1. Consider the following system with a sampling period of 0.1 second.

\[ G(z) = \frac{z + 0.5}{z^3 - z^2 - 0.25z + 0.25} \]

Determine the transfer function \( G(z)/R(z) \). Simplify the result as much as possible. (25pts)

2. Consider a negative feedback discrete-time control system, where the loop gain is given by

\[ G(z)H(z) = K \frac{z + 0.5}{z^3 - z^2 - 0.25z + 0.25} \]

Determine the range of stability in terms of the gain \( K \). (25pts)

3. Consider the following system with a sampling period of 1 second.

Design the simplest controller \( D(z) \) that satisfies the following requirements.

- The 5% settling-time is less than or equal to 3 seconds.
- The maximum percent-overshoot is between 15% and 25% for the unit-step input.
- The steady-state error is zero for a step input. (25pts)

4. Consider the following feedback control system, where

\[ G(z) = \frac{4950.5(z + 1)(10001z + 9999)^2}{11z^3 - 12.7822z^2 - 7.0396z + 8.8218} \]

Determine the gain and phase margins of the closed-loop system. Design the simplest controller \( D(z) \), such that gain margin of the system is increased by 100 dB. Assume the sampling period \( T = 2s \). (25pts)

\[ W[G(j\omega)] = W[G(z)] \bigg|_{z=e^{j\omega T}} = \frac{4950.5(z + 1)(10001z + 9999)^2}{11z^3 - 12.7822z^2 - 7.0396z + 8.8218} \bigg|_{z=e^{j\omega T}} = \frac{5000(\omega + 0.0001)^2}{\omega(\omega + 0.01)(\omega + 10)} \]
1. Consider the following system with a sampling period of 0.1 second.

\[ \begin{array}{c}
\text{r} \\
\uparrow \\
\text{c}
\end{array} \rightarrow \begin{array}{c}
\frac{z}{z-1} \\
\text{ZOH}
\end{array} \rightarrow \begin{array}{c}
\frac{2}{s} \\
\text{ } \\
\frac{1}{s+2}
\end{array} \rightarrow \text{c} \]

Determine the transfer function \( C(z)/R(z) \). Simplify the result as much as possible.

**Solution:** In order to be able to take the z-transforms of signals, they need to be sampled or pseudo-sampled. Denoting the transfer function of the zero-order hold (ZOH) by \( G_{ZOH} \), we have

\[ E(s) = R(s) - \left( \frac{1}{s+2} \right) C(s) = R(s) - \left( \frac{1}{s+2} \right) \left( \frac{2}{s} \right) G_{ZOH}(s) \left( \frac{z}{z-1} \right) E^*(s), \]

where \( E^*(z) \) represents the ideally-sampled \( E(s) \). When we take the z-transforms of the inverse Laplace transforms in the above equation, we get

\[ E(z) = R(z) - Z \left[ \mathcal{L}_z^{-1} \left[ \left( \frac{2}{s(s+2)} \right) G_{ZOH}(s) \right] \right](z) \left( \frac{z}{z-1} \right) E(z). \]

To simplify the notation, we let

\[ (G_{ZOH}G_1G_2)(z) = Z \left[ \mathcal{L}_z^{-1} \left[ \left( \frac{2}{s(s+2)} \right) G_{ZOH}(s) \right] \right](z) \]

\[ = \left( \frac{z-1}{z} \right) Z \left[ \mathcal{L}_z^{-1} \left[ \frac{2}{s^2(s+2)} \right] \right](z) \]

\[ = \left( \frac{z-1}{z} \right) Z \left[ \mathcal{L}_z^{-1} \left[ \frac{-1/2}{s} + \frac{1/2}{s+2} \right] \right](z) \]

\[ = \left( \frac{z-1}{z} \right) \left( \frac{-(1/2)\frac{z}{z-1} + \frac{1/2}{z+2}}{2(z-e^{-2T})} \right) \]

\[ = \frac{(2T-1+e^{-2T})z + (1 - (1 + 2T)e^{-2T})}{2(z-e^{-2T})(z-1)}. \]

Then,

\[ E(z) = R(z) - \left( \frac{(2T-1+e^{-2T})z + (1 - (1 + 2T)e^{-2T})}{2(z-e^{-2T})(z-1)} \right) \left( \frac{z}{z-1} \right) E(z), \]

or

\[ \left( 1 + \left( \frac{(2T-1+e^{-2T})z + (1 - (1 + 2T)e^{-2T})}{2(z-e^{-2T})(z-1)} \right) \left( \frac{z}{z-1} \right) \right) E(z) = R(z); \]

1
and for \( T = 0.1 \) s, we get

\[
\left( \frac{z^3 - 2.8094z^2 + 2.6462z - 0.8187}{z - 0.8187}(z - 1)^2 \right) E(z) = R(z),
\]

or

\[
E(z) = \left( \frac{z - 0.8187}(z - 1)^2 \right) R(z).
\]

The z-transform of the inverse Laplace transform on the pseudo-sampled output gives

\[
C(z) = \mathcal{Z}^{-1} \left[ \frac{2}{s} \mathcal{G}_{\text{row}}(s) \right] \left( \frac{z}{z - 1} \right) \left( \frac{z - 1}{z} \right) E(z)
\]

\[
= \frac{(z - 1)}{z} \mathcal{Z}^{-1} \left[ \frac{1}{s} \right] \left( \frac{z}{z - 1} \right) E(z)
\]

\[
= \frac{2T}{(z - 1)^2} E(z) = \frac{0.2z}{(z - 1)^2} E(z).
\]

Substituting the expression for \( E(z) \) in the previous equation, we get

\[
C(z) = \left( \frac{0.2z}{(z - 1)^2} \right) \left( \frac{z - 0.8187(z - 1)^2}{z^3 - 2.8094z^2 + 2.6462z - 0.8187} \right) R(z),
\]

or

\[
\frac{C(z)}{R(z)} = \frac{0.2z(z - 0.8187)}{z^3 - 2.8094z^2 + 2.6462z - 0.8187}.
\]

2. Consider a negative feedback discrete-time control system, where the loop gain is given by

\[
G(z)H(z) = K \frac{z + 0.5}{z^3 - z^2 - 0.25z + 0.25}.
\]

Determine the range of stability in terms of the gain \( K \).

**Solution:** For \( G(z)H(z) = K ((z + 0.5)/(z^3 - z^2 - 0.25z + 0.25)) \), the characteristic equation is

\[
1 + K \frac{z + 0.5}{z^3 - z^2 - 0.25z + 0.25} = 0,
\]

or

\[
z^3 - z^2 - 0.25z + 0.25 + K (z + 0.5) = 0.
\]

Therefore, the characteristic polynomial is

\[
q(z) = z^3 - z^2 + (K - 0.25)z + (0.5K + 0.25).
\]

To determine the range of stability for all \( K \), we can use Jury’s stability test criteria. In our case, the order of the system \( n = 3 \). The two boundary conditions are

\[
q(1) > 0,
\]

\[
(1)^3 - (1)^2 + (K - 0.25)(1) + (0.5K + 0.25) > 0,
\]

\[
K > 0,
\]

(2.1)
and
\[ (-1)^3 \varphi(-1) > 0. \]
\[ (-1)^3 \left[ ( -1 )^2 - ( -1 )^2 + (K - 0.25)(-1) + (0.5K + 0.25) \right] > 0, \]
\[ K > -2.5. \quad (2.2) \]

The pole-product condition is
\[ |a_p| < a_n, \]
\[ |0.5K + 0.25| < 1, \]
\[ -1 < 0.5K + 0.25 < 1, \]
\[ -1.25 < 0.5K < 0.75, \]
\[ -2.5 < K < 1.5. \quad (2.3) \]

The rest of the conditions is to be obtained from the Jury's table.

<table>
<thead>
<tr>
<th>( z^1 )</th>
<th>( z^2 )</th>
<th>( z^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 = 0.5K + 0.25 )</td>
<td>( a_1 = K - 0.25 )</td>
<td>( a_2 = -1 )</td>
</tr>
<tr>
<td>( a_3 = 1 )</td>
<td>( a_4 = 0 )</td>
<td>( a_5 = 1 )</td>
</tr>
</tbody>
</table>

Since we have a third-order system, the table will only give one more additional condition.
\[ |a_3| > |a_{n-1}|. \]
\[ |0.25(K - 1.5)(K + 2.5)| > | -1.5K|, \]
\[ |(K - 1.5)(K + 2.5)| > 6|K|. \]

From the Inequality 2.1, we know that \( K > 0 \), therefore we have
\[ |(K - 1.5)(K + 2.5)| > 6K > 0. \]

\((K - 1.5)(K + 2.5) > 6K > 0\) Case:
In this case,
\[ K^2 + K - 3.75 > 6K > 0, \]
\[ K^2 - 5K - 3.75 > 0, \]
\[ (K + 0.662278)(K - 5.662278) > 0, \]
\[ K < -0.662278, \text{ or } K > 5.662278. \quad (2.4) \]

However in this case, the intersection of the regions described by Inequalities 2.1–2.3 and 2.4 is empty.
\[-(K - 1.5)(K + 2.5) > 6K > 0 \text{ Case:} \]

In this case,

\[-K^2 - K + 3.75 > 6K > 0,\]

\[K^2 + 7K - 3.75 < 0,\]

\[(K + 7.5)(K - 0.5) < 0,\]

\[-7.5 < K < 0.5.\]

(2.5)

From the intersection of the regions described by Inequalities 2.1–2.3 and 2.5, we conclude that the system will be asymptotically stable, when

\[0 < K < 0.5.\]

3. Consider the following system with a sampling period of 1 second.

Design the simplest controller \(D(z)\) that satisfies the following requirements.

- The 5% settling-time is less than or equal to 3 seconds.
- The maximum percent-overshoot is between 15% and 25% for the unit-step input.
- The steady-state error is zero for a step input.

**Solution:** We determine the restrictions on the location of the desired-pole locations from the performance specifications.
<table>
<thead>
<tr>
<th>Given Requirements</th>
<th>General System Restrictions</th>
<th>Specific System Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum percent-overshoot for the unit-step input</td>
<td>$15% \leq M_p \leq 25%$.</td>
<td>From the $\alpha$-$M_p$ curves, $\zeta = 0.5$ provides the broadest range of $\alpha$ values.</td>
</tr>
<tr>
<td>Settling time for the unit-step input</td>
<td>$\rho \leq (0.05)^{1/(b_{1.5} - 1)}$.</td>
<td>For $t_{50%} = k_{1.5} T \leq 3 \text{ s}$, and $k_{1.5} \leq 3/1 = 3$, when $T = 1 \text{ s}$: $\rho \leq (0.05)^{1/(0.5 - 1)} = 0.2236$.</td>
</tr>
<tr>
<td>The steady-state error is zero for a step input.</td>
<td>Open-loop gain has a pole at 1.</td>
<td>Open-loop gain $D(z) = \frac{z + 0.1}{(z + 0.8)(z - 1)}$ has to have a pole at 1. Since the open-loop gain already has a pole at 1, as long as $D(z)$ doesn't cancel it, this requirement is satisfied.</td>
</tr>
</tbody>
</table>

When we mark these restrictions on the $z$-plane, we determine that a possible set of desired-pole locations is at $z_d = -0.19 \pm j0.1$. 

$$z = R(z) + j\Im(z) = e^{-\tau T_e \sqrt{1 - \sigma^2}}$$
The deficiency angle, \( \phi \), needed at the desired location to ensure that one of the root-locus branches goes through the location, can be determined from the angular condition:

\[
\phi + \angle(z_d - (-0.1)) - \angle(z_d - (-0.8)) + \angle(z_d - (1)) = (2k + 1)\pi,
\]

for an integer \( k \). For \( z_d = -0.19 + j0.1 \),

\[
\phi + \tan^{-1} \left( \frac{(0.1) - (0)}{(-0.19) - (-0.1)} \right) - \tan^{-1} \left( \frac{(0.1) - (0)}{(-0.19) - (-0.8)} \right) + \tan^{-1} \left( \frac{(0.1) - (0)}{(-0.19) - (1)} \right) = 180^\circ + k360^\circ,
\]

\[
\phi = 131.99^\circ - 9.31^\circ - 175.20^\circ = 180^\circ + k360^\circ,
\]

or \( \phi = -127.48^\circ \).

In order to preserve the system order so that transient specifications stay accurate, we need to cancel a pole or zero and place another one in such a way that the pole-zero combination provides the necessary deficiency angle at \( z_d \). The best choice for cancellation is the pole at \(-0.8\), since it is the slowest, and since canceling the other pole at 1 prevents the satisfaction of the steady-state requirement.

\[
\tan^{-1} \left( \frac{-0.19 - (-0.8)}{0.1 - (0)} \right) = 89.69^\circ
\]

\[
137.45^\circ - 60.69^\circ = 66.76^\circ
\]

From the above analysis,

\[
D(z) = K \frac{z + 0.8}{z + 0.08}.
\]

And the magnitude \( K \) is obtained from the magnitude condition at \( z_d \).

\[
\left| D(z)G(z) \right|_{z=z_d} = 1,
\]

\[
\left| K \frac{z + 0.1}{(z + 0.08)(z - 1)} \right|_{z=-0.19+j0.1} = 1,
\]

or \( K = 1.3196 \). Therefore,

\[
D(z) = 1.3196 \frac{z + 0.8}{z + 0.08}
\]

is one possible controller.
4. Consider the following feedback control system, where

\[
G(z) = \frac{4950.5(z + 1)(10001z + 9999)^2}{11z^3 - 12.7822z^2 - 7.0396z + 8.8218}
\]

Determine the gain and phase margins of the closed-loop system. Design the simplest controller \( D(z) \), such that gain margin of the system is increased by 100 dB. Assume the sampling period \( T = 2s \).

**Hint:**

\[
W[G(e^s)] = \left[ G(s) \right]_{s = j\omega / T} = \left[ \frac{4950.5(z + 1)(10001z + 9999)^2}{11z^3 - 12.7822z^2 - 7.0396z + 8.8218} \right]_{s = j\omega / T} = \frac{5000(w + 10000)^2}{w(w + 0.01)(w + 10)}
\]

**Solution:** From the \( w \)-transform of \( G \)

\[
W [G](w) = \frac{5000(w + 10000)^2}{w(w + 0.01)(w + 10)} = \frac{5 \times 10^{12}(1 + w/10000)^2}{w(1 + w/0.01)(1 + w/10)}
\]

we determine the cut-off frequencies and the \( \text{Gain}_{dB} = 20 \log(5 \times 10^{12}) \text{dB} = 253.98 \text{dB} \) to plot the asymptotic bode plots.
In order to determine the gain and phase margins, we first need to determine the phase and the gain crossover frequencies.

The phase crossover frequency is determined from the bode plots, when phase angle becomes $-180^\circ$ for the first time. From the asymptotic phase-bode plot, we observe that the $-180^\circ$ crossing is at the mid point of $\omega = 0.1$ and $\omega = 1$. Since the horizontal scale is logarithmic, the mid point is such that

$$\log(\omega_0) = \frac{1}{2} \left( \log(0.1) + \log(1) \right) = -\frac{1}{2};$$

or

$$\omega_0 = 10^{-1/2} \approx 0.32 \text{ rad/s}.$$  

At the mid point of $\omega = 0.1$ and $\omega = 1$, the gain is approximately $-20 \text{ dB} + 253.98 \text{ dB} = 233.98 \text{ dB}$. As a result, the gain margin is approximately $-233.98 \text{ dB}$.

The gain crossover frequency is determined from the bode plots, when the gain is 0 dB for the first time. From the asymptotic gain-bode plot, we observe that the 0 dB gain is between $\omega = 1000$ and $\omega = 10000$. At $\omega = 1000$, the gain is $-200 \text{ dB} + 253.98 \text{ dB} = 53.98 \text{ dB}$; and at $\omega = 10000$, the gain is $-200 \text{ dB} + 253.98 \text{ dB} = -6.02 \text{ dB}$. So, using the straight-line equation, we get the gain crossover frequency such that

$$\log(\omega_c) - \log(1000) = \left( \log(10000) - \log(1000) \right) \left( 0 - (53.98) \right) = \frac{53.98}{60},$$

or

$$\omega_c = 10^{53.98/60+3} \approx 7937 \text{ rad/s}.$$  

Again using the straight-line approximation, we get

$$\text{Phase}\mid_{\omega=7937} = (-270^\circ) = \left( \frac{-180^\circ - (-270^\circ)}{\log(10000) - \log(1000)} \right) \left( \log(7937) - \log(1000) \right) = 80.97^\circ,$$

or

$$\text{Phase}\mid_{\omega=7937} = -189.03^\circ.$$
As a result, the phase margin is approximately \(-189.03^\circ + 180^\circ = -9.03^\circ\).

The system is unstable, since the gain margin is \(-233.98\, \text{dB}\) and the phase margin is \(-9.03^\circ\).

We may proceed to design the desired compensator as a lead or a lag compensator. However, in this case, the lag compensator design is a lot simpler; since all we need to do is to supply \(-100\, \text{dB}\) gain at a low enough frequency, so that the phase-angle of the lag compensator does not interfere with the phase cross-over frequency. The lag compensator is given by

\[
W[D](j\omega) = \frac{1 + j\omega/\omega_L}{1 + j\omega/(\omega_L/\beta)}
\]

Choosing the cut-off frequency \(\omega_L\) of the compensator at least 10 times slower than the gain-crossover frequency \(\omega_c = 0.32\, \text{rad}/\text{s}\), so that the negative phase angle of the compensator doesn’t affect the phase angle directly, we get the lag compensator in the \(\omega\)-transform domain as

\[
W[D](j\omega) = \frac{1 + j\omega/0.01}{1 + j\omega/(0.01/\beta)}
\]

Since this gain is to be reduced by 100 dB by the lag compensator, the compensator gain \(\beta\) is such that

\[
|\beta|_{\text{dB}} = 20 \log(\beta) = 100,
\]

or \(\beta = 10^4\). However, we need to remember that this value of \(\beta\) is too large to be practically feasible.

Finally, \(D(z)\) is determined from

\[
D(z) = \left[ W[D](\omega) \right]_{\omega = \frac{\pi}{T}} = \left[ \frac{1 + w/10^{-2}}{1 + w/10^{-7}} \right]_{w = \frac{\pi}{T}}^{-1},
\]

or

\[
D(z) = 10^{-5} \left( \frac{z - 0.98}{z - 1} \right).
\]