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# An overview of Modern Portfolio Theory

The major idea behind modern portfolio theory (MPT) is that by carefully selecting proportions of different assets, we can develop investment tools that maximize the expected return under a given portfolio risk. Although the basic assumptions have been challenged literally since its first proposal, MPT is used widely in the industry of finance. Since the first advancement proposed by Markowitz ([Markowitz, 1959]), MPT theory has been a major financial evaluation tool for investment professionals. Many market makers and investors have been working under the MPT framework trying to keep risks low and the returns maximized.

## *1. Introduction*

The basis of the modern portfolio theory concentrates on the optimization of the investment decision under uncertainty, since the future returns are not established with certainty, so we have to estimate the future performance of the available securities. The fundamental theorem of the “Mean-Variance Analysis”, which is proposed by Harry Markowitz, is to maximize the expected return given a constant variance, or to minimize the variance with the expected return held constant (Markowitz, 1959). His theory shows the importance of diversification to reduce the unsystematic risks, i.e. the risks can be affected by investor’s actions, and he also argues that the risks can be minimized but not totally eliminated. In the discussion of the risks, it should be pointed out that we essentially have two kinds of risks, unsystematic risks and systematics risks. In the following paper, we will first introduce the concept of the expected return and risks, then demonstrate geometrically the efficient frontier, and finally, discuss the mathematical details of the mechanism of diversification, which is a way to reduce the unsystematic risks to any arbitrary low level.

Before we set up the problems, we need to build those five basic assumptions that are fundamentals upon which the MPT is constructed:

- *Assumption 1:* The expected return and the variance are the only parameters that affect an investor's decision.
- Assumption 2*:* Investors are generally rational and risk adverse. They are completely aware of all the risk contained in investment and actually take positions based on the risk determination demanding a higher return for accepting greater volatility.
- *Assumption 3:* For buying and selling securities there are no transaction costs.
- *Assumption 4:* All investors have the same expectations concerning the expected return, variance, and covariance.
- *Assumption 5:* Analysis is based on a single period model of investment.

## *2. Review of Expectation and Variance*

Before I introduce the concepts of expected return and risk, I shall give a review of the expectation and variance.

Let R be a random variable, which has M possible outcomes $r_1 \ldots r_M$, each outcomes with probability $p_i$, where $i = 1,2 \ldots M$. Then the expectation of the random variable R is denoted as $E\ (R)$, where

$$E(R) = \sum_{i=1}^{M} p_i r_i$$

And the variance of X is denoted by $D(X)$, where

$$D(R) = \sigma^2 = \sum_{i=1}^{M} [p_i \left(R_i - E(R)\right)^2]$$

Now consider a collection of random variable $R_1 \ldots R_N$, and R is the weighted sum of $R_i$, let

$$b = (b_1, \ldots, b_N)$$

be the weights of the each random variable. Then

$$R = b_1 R_1 + \cdots + b_N R_N$$

which is also a random variable. Take the expectation of R, we get

$$E(R) = b_1 E(R_1) + \cdots + b_N E(R_N)$$

While, the variance of the random variable R is somehow complicated, since each random variable may be correlated, we have to take those correlations into account of our total variance. We define covariance to measure the linear relationship between each random variable. In order to derive the general formula for the variance of R, i.e. D(R), let me first explicitly derive the formula for 3 random variables.

Assume 3 random variables, $R_1$ $R_2$ $R_3$, and $X_i$ be the weights assign to the each of the random variables, where $R = X_1 R_1 + X_2 R_2 + X_3 R_3$, then

$$\sigma^2 = E\big(R - E(R)\big)^2 = E[X_1 R_{1j} + X_2 R_{2j} + X_3 R_{3j} - (X_1 E(X_1) + X_2 E(X_2) + X_3 E(X_3))]^2$$

$$\begin{aligned} &= X_1^2 E[(R_{1j} - E(R_1)]^2 + X_2^2 E[(R_{2j} - E(R_2)]^2 + X_3^2 E[(R_{3j} - E(R_3)]^2 \\ &\qquad + 2X_1 X_2 E\left[\left(R_{1j} - E(X_1)\right)\left(R_{2j} - E(X_2)\right) + 2X_2 X_3 E\left[\left(R_{2j} - E(X_2)\right)\left(R_{3j} - E(X_3)\right)\right.\right. \\ &\qquad + 2X_1 X_3 E\left[\left(R_{1j} - E(X_1)\right)\left(R_{3j} - E(X_3)\right)\right] \end{aligned}$$

where $\sigma_{ij} = E[\big(R_i - E(X_i)\big)\left(R_j - E\left(X_j\right)\right)]$ define the covariance between two random variable $i, j$, and when $i = j$, $\sigma_{ii}$ denote the variance of a single variable $R_i$. Hence, the above equation can be written as

$$\sigma^2 = X_1^2 \sigma_1{}^2 + X_2^2 \sigma_2{}^2 + X_3^2 \sigma_3{}^2 + 2X_1 X_2 \sigma_{12} + 2X_2 X_3 \sigma_{23} + 2X_1 X_3 \sigma_{12}$$

Now, we can see that the total variance has two parts, the first part can be written as

$$\sum_{i=1}^{N} (X_i \sigma_i)^2$$

and the second part can be written as

$$\sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} (X_i X_j \sigma_{ik})$$

Hence, the variance of the total variance of the random variables $R_1 \ldots$ $R_N$ is

$$D(R) = \sum_{i=1}^{N} (X_i \sigma_i)^2 + \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} (X_i X_j \sigma_{ik})$$

Now, let us get back to the return and risk of a portfolio.

### *3. Expected return and risk*

Return is the basic motivating force and the principle reward in any investment process. It is defined as

$$Return = \frac{S_t - S_0}{S_0}$$

where $S_0, S_t$ denote the stock price at $T = 0$ and $T = t$ respectively.

- ***Return of the portfolio***

To illustrate the main idea of the MPT, we only consider the investment problem in a single-period. Assume that the investor can allocate his starting capital $x$ at the time $t = 0$ among the stock $A_1, \ldots$ $A_N$ at the price $P_o(A_1), \ldots$ $P_o(A_N)$, respectively. Let $b = (b_1 \ldots b_N)$ be the investment portfolio, where $b_i$ is the corresponding shares of stock that investor invested in the stock $A_i$. Then, the price of the portfolio at time $T = 0$ is

$$S_0(b) = b_1 P_o(A_1) + \cdots + b_N P_0(A_N)$$

and $S_0(b) = x$, i.e. the initial capital.

Assume the price of the assets at $T = 1$, are given by the random interest rate $\rho\,(A_i)$ of $A_i$, that is

$$P_1(A_i) = (1 + \rho\,(A_i)) P_0(A_i)$$

for $i = 1,2,3, \ldots, n$. Then the price of the portfolio at time $T = 1$ is

$$S_1(b) = b_1 P_1(A_1) + \cdots + b_N P_1(A_N)$$

Hence, the return of the portfolio is

$$R(b) = \frac{S_1(b) - S_0(b)}{S_0(b)} = \frac{S_1(b)}{x} - 1 = \frac{\sum b_i P_1(A_i)}{x} - 1 = \sum \frac{b_i P_0(A_i)}{x} \frac{P_1(A_i)}{P_0(A_i)} - 1$$

Let $d_i = \frac{b_i P_0(A_i)}{x}$, which can be interpreted as the fraction of money invested in stock $A_i$ among the initial capital $x$. So $d = (d_1 \ldots d_N)$ and observe that $\sum d_i = 1$. Hence the above $R(b)$ can be written as

$$R(b) = \sum d_i \frac{P_1(A_i)}{P_0(A_i)} - 1 = \sum d_i (\frac{P_1(A_i)}{P_0(A_i)} - 1) = \sum d_i \rho\,(A_i) \qquad (1)$$

Denote $R(b) = \rho\,(d) = \sum d_i \rho\,(A_i)$, which is the return of the portfolio.

- ***Risk of the portfolio***

Risk is the unpredictability of the future returns from an investment (Elton, E. J., Martin J. G., pp. 19-20). To quantify the concept of risk, Markowitz used the statistical measures of variance and covariance. Let $b = (b_1 \ldots b_N)$ be the investment portfolio, then the risk of a portfolio comprised of more than two assets can be expressed by:

$$var(R) = \text{var}\left(\sum_{i=1}^{N} b_i R_i\right) = \sum_{i=1}^{N} (b_i \sigma_i)^2 + \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} (b_i b_j \sigma_{ik})$$

As we can see from above, the variance of a portfolio depends not only on the variances of the each individual asset but also upon the covariance of any two assets (Ross and Jaffrey, pp.401-416). This measures how closely the returns on every two assets in the portfolio move with respect to each other, and this is mainly due to the fact that financial markets interact, meaning that the investments do not vary independently. The covariance on any two assets can be calculated as:

$$cov(R_i, R_k) = \sigma_{ik} = E[\big(R_i - E(X_i)\big)\big(R_k - E(X_k)\big)]$$

Now, we formally define two different types of risks in the financial market:

- Unsystematic risk, which can be reduced by diversification. It is the total variance of the each individual asset.
- Systematic risk, which cannot be eliminated. It is affected by the covariance between the assets.

## 4. *Efficient frontier*

- ***Efficient portfolios***

Efficient portfolios may contain any number of asset combinations. To illustrate the main idea of the efficient portfolios, we would construct a portfolio, which contains two risky assets. After we understand the properties of portfolio formed by mixing two risky assets, it will be easy to see how a portfolio of many risky assets might best be constructed.

Consider a portfolio, and $b = (b_1, b_2)$ represents fraction of fraction of money allocated to asset 1 and asset 2, and $b_1 + b_2 = 1$. Then expected return is

$$E(R_p) = b_1 E(R_1) + b_2 E(R_2) = b_1 \mu_1 + (1 - b_1)\mu_2 \qquad (2)$$

Where $\mu_1, \mu_2$ denote the expected returns of Asset 1 and Asset 2. The variance of portfolio is

$$Var(R_p) = \sigma^2 = b_1^2 {\sigma_1}^2 + b_2^2 {\sigma_2}^2 + 2 b_1 b_2 \sigma_{12} = (b_1 \sigma_1 + (1 - b_1)\sigma_2)^2 \qquad (3)$$

The covariance between the two assets $\sigma_{12}$ can be expressed as $\rho_{12}\sigma_1\sigma_2$.

***Case 1*$-(\boldsymbol{\rho} = +\mathbf{1})$**

Since $\rho_{12} = +1$, then $\sigma = b_1\sigma_1 + (1 - b_1)\sigma_2$. Both risk and return are simply weighted average of the risk and return of each security. Solving $b_1$ in the expression (3), we can get

$$b_1 = \frac{\sigma - \sigma_2}{\sigma_1 - \sigma_2}$$

Substituting $b_1$ in (2), we can get the return of the portfolio in terms of the its standard deviation

$$R_p = (\mu_2 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}\sigma_2) + (\frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2})\sigma$$

Which means two securities that are perfectly correlated will lie on a straight line in E-V space.

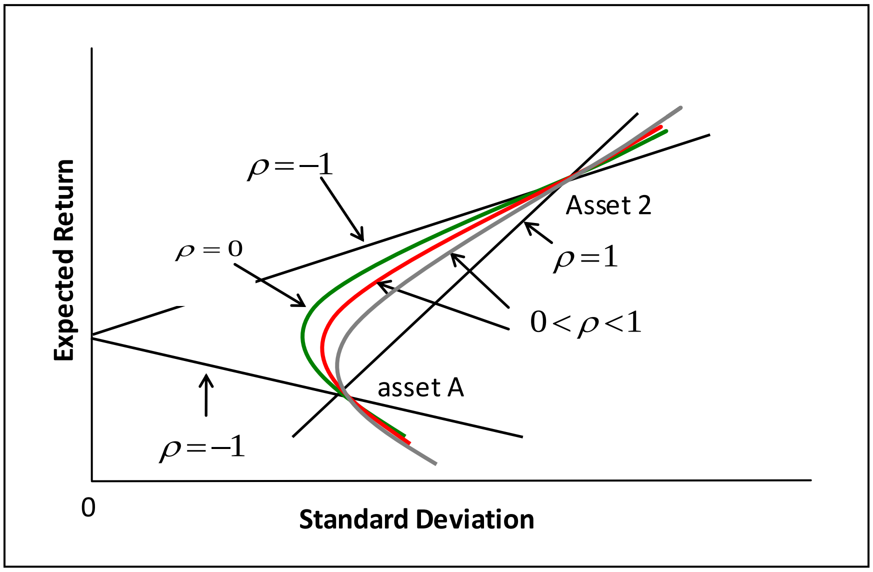
***Case 2*$-(\boldsymbol{\rho} = -\mathbf{1})$**

Since $\rho_{12} = -1$, then $\sigma = b_1\sigma_1 - (1 - b_1)\sigma_2$, or $\sigma = -b_1\sigma_1 + (1 - b_1)\sigma_2$. By the same manipulation of the formula, as shown above, we also can express the return of the portfolio as a linear function of its standard deviation.

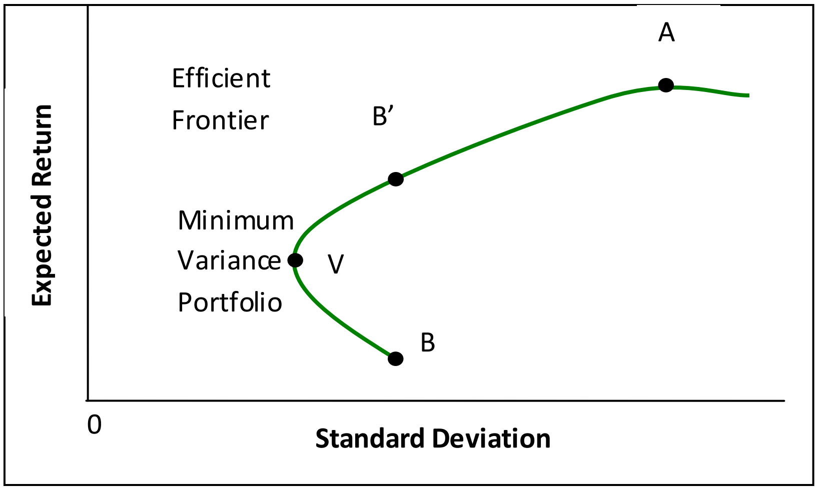
By those two extreme cases($\rho = +1$ & $\rho = -1$). It is sufficient for us to draw the possible frontier of the two risky assets in E-V space. Notice that when $\rho = +1$ $\rho = -1$, the frontier are straight lines.

- ***Markowitz efficient frontier***

Every possible asset combination can be plotted in E-V space, and the collection of all such possible portfolio defines a region in this space. It is clear from the picture, in order to achieve the maximum mean value of the returns, given a value of the variance, or conversely, achieve the minimum variance given a certain value of expected returns, we must choose the portfolios such that the points lie on the upper edge of this region.




For example, comparing point $B$ and point $B'$, the rational investors will choose $B'$, since they always prefer more return of portfolio given a certain risk. At the same time, when comparing point $V$ and $B$, since V has lower risk and higher return, the investors will choose V definitely. The same argument will also apply on the points either below or above efficient frontier by the same criterion. However, when comparing the points on the upper edge VA, different investors will choose different combinations of assets based on their own preferences.




### *5. Diversification*

The mean-variance analysis revealed the essential role of covariance of the prices, which is important for us to reduce the unsystematic risk in the portfolio.

Consider the Two-risky-assets portfolio,

$$E(R_p) = b_1 E(R_1) + b_2 E(R_2) = b_1 \mu_1 + b_2 \mu_2$$

$$Var(R_p) = \sigma^2 = b_1^2 {\sigma_1}^2 + b_2^2 {\sigma_2}^2 + 2b_1 b_2 \sigma_{12} = (b_1\sigma_1 - b_2\sigma_2)^2 + 2b_1 b_2 \sigma_1 \sigma_2 (1 + \rho_{12})$$

- If $\rho_{12} = +1$, there are no gains form diversification. Whatever the proportions of asset A and asset B, both the portfolios' mean and the standard deviation are simple weighted averages. This is the only case in which there is no benefit from the diversification.
- If $\rho_{12} = -1$, and choose a specific weights of portfolio, $b = (b_1, b_2)$, such that $b_1\sigma_1 = b_2\sigma_2$. Then $Var(R_p) = 0$.

By carefully examining the above two extreme cases, it is clear that if $\rho_{12} < 1$, then there will be some diversification effects, for example, the portfolio standard deviation, e.g. $\sqrt{Var(R_p)}$, is less than the weighted average of the standard deviations of the two securities. In fact, by carefully assigning the weights in a portfolio, we will always get some benefits through diversification whenever all asset returns are not perfectly correlated, and we can make $E(R_p)$ larger and $Var(R_p)$ smaller by choosing the securities whose correlation is as close to $-1$ as possible.

There is one more case that we haven't talked about yet. That is when all assets in a given portfolio are uncorrelated. To illustrate the main idea, we only consider the variance of the each security is finite, that is

$$var(R_i) = {\sigma_i}^2 \leq M, for\,\forall i = 1,2,\dots,N, M \in R$$

In this case, when we have sufficient large number of securities N, then

$$var(R) = \mathrm{var}\left(\sum_{i=1}^{N} b_i R_i\right) = \sum_{i=1}^{N} (b_i\sigma_i)^2 \leq M\sum_{i=1}^{N} {b_i}^2$$

By choosing $b_i = 1/N$ so that

$$var(R) \leq M\sum_{i=1}^{N} {b_i}^2 = \frac{N}{N^2}M = \frac{1}{N}M \to 0, as\, N \to \infty$$

These three particular examples show that in building an investment portfolio, we have to either invest in many negatively correlated securities, or make our the portfolio sufficiently large, when each securities is uncorrelated.

Now, move to a general setting. Suppose we have N securities in a given portfolio, by previous setting in return of the portfolio, we have already defined that

$$R(b) = \sum d_i \frac{P_1(A_i)}{P_0(A_i)} - 1 = \sum d_i (\frac{P_1(A_i)}{P_0(A_i)} - 1) = \sum d_i \rho\,(A_i) \qquad (1)$$

and $R(b) = \rho\,(d) = \sum d_i \rho\,(A_i)$. Hence

$$S_1(b) = S_0(b)\big(1 + R(b)\big) = S_0(b)\big(1 + \rho\,(d)\big)$$

In order to maximize $S_1(b)$, we only need to maximize $\rho\,(d)$. Given $d_i = 1/N$, that is each asset is equally weighted in the portfolio.

$$var\big(\rho\,(d)\big) = \mathrm{var}\left(\sum_{i=1}^{N} d_i\rho\,(A_i)\right) = \sum_{i=1}^{N} (d_i Var(\rho\,(A_i)))^2 + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N} d_i d_j Cov(\rho\,(A_i), \rho\,(A_i))$$

Consider the finite $Var\ (\rho\ (A_i)$, that is $Var\ (\rho\ (A_i) \leq M, for\ \forall i = 1,2 \ldots N, M \in R$. Hence, by the same argument, the first term in the above function can be eliminated given sufficiently large N.

For the second term,

$$\sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq j}}^{N} d_i d_j Cov\big(\rho\ (A_i), \rho\ (A_i)\big) = \left(\frac{1}{N}\right)^2 (N^2 - N)\frac{1}{N^2 - N}\sum_{\substack{i,j=1\\i\neq j}}^{N} Cov\big(\rho\ (A_i), \rho\ (A_i)\big)$$

$$= \left(1 - \frac{1}{N}\right) E\left(Cov\big(\rho\ (A_i), \rho\ (A_i)\big)\right) \rightarrow E\left(Cov\big(\rho\ (A_i), \rho\ (A_i)\big)\right) as\ N \rightarrow\ \infty$$

Where $E\left(Cov\big(\rho\ (A_i), \rho\ (A_i)\big)\right)$ represents the average of the total covariance among all N securities.

Hence,

$$var\big(\rho\ (d)\big)\ \rightarrow E\left(Cov\big(\rho\ (A_i), \rho\ (A_i)\big)\right), as\ N \rightarrow\ \infty$$

In the real-world financial market, the prices of securities are usually positively correlated (Shiryaev, pp.50). By simply adding the number of securities in the asset, we may not be able to reduce the total risk because the existence of the average of covariance.

Indeed, the first term is the so-called unsystematic risk, which can be reduced by picking a large number of assets, while the limit value of the second term is the so-called systematic risk, which is inherent in the market, and cannot be eliminated by individual investor.

## *6. Conclusion*

To conclude, this thesis focuses on the Modern Portfolio Theory (MPT) and its cornerstones, the mean-variance analysis and the efficient frontier. Markowitz’s work leads to a revolution in the way of operating capital markets. Prior to Markowitz's work, security-selection models focused primarily on maximizing returns generated by investment opportunities. However, MPT not only retains the emphasis on return, but it also brings the concept of risk to the same level of importance. Moreover, Markowitz’s mean-variance analysis was the first mathematically rigorous theory showing that the variance of a portfolio can be reduced through diversification. He proposed that the selection of the portfolio should be based on their overall risk-return characteristics, rather than just to construct portfolios by adding assets based “only on the characteristics that were unique to the security” (Edwin and Martin, pp.1744-1745).

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