possess equilibrium points that belong to an infinite dimensional "manifold" of other equilibria.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- We come back to our system

$$\begin{cases} d\mathcal{X}_t^\varepsilon = -\mathcal{X}_t^\varepsilon \mathcal{Y}_t^\varepsilon dt - \varepsilon \mathcal{X}_t^\varepsilon dt + \sqrt{\varepsilon} dW_t^1 \ , \ \mathcal{X}_0^\varepsilon = x_0 \ , \\ d\mathcal{Y}_t^\varepsilon = (\mathcal{X}_t^\varepsilon)^2 dt - \varepsilon \mathcal{Y}_t^\varepsilon dt + \sqrt{\varepsilon} dW_t^2 \ , \ \mathcal{Y}_0^\varepsilon = y_0 \ . \end{cases}$$

- We do a time rescaling $t \to \dfrac{t}{\varepsilon}$ and we let

$$(X_t^\varepsilon, Y_t^\varepsilon) = (\mathcal{X}_{t/\varepsilon}^\varepsilon, \mathcal{Y}_{t/\varepsilon}^\varepsilon) \ .$$

- Then

$$\begin{cases} dX_t^\varepsilon = -\dfrac{1}{\varepsilon} X_t^\varepsilon Y_t^\varepsilon dt - X_t^\varepsilon dt + dW_t^1 \ , \ X_0^\varepsilon = x_0 \ , \\ dY_t^\varepsilon = \dfrac{1}{\varepsilon}(X_t^\varepsilon)^2 dt - Y_t^\varepsilon dt + dW_t^2 \ , \ Y_0^\varepsilon = y_0 \ . \end{cases}$$

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- System $(X_t^\varepsilon, Y_t^\varepsilon)$:

$$\begin{cases} dX_t^\varepsilon = -\frac{1}{\varepsilon} X_t^\varepsilon Y_t^\varepsilon dt - X_t^\varepsilon dt + dW_t^1 \ , \ X_0^\varepsilon = x_0 \ , \\ dY_t^\varepsilon = \frac{1}{\varepsilon}(X_t^\varepsilon)^2 dt - Y_t^\varepsilon dt + dW_t^2 \ , \ Y_0^\varepsilon = y_0 \ . \end{cases}$$

- Separation of slow and fast motions.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- In the limit as $\varepsilon \downarrow 0$, the process $(X_t^\varepsilon, Y_t^\varepsilon)$ is pushed by the flow onto Oy_A, and will be close to $\pi(x_0, y_0)$ in short time.
- There, the Y–component Y_t^ε behaves as a 2–dimensional linearly damped radial Bessel process (*damped–BES(2)*) on Oy_A :

$$dY_t = \left(\frac{1}{2Y_t} - Y_t\right) dt + dW_t^2 \ , Y_0 = y^\pi(x_0, y_0) \ .$$

- What is the heuristic ?

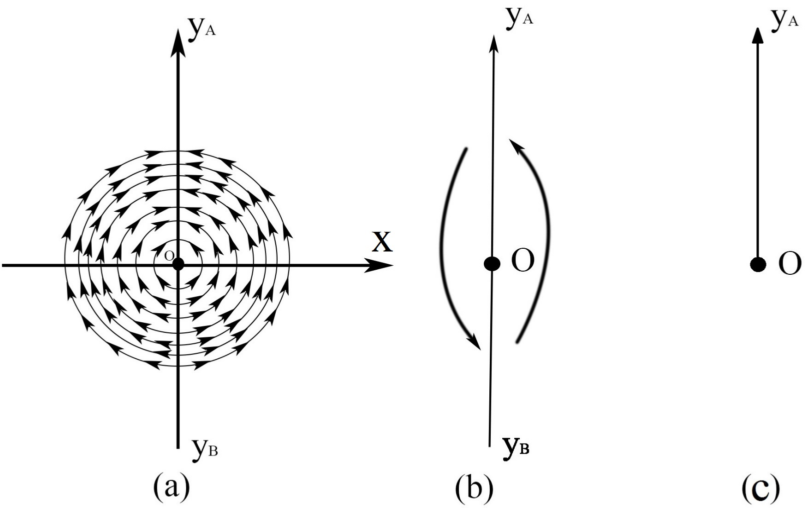

FIGURE: The *AB* model.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- The radial process $r_t^\varepsilon = \sqrt{(X_t^\varepsilon)^2 + (Y_t^\varepsilon)^2}$.
- By applying Itô's formula we see that

$$dr_t^\varepsilon = \left(\frac{1}{2r_t^\varepsilon} - r_t^\varepsilon\right) dt + dW_t^r \ , \ r_0^\varepsilon = \sqrt{(X_0^\varepsilon)^2 + (Y_0^\varepsilon)^2}$$

- As $\varepsilon \downarrow 0$ the process X_t^ε is pushed by the fast flow to be close to 0 when $Y_t^\varepsilon \geq \delta > 0$.
- $\delta = \varepsilon^{1/10}$.
- Near the Oy_A axis we have $r_t^\varepsilon \approx Y_t^\varepsilon$ as $\varepsilon \downarrow 0$.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- What happened when $Y_t^{\varepsilon} < \delta$?
- The above comparison with the radial process will not work.
- If $(X_t^{\varepsilon}, Y_t^{\varepsilon})$ is close to the origin $O = (0,0)$, we look at our system

$$\begin{cases} dX_t^{\varepsilon} = -\dfrac{1}{\varepsilon} X_t^{\varepsilon} Y_t^{\varepsilon} dt - X_t^{\varepsilon} dt + dW_t^1 \ , \ X_0^{\varepsilon} = x_0 \ , \\ dY_t^{\varepsilon} = \dfrac{1}{\varepsilon}(X_t^{\varepsilon})^2 dt - Y_t^{\varepsilon} dt + dW_t^2 \ , \ Y_0^{\varepsilon} = y_0 \ . \end{cases}$$

- In the limit as $\varepsilon \downarrow 0$ the positive drift $\dfrac{1}{\varepsilon}(X_t^{\varepsilon})^2 \to \dfrac{1}{2Y_t}$.
- Recall that the damped BES(2) process takes the form

$$dY_t = \left(\frac{1}{2Y_t} - Y_t\right) dt + dW_t^2 \ , Y_0 = y^{\pi}(x_0, y_0) \ .$$

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- This suggests that the origin O is inaccessible and as $\varepsilon \to 0$ the limit process $Y_t^\varepsilon \to Y_t$ lives only on Oy_A axis.
- Also supported by simulation results.

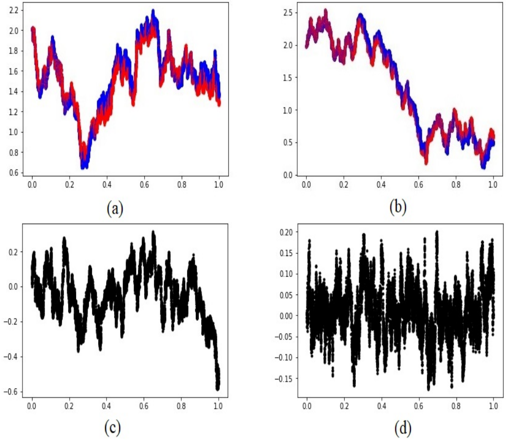

FIGURE: Sample pathes of the X_t^ε and Y_t^ε processes, as well as the limiting Y–process (driven by W_t^2) starting from $(X, Y) = (0, 2)$ in 15000 steps for stepsize= 0.0001, that is rescaled to $[0, 1]$. (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.01$; the red curves are the sample pathes for Y_t, the blue curves are the sample pathes for Y_t^ε. (c) $\varepsilon = 0.1$; (d) $\varepsilon = 0.01$; the black curves are the sample pathes for X_t^ε.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- To prove this we have to carefully analyze the behavior of the process $(X_t^\varepsilon, Y_t^\varepsilon)$ near the origin $O = (0,0)$.
- We introduce the angular variable $\theta_t^\varepsilon = \arctan\left(\dfrac{Y_t^\varepsilon}{X_t^\varepsilon}\right)$.
- $$\begin{cases} d\theta_t^\varepsilon = \dfrac{1}{\varepsilon} X_t^\varepsilon dt + \dfrac{1}{(r_t^\varepsilon)^2} dW_t^\theta \ , \theta_0^\varepsilon = \arctan\left(\dfrac{Y_0^\varepsilon}{X_0^\varepsilon}\right) \ , \\ dr_t^\varepsilon = \left(\dfrac{1}{2r_t^\varepsilon} - r_t^\varepsilon\right) dt + dW_t^r \ , r_0^\varepsilon = \sqrt{(X_0^\varepsilon)^2 + (Y_0^\varepsilon)^2} \ . \end{cases}$$

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- $\delta = \varepsilon^{1/10}$.
- Set the slow time clock $\mathrm{t} = (\delta/\varepsilon)t$ and let us consider a time–rescaled pair of processes $\Theta_{\mathrm{t}}^{\varepsilon} = \theta_{(\varepsilon/\delta)\mathrm{t}}^{\varepsilon}$ and $R_{\mathrm{t}}^{\varepsilon} = r_{(\varepsilon/\delta)\mathrm{t}}^{\varepsilon}$.
- Then the stochastic differential equations satisfied by $(\Theta_{\mathrm{t}}^{\varepsilon}, R_{\mathrm{t}}^{\varepsilon})$ are given by

$$\begin{cases} d\Theta_{\mathrm{t}}^{\varepsilon} = \dfrac{X_{(\varepsilon/\delta)\mathrm{t}}^{\varepsilon}}{\delta} d\mathrm{t} + \sqrt{\dfrac{\varepsilon}{\delta}} \cdot \dfrac{1}{(R_{\mathrm{t}}^{\varepsilon})^2} dW_{\mathrm{t}}^{\theta} \ , \ \Theta_0^{\varepsilon} = \theta_0^{\varepsilon} \\ dR_{\mathrm{t}}^{\varepsilon} = \dfrac{\varepsilon}{\delta} \left(\dfrac{1}{2R_{\mathrm{t}}^{\varepsilon}} - R_{\mathrm{t}}^{\varepsilon} \right) d\mathrm{t} + \sqrt{\dfrac{\varepsilon}{\delta}} dW_{\mathrm{t}}^{r} \ , R_0^{\varepsilon} = r_0^{\varepsilon} \ . \end{cases}$$

- $\Theta_{\mathrm{t}}^{\varepsilon}$ is fast motion and $R_{\mathrm{t}}^{\varepsilon}$ is slow motion.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- This analysis enables us to conclude that for any initial condition $|X_0^\varepsilon| \geq 2\delta$, the flow will quickly bring the particle back to the region $Y \geq \delta$, and during this process the $|X|$–value is less or equal than 3δ.
- In particular, this implies that

$$\mathbf{P}\left(|X_t^\varepsilon| \leq 3\delta \text{ for } 0 \leq t \leq T\right) \to 1$$

  as $\varepsilon \downarrow 0$.
- As $\varepsilon \to 0$ the process X_t^ε will be localized near 0, and the process Y_t^ε lives on $\{Y \geq \delta\}$.
- We also need to do some exit time analysis and the proof of tightness for $\{Y_t^\varepsilon : 0 \leq t \leq T\}_{\varepsilon > 0}$.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

Theorem

Let $T > 0$ and initial condition $(x_0, y_0) \in \mathbb{R}^2$. Then
(a) For any bounded continuous function $F : \mathbb{R}^2 \to \mathbb{R}$ that is uniformly Lipschitz continuous with a Lipschitz constant $Lip(F) < \infty$ we have

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}\left[F(X_T^\varepsilon, Y_T^\varepsilon) - F(0, Y_T^\varepsilon)\right] = 0 \ .$$

(b) The measures on $\mathbf{C}_{[0,T]}(\mathbb{R})$ induced by the process Y_t^ε converge weakly as $\varepsilon \downarrow 0$ to the measure induced by Y_t with $Y_0 = y^\pi(x_0, y_0)$.

Limit as $\varepsilon \to 0$ of the perturbed AB–model.

- The proof makes use of the martingale problem framework for Markov processes.

Metastable behavior.

- For fixed $\varepsilon > 0$, at a subexponential time scale, excursions of the process $(X_t^\varepsilon, Y_t^\varepsilon)$ moving from Oy_A towards a level set $y = -a$ will be observed.
- This induces jumps from points in Oy_B to points in Oy_A.
- As ε becomes smaller, motions of the process $(X_t^\varepsilon, Y_t^\varepsilon)$ to Oy_B and jumping back become more and more rare, and in the limit no more such jumps appear, so that we come to the limiting process Y_t which cannot penetrate through O.
- Thus as $\varepsilon > 0$ is close to 0, the description of the "metastable" behavior of system involves both a diffusion part and a jump part.

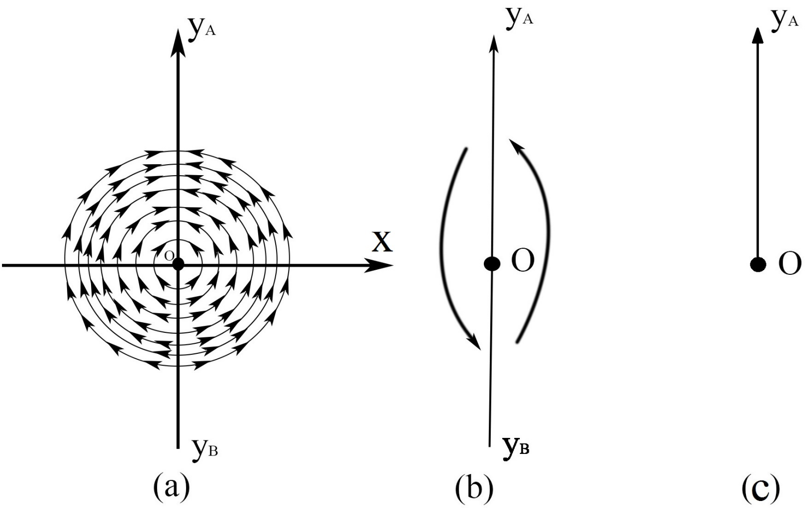

FIGURE: The *AB* model.

Final Remarks.

- Motion on the cone of invariant measures of the unperturbed system.
- One can use this result to analyze behavior of elliptic operator $L = \frac{1}{\varepsilon} L_0 + L_1$, where

$$L_0 = -xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$$

  and

$$L_1 = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{\partial^2}{\partial y^2} \, .$$

- The underlying 2d system is related to Euler equation - it is, in fact, the Euler-Arnold equation on a Lie algebra of the 2d Lie group of affine transformations of the line.

Thank you for your attention !