Lotka-Volterra models of Predator-Prey Relationships

Author: Tyler Blaszak

Professor: Dr. Wenqing Hu

Course: Math 3304 – Ordinary Differential Equations

Description of the Model

The Lotka-Volterra equations were developed to describe the dynamics of biological systems. This system of non-linear differential equations can be described as a more general version of a Kolmogorov model because it focuses only on the predator-prey interactions and ignores competition, disease, and mutualism which the Kolmogorov model includes.

The Lotka-Volterra equations can be written simply as a system of first-order non-linear ordinary differential equations (ODEs). Since the equations are differential in nature, the solutions are deterministic (no randomness is involved, and the same initial conditions will produce the same outcome), and the time is continuous (the generations of predators and prey are continually overlapping). As with many other mathematical models, many assumptions were made in the creation of the Lotka-Volterra equations. Such assumptions include:

1. There is no shortage of food for the prey population.
2. The amount of food supplied to the prey is directly related to the size of the prey population.
3. The rate of change of population is directly proportional to its size.
4. The environment is constant and genetic adaptation is not assumed to be negligible
5. Predators will never stop eating.

After such assumptions are made, the Lotka-Volterra equations can be written as:

$$\boldsymbol{\frac{dx}{dt} = \alpha x - \beta xy}$$

$$\boldsymbol{\frac{dy}{dt} = \delta xy - \gamma y}$$

where

x	= number of prey
y	= number of predators
$\frac{dx}{dt}$ and $\frac{dy}{dt}$	= the instantaneous rates of the prey and predators, respectively.
t	= time
α, β, δ, γ	= positive real constants

Taking a closer look at the prey equation we can see that the prey are assumed to reproduce exponentially which is represented by the term αx. The equation also shows that the rate at which predators kill prey is proportional to the product of the number of prey and the number of predators, or in other terms, how often the two populations meet. This is represented by the term βxy. Therefore, if there is no population of prey or no population of predators, no decrease in the population of prey (also known as predation) can occur. The equation for prey can be summed up as: the rate at which new prey is born, minus the rate at which prey is killed off.

Looking now at the predator equation we can see that the growth of the predator population is proportional to the amount of times the two populations meet. This is similar to the rate at which predators kill prey except that a different constant is used to describe this relationship since the rate at which predators kill and the rate at which they reproduce are not identical. This term is represented by the term δxy. Since prey cannot kill the predators, the decrease in predator population is due to death by natural causes or by emigration. This is assumed to be an exponential decay which is represented by the term γy. The equation for predators can be summed up as: the rate at which they consume prey, minus the natural death rate of the population.

Tips to Develop the Lotka-Volterra Equations

Let us now look at how to implement the equations in MATLAB. This code uses MATLAB's `ode45` and `deval` commands to solve the system of equations. The `ode45` command is an integrated six-stage, fifth-order, Runge-Kutta method of solving differential equations. The `deval` is a command which evaluates solutions of differential equations at points contained in a separate solution, thus making it possible to solve the system of equations.

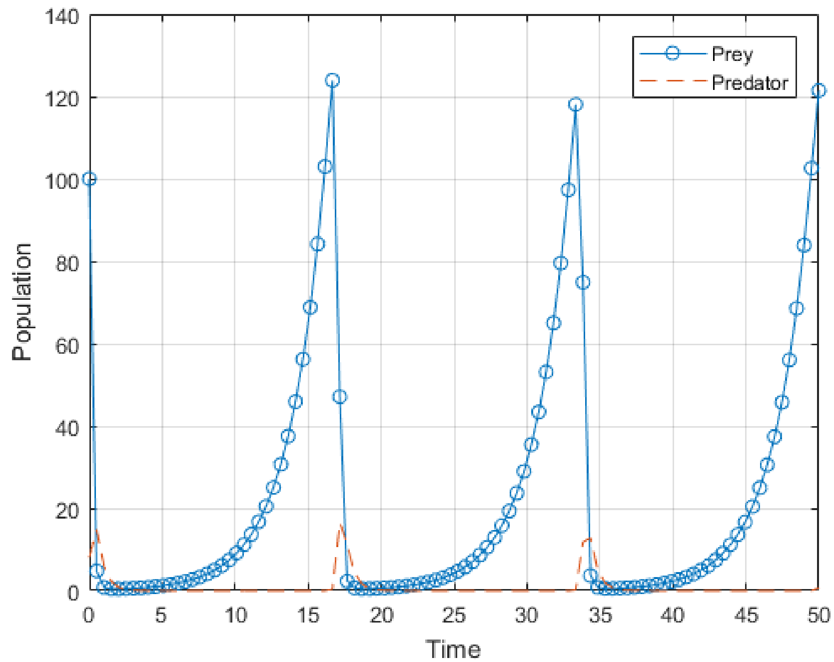
Since the parameters α, β, δ, γ are biologically determined, we will consider a system where $\alpha = 0.4$, $\beta = 0.4$, $\delta = 0.09$, and $\gamma = 2$. The initial conditions of the system of equations is such: Initial population of prey is 100, and the initial population of predators is 8. We will consider a time frame of 50 units. The MATLAB code is as follows:

```
y0 = [100;8];
soln = ode45(@f2,[0 100],y0)
t = linspace(0,50,100);
y(:,1) = deval(soln,t,1); %Prey
y(:,2) = deval(soln,t,2); %Predator

%Predator-prey function
function dxdt = f2(t,x)
dxdt = [0;0];
alpha = 0.4; beta = 0.4; delta = 0.09; gamma = 2.0;
dxdt(1) = alpha*x(1) - beta*x(1)*x(2); %prey
dxdt(2) = delta*x(1)*x(2) - gamma*x(2); %predators

figure
plot(t,y(:,1),'-o',t,y(:,2),'--');
grid on;
legend('Prey','Predator');
xlabel('Time');
ylabel('Population');
```

The plot of the solution looks as such:



Solutions to the Equations

Looking at the results of the simulation we can see that as the population of the prey begins the rise, the number of predators also begins to rise till the point at which predators kill off the prey faster than they can reproduce. Then the numbers begin to fall for the prey which thus causes a lack of food for the predators which numbers also begin to decline. The solution to this simulation is periodic meaning that the cycle will continue *ad infinitum* with the rise and fall of both populations. This looks very similar to the solution of simple harmonic motion, such as an un-damped spring-mass system except for the addition of a secondary plot.

Equilibrium Solutions to the Equations

Since we know that two populations can be in equilibrium with one another, we can solve the Lotka-Volterra equations to determine when this occurs. Since equilibrium means that the populations are not changing with respect to time the system of equations can be written as:

$$\mathbf{0 = \alpha x - \beta xy}$$

$$\mathbf{0 = \delta xy - \gamma y}$$

Which can then be solved for solutions of *x* and *y*, giving us the following two solutions:

1. $\mathbf{x = 0}$ and $\mathbf{y = 0}$
2. $\mathbf{x} = \frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}$ and $\mathbf{y} = \frac{\boldsymbol{\gamma}}{\boldsymbol{\delta}}$

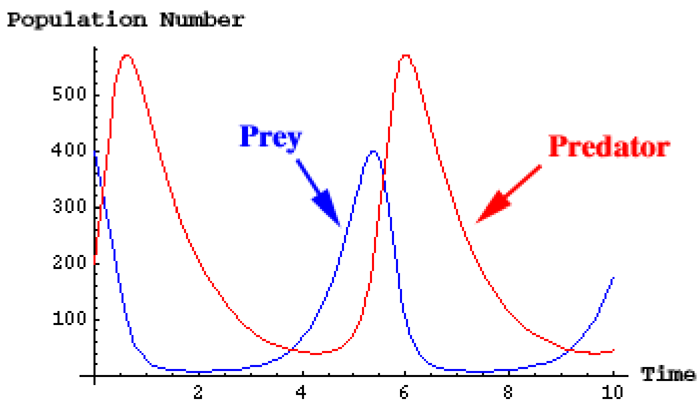
The first solution shows that if both populations are extinct, then they will continue be extinct until an outside factor can change that. The second solution describes an equilibrium where the population of the prey is equal to the ratio of the constants α and β, and the population of the predator is a ratio of δ and γ. Since both equations of the second solution depend on the biologically determined parameters, the population at which each equilibrium occurs will depend of the chosen values of the constants.

Problems with the Lotka-Volterra Equations

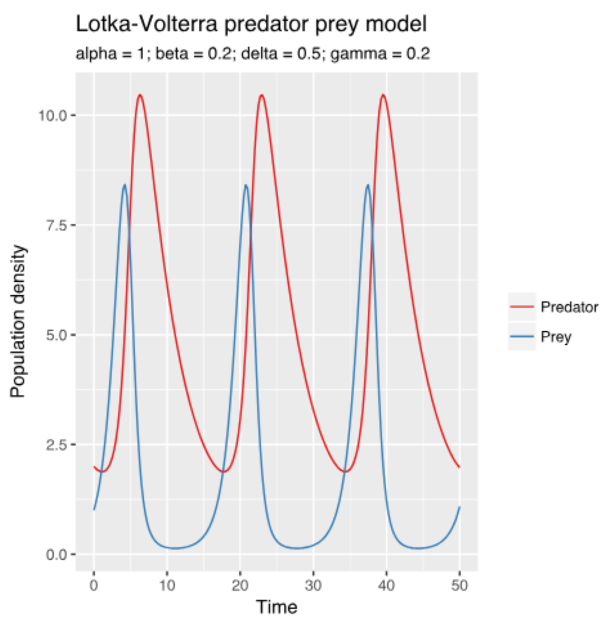
Since the Lotka-Volterra equations are a simplified and more general example of the Kolmogorov model, some problems can arise. The most significant problem of the Lotka-Volterra equations as a biological model is the ability of a prey population to "bounce back" even when subjected to extremely low population numbers. This is very rarely seen in real-life scenarios as the likelihood would be that the prey population would go extinct which would then cause the extinction of the predator population very soon after. This problem is not limited to just the Lotka-Volterra. It appears in many other simplified biological models and has been called the "atto-fox problem" (Mollison 1991).

This inability for a population to become extinct can in this model can be explained when looking at the stability of the point (0,0). This point produces a saddle point and as such is unstable. This means that it is very difficult for a population to reach a population of zero. Because of this the population can reach values very close to zero (in the case of atto-fox this can be 10^{-18}) and still increase in population size.

Further Examples



www.bio.miami.edu



www.strimas.com

References

Arditi, R., & Ginzburg, L. R. (1989). Coupling in predator-prey dynamics: Ratio-Dependence. *Journal of Theoretical Biology, 139*(3), 311-326. doi:10.1016/s0022-5193(89)80211-5

Mollison, D. (1991). Dependence of epidemic and population velocities on basic parameters. *Mathematical Biosciences, 107*(2), 255-287. doi:10.1016/0025-5564(91)90009-8

Rosenzweig, M. L., & Macarthur, R. H. (1963). Graphical Representation and Stability Conditions of Predator-Prey Interactions. *The American Naturalist, 97*(895), 209-223. doi:10.1086/282272

Volterra, V. (1928). Variations and Fluctuations of the Number of Individuals in Animal Species living together. *ICES Journal of Marine Science, 3*(1), 3-51. doi:10.1093/icesjms/3.1.3