1. Consider the following system with a sampling period of 0.1 second.

\[
\begin{align*}
\begin{array}{c}
\quad 
\end{array}
\end{align*}
\]

Determine the transfer function \(C(z)/R(z)\). Simplify the result as much as possible. (20pts)

2. The transfer function of a discrete-time control system is given by

\[
\frac{C(z)}{R(z)} = \frac{2(z - 0.2)}{z^2 - 0.4z + 0.4}.
\]

(a) Determine the unit-step response. (10pts)
(b) Determine the approximate maximum percent overshoot. (10pts)

3. Consider the following system with a sampling period of 1 second.

\[
\begin{align*}
\begin{array}{c}
\quad 
\end{array}
\end{align*}
\]

Design the simplest controller \(D(z)\) that satisfies the following requirements. (30pts)

- The 2% settling-time is approximately 6 seconds.
- The maximum percent-overshoot is about 10% for a unit-step input.
- The steady-state error is less than 0.2 for the unit-ramp input.

4. Consider the following feedback control system.

\[
\begin{align*}
\begin{array}{c}
\quad 
\end{array}
\end{align*}
\]

Design the simplest controller \(D(z)\), such that the steady-state error for a unit-ramp input \(e(\infty) \leq 0.0001\), and the phase margin of the system is about 45°. Assume the sampling period \(T = 2s\). (30pts)
1. Consider the following system with a sampling period of 0.1 second.

```
\[ r^* \rightarrow \frac{1}{s+1} \rightarrow u \rightarrow \text{ZOH} \rightarrow \frac{5}{s+2} \rightarrow e \]
```

Determine the transfer function \( C(z)/R(z) \). Simplify the result as much as possible.

**Solution:** In order to be able to take the \( z \) transforms of signals, they need to be sampled or pseudo-sampled. Denoting the transfer function of the zero-order hold (ZOH) by \( G_{ZOH} \), we have

\[
U(s) = \left( \frac{1}{s + 1} \right) \left( R^*(s) - \left( \frac{5}{s + 2} \right) G_{ZOH}(s) U^*(s) \right)
\]

\[
= \left( \frac{1}{s + 1} \right) R^*(s) - \left( \frac{1}{s + 1} \right) \left( \frac{5}{s + 2} \right) G_{ZOH}(s) U^*(s)
\]

Simplifying the above expression, we get

\[
U^*(s) = \left( \frac{1}{s + 1} \right) R^*(s) - \frac{5}{(s + 1)(s + 2)} G_{ZOH}(s) U^*(s),
\]

or

\[
(1 + \frac{5}{(s + 1)(s + 2)} G_{ZOH}(s)) U^*(s) = \left( \frac{1}{s + 1} \right) R^*(s).
\]

When we take the \( z \) transforms of the inverse Laplace transforms in the above equation, we get

\[
\left( 1 + Z \left[ L^{-1}_s \left[ \frac{5}{((s + 1)(s + 2))} G_{ZOH}(s) \right] \right] (z) \right) U(z) = \left( Z \left[ L^{-1}_s \left[ \frac{1}{(s + 1)} \right] \right] \right) R(z).
\]

To simplify the notation, we let

\[
G_1(z) = Z \left[ L^{-1}_s \left[ \frac{1}{(s + 1)} \right] \right] (z),
\]

\[
(G_1 G_{ZOH} G_2)(z) = Z \left[ L^{-1}_s \left[ \frac{5}{((s + 1)(s + 2))} G_{ZOH}(s) \right] \right] (z).
\]

Then,

\[
U(z) = \frac{G_1(z)}{1 + (G_1 G_{ZOH} G_2)(z)} R(z).
\]

We also have

\[
C(s) = \left( \frac{5}{s + 2} \right) G_{ZOH}(s) U^*(s),
\]

and the \( z \) transform of the inverse Laplace transform on the pseudo-sampled output gives

\[
C(z) = Z \left[ L^{-1}_s \left[ \frac{5}{(s + 2)} G_{ZOH}(s) \right] \right] (z) U(z)
\]

\[
= (G_2 G_{ZOH})(z) U(z),
\]

1
where
\[(G_2 G_{ZOH})(z) = Z \left[ \mathcal{L}_s^{-1} \left[ \left( \frac{5}{s + 2} \right) G_{ZOH}(s) \right] \right](z).\]

Substituting the expression for \(U(z)\) in the previous equation, we get
\[C(z) = \frac{(G_2 G_{ZOH})(z)G_1(z)}{1 + (G_1G_{ZOH}G_2)(z)}R(z).\]

Next, we need to determine the \(z\)-transform terms.

\[G_1(z) = Z \left[ \mathcal{L}_s^{-1} \left[ \frac{1}{s + 1} \right] \right](z) = \frac{z}{z - e^{-T}} = \frac{z}{z - e^{-0.1}} = \frac{z}{z - 0.9048}.\]

\[(G_2 G_{ZOH})(z) = \left( \frac{z - 1}{z} \right) Z \left[ \mathcal{L}_s^{-1} \left[ \left( \frac{1}{s} \right) \left( \frac{5}{s + 2} \right) \right] \right](z)\]
\[= \left( \frac{z - 1}{z} \right) Z \left[ \mathcal{L}_s^{-1} \left[ \frac{5/2}{s} - \frac{5/2}{s + 2} \right] \right](z) = \left( \frac{z - 1}{z} \right) \left( \frac{(5/2)z}{z - 1} - \frac{(5/2)z}{z - e^{-2T}} \right)\]
\[= 5/2 - \frac{(5/2)(z - 1)}{z - e^{-2T}} = \frac{5/2}{z - 0.8187}.\]

\[(G_1G_{ZOH}G_2)(z) = \left( \frac{z - 1}{z} \right) Z \left[ \mathcal{L}_s^{-1} \left[ \left( \frac{1}{s} \right) \left( \frac{5}{(s + 1)(s + 2)} \right) \right] \right](z)\]
\[= \left( \frac{z - 1}{z} \right) Z \left[ \mathcal{L}_s^{-1} \left[ \frac{5/2}{s} - \frac{5/2}{s + 1} + \frac{5/2}{s + 2} \right] \right](z)\]
\[= \left( \frac{z - 1}{z} \right) \left( \frac{(5/2)z}{z - 1} - \frac{5z}{z - e^{-T}} + \frac{(5/2)z}{z - e^{-2T}} \right)\]
\[= 5/2 - \frac{5(z - 1)}{z - e^{-0.1}} + \frac{(5/2)(z - 1)}{z - e^{-0.2}}\]
\[= 2.5 - \frac{5(z - 1)}{z - 0.9048} + \frac{2.5(z - 1)}{z - 0.8187} = \frac{0.0226(z + 0.9048)}{(z - 0.9048)(z - 0.8187)}.\]

Substituting these expressions in the output expression, we get
\[C(z) = \frac{(G_2 G_{ZOH})(z)G_1(z)}{1 + (G_1G_{ZOH}G_2)(z)}R(z) = \frac{0.4532z}{(z - 0.9048)(z - 0.8187) + 0.0226(z + 0.9048)},\]

or
\[\frac{C(z)}{R(z)} = \frac{0.4532z}{z^2 - 1.7z + 0.72}.\]
2. The transfer function of a discrete-time control system is given by

\[
\frac{C(z)}{R(z)} = \frac{2(z - 0.2)}{z^2 - 0.4z + 0.4}.
\]

(a) Determine the unit-step response.

**Solution:** The unit-step response for a second-order system with a zero in the form

\[
\frac{C(z)}{R(z)} = \frac{K_n(z - \sigma_z)}{(z - \rho e^{j\phi})(z - \rho e^{-j\phi})},
\]

where \(K_n\) is such that \(C(1)/R(1) = 1\), is given by

\[
c(k) = 1 - \frac{\rho^k}{\cos(\alpha)} \cos(k\phi + \alpha)
\]

for \(k \geq 0\), where the auxiliary angle \(\alpha\) is as shown in the following figure with respect to the pole location \(\rho e^{j\phi}\) marked with the cross and the zero location \(\sigma_z\) marked with the circle.

![Diagram showing the auxiliary angle \(\alpha\) with respect to the pole and zero locations.]

The expression for the angle \(\alpha\) can be determined as

\[
\alpha = \frac{\pi}{2} + \tan^{-1}\left(\frac{\omega_p}{1 - \sigma_p}\right) - \tan^{-1}\left(\frac{\omega_p}{\sigma_z - \sigma_p}\right),
\]

where \(\rho e^{j\phi} = \sigma_p + j\omega_p\).

In our case,

\[
\frac{C(z)}{R(z)} = \frac{2(z - 0.2)}{z^2 - 0.4z + 0.4} = \frac{1.25(z - 0.2)}{(z - 0.632456e^{j1.249046})(z - 0.632456e^{j1.249046})}
\]

\[
= \frac{1.25(z - 0.2)}{(z - 0.632456e^{j1.5651\circ})(z - 0.632456e^{j1.5651\circ})}
\]

In other words, \(\rho = 0.632456, \phi = 1.249046 = 71.5651\circ, \sigma_p = 0.2, \omega_p = 0.6, \text{ and } \sigma_z = 0.2\). The auxiliary angle

\[
\alpha = \frac{\pi}{2} + \tan^{-1}\left(\frac{0.6}{1 - 0.2}\right) - \tan^{-1}\left(\frac{0.6}{0.2 - 0.2}\right)
\]

\[
= 1.570796 + 0.643501 - 1.570796 = 0.643501 = 36.8699\circ.
\]
Therefore, the unit-step response is
\[
c(k) = K \left( 1 - \frac{\rho^k}{\cos(\alpha) \cos(k\phi + \alpha)} \right) = 1.6 \left( 1 - \frac{(0.632456)^k}{\cos(0.643501) \cos(1.249046k + 0.643501)} \right),
\]
or
\[
c(k) = 1.6 \left( 1 - 1.25(0.6325)^k \cos(1.2490k + 0.6435) \right) = 1.6 \left( 1 - 1.25(0.6325)^k \cos(71.5651^\circ k + 36.8699^\circ) \right) \text{ for } k \geq 0.
\]

(b) Determine the approximate maximum percent overshoot.

**Solution:** We may determine the maximum overshoot $M_p$ for the system from the $\alpha$-versus-$M_p$ curves for different values of the damping constant $\zeta$. Since for the poles that are located at
\[
p_{1,2} = \rho e^{\pm j\phi} = e^{-\zeta \omega_n T} e^{\pm j \sqrt{1-\zeta^2} \omega_n T},
\]
the damping constant is
\[
\zeta = \frac{-\ln(\rho)}{\sqrt{\ln^2(\rho) + \phi^2}} = \frac{-\ln(0.632456)}{\sqrt{\ln^2(0.632456) + (1.249046)^2}} = 0.3444.
\]

From the middle of the $\alpha$-versus-$M_p$ curves for $\zeta = 0.3$ and $\zeta = 0.4$, we get $M_p \approx 50\%-55\%$ when $\alpha = 0.6435$. Indeed, the actual expression for the maximum overshoot gives
\[
M_p = \sqrt{1 - \zeta^2} e^{\left( \frac{\zeta}{\sqrt{1-\zeta^2}} \right) \left( \pi - \alpha - \sin^{-1}(\zeta) \right)} = \sqrt{1 - (0.3444)^2} e^{\left( \frac{0.3444}{\sqrt{1-(0.3444)^2}} \right) \left( \pi - 0.6435 - \sin^{-1}(0.3444) \right)},
\]
or $M_p = 0.5340 = 53.4\%$.

3. Consider the following system with a sampling period of 1 second.

![Control System Diagram](image)

Design the simplest controller $D(z)$ that satisfies the following requirements.

- The 2% settling-time is approximately 6 seconds.
- The maximum percent-overshoot is about 10% for a unit-step input.
- The steady-state error is less than 0.2 for the unit-ramp input.
Solution: We determine the restrictions on the location of the desired-pole locations from the performance specifications.

<table>
<thead>
<tr>
<th>Given Requirements</th>
<th>General System Restrictions</th>
<th>Specific System Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum percent-overshoot</td>
<td>$M_p \approx 10% = 0.10.$</td>
<td>From the $\alpha-M_p$ curves,</td>
</tr>
<tr>
<td>for a unit-step input</td>
<td></td>
<td>$\zeta = 0.6$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>provides the broadest range of $\alpha$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>values.</td>
</tr>
<tr>
<td>Settling time for a unit-step</td>
<td>$\rho \leq (0.02)^{1/(k_{2%s} - 1)}.$</td>
<td>For $t_{2%s} = k_{2%s} T \leq 6$ s, and</td>
</tr>
<tr>
<td>input</td>
<td></td>
<td>$k_{2%s} \leq 6/1 = 6$, when $T = 1$ s;</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\rho \leq (0.02)^{1/(6-1)} = 0.4573.$</td>
</tr>
<tr>
<td>The steady-state error is finite</td>
<td>Open-loop gain has a pole at 1, and the</td>
<td>Open-loop gain</td>
</tr>
<tr>
<td>for a ramp input.</td>
<td>static error-coefficient $K_v \geq 1/e_{\text{desired}}(\infty)$.</td>
<td>$= D(z) \frac{z + 0.3}{z(z - 0.7)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>has to have a pole at 1. The static error-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>coefficient needs to be considered after all</td>
</tr>
<tr>
<td></td>
<td></td>
<td>the other design criteria.</td>
</tr>
</tbody>
</table>

When we mark these restrictions on the $z$-plane, we determine that a possible set of desired-pole locations is at $z_d \approx 0.19 \pm j0.39$. 
In order to be able to satisfy the steady-state error criterion, we have to have a pole at 1, which needs to be supplied by the controller. We also would like to keep the order of the system the same, since most of the relationships used obtain system parameters are derived for a second-order system. As a result, we choose

\[ D(z) = \frac{z - 0.7}{z - 1} D'(z), \]

where \( D'(z) \) is additional controller to be designed to satisfy the other criteria and the slowest pole of the system at 0.7 is cancelled to keep the order the same. After this initial design, we have

\[
\text{Open-loop gain} = D(z)G(z) = \left( \frac{z - 0.7}{z - 1} D'(z) \right) \left( \frac{z + 0.3}{z(z - 0.7)} \right) = \frac{z + 0.3}{z(z - 1)} D'(z).
\]

The deficiency angle, \( \phi \), needed at the desired location to ensure that one of the root-locus branches goes through the location, can be determined from the angular condition.

\[ \phi + \angle(z_d - (-0.3)) - \angle(z_d - (0)) - \angle(z_d - (1)) = (2k + 1)\pi, \]

for an integer \( k \). For \( z_d = 0.19 + j0.39 \),

\[ \phi + \tan^{-1}\left( \frac{(0.39) - (0)}{(0.19) - (-0.3)} \right) - \tan^{-1}\left( \frac{(0.39) - (0)}{(0.19) - (0)} \right) - \tan^{-1}\left( \frac{(0.39) - (0)}{(0.19) - (1)} \right) = 180^\circ + k360^\circ, \]

\[ \phi + 38.52^\circ - 64.03^\circ - 154.29^\circ = 180^\circ + k360^\circ, \]

or \( \phi = -0.2^\circ \).

The size of the deficiency angle suggests that one of the root-locus branches almost goes through the desired location. As a result, we only need to determine the constant that would set the closed-loop poles to the desired location. In order words \( D'(z) = KD''(z) \), where \( K \) is obtained from the magnitude condition at \( z_d \), and \( D''(z) \) is to be determined from the steady-state requirement.

\[ |D(z)G(z)|_{z=z_d} = 1, \]
Assuming, $D''(z) = 1$, we get
\[
\left| \frac{Kz + 0.3}{z(z - 1)} \right|_{z=0.19 + j0.39} = 1,
\]
or $K = 0.6228$. Therefore, the controller
\[
D(z) = 0.6228 \frac{z - 0.7}{z - 1} D''(z)
\]
satisfies the settling time and the maximum overshoot requirements, when $D''(z) = 1$.

The controlled system is type 1, so the steady-state error
\[
e(\infty) = \frac{1}{K_v},
\]
where
\[
K_v = \lim_{z \to \infty} \left( \frac{z - 1}{T} \right) \left( 0.6228 \frac{z - 0.7}{z - 1} D''(z) \right) = 0.8096D''(1);
\]
or $e(\infty) = 1/K_v = 1.2352/D''(1)$.

Since $e_{\text{desired}}(\infty) \leq 0.2$,
\[
\frac{1.2352}{D''(1)} \leq 0.2,
\]
or
\[
D''(1) \geq 6.1761.
\]
In other words, we need a lag compensator $D''(z)$ with a gain at $z = 1$ greater than 6.1761 without changing the existing desired closed-loop pole location. To have the minimal effect on the existing poles, the pole and the zero of the compensator should be as close as possible to each other yet provide the necessary value at $z = 1$. These two requirements may be satisfied, if the pole-zero pair is chosen close to 1. Assuming
\[
D''(z) = \frac{z - a}{z - b},
\]
we have
\[
D''(1) = \frac{1 - a}{1 - b} \geq 6.1761,
\]
or $a \leq 6.1761b - 5.1761$. Letting $b = 0.99$, we get $a \leq 0.9382$. Choosing $a = 0.93$ satisfies all the requirements. Therefore,
\[
D(z) = 0.6228 \left( \frac{z - 0.7}{z - 1} \right) \left( \frac{z - 0.93}{z - 0.99} \right)
\]
is one possible compensator.

4. Consider the following feedback control system.

Design the simplest controller $D(z)$, such that the steady-state error for a unit-ramp input $e(\infty) \leq 0.0001$, and the phase margin of the system is about 45°. Assume the sampling period $T = 2s$. 
Solution: For \( G(z) = \frac{(10001z + 9999)(z+1)}{(20z(z-1))} \), the steady-state error for a unit-ramp input is given by
\[
e(\infty) = \frac{1}{K_v},
\]
where \( K_v \) is the velocity steady-state error coefficient; and
\[
K_v = \lim_{z \to 1} \left( \left( \frac{z-1}{T} \right) D(z)G(z) \right) = \lim_{z \to 1} \left( \left( \frac{z-1}{T} \right) D(z) \left( \frac{(10001z + 9999)(z+1)}{20z(z-1)} \right) \right)
\]
\[
= \lim_{z \to 1} \left( D(z) \frac{(10001z + 9999)(z+1)}{20Tz} \right) = \left( D(1) \frac{20000}{20(2)} \right) = 1000D(1).
\]
Since \( e_{desired}(\infty) \leq 0.0001 \),
\[
K_{v_{desired}} = \frac{1}{e_{desired}(\infty)} \geq 10000,
\]
and setting \( 10K_v = K_{v_{desired}} \), we get \( D(1) = 10 \). To incorporate the gain into the system, let \( D(z) = 10D'(z) \) and \( G'(z) = 10G(z) \). The \( w \)-transform of \( G' \)
\[
\mathcal{W} \left[ G'(z) \right] = \mathcal{W} \left[ \left. G'(z) \right|_{z=\frac{1+w}{1-w}} \right] = \left[ \frac{(10001z + 9999)(z+1)}{2z(z-1)} \right]_{z=\frac{1+w}{1-w}}
\]
\[
= \left( \frac{10001 \frac{1+w}{1-w} + 9999}{1-w} \right) \left( \frac{1+w}{1-w} + 1 \right) = \frac{2(10000 + w)}{1-w} \left( \frac{1+w}{1-w} \right) \frac{2}{1-w}
\]
\[
= \frac{10000 + w}{w(1+w)} = \frac{10000(1 + w/10000)}{w(1+w)}.
\]
We may proceed to design the desired compensator as a lead or a lag compensator. Here, we will consider both designs.

**Lead compensator design**

The gain-crossover frequency for \( D'(z) = 1 \) or \( \mathcal{W} \left[ D' \right](w) = 0 \) is when
\[
\left| \mathcal{W} \left[ G' \right](j\omega_g) \right|_{dB} = 0,
\]
or
\[
\left| \mathcal{W} \left[ G' \right](j\omega_g) \right| = 1.
\]
In our case,
\[
\frac{10000 + j\omega_g}{j\omega_g(1 + j\omega_g)} = 1,
\]
\[
\frac{(10000^2 + \omega_g^2)}{\omega_g^2(1^2 + \omega_g^2)} = 1,
\]
or $\omega_g^4 = 10^8$. So, the gain-crossover frequency $\omega_g = 100 \text{ rad/s}$.

\[
\text{Phase-Margin} = \angle (W [G'] (j\omega_g))_{\omega_g=100} + 180^\circ = \angle \left( \frac{10000 + j\omega_g}{j\omega_g(1 + j\omega_g)} \right)_{\omega_g=100} + 180^\circ
\]

\[
= \angle (10000 + j100) - \angle (j100) - \angle (1 + j100) + 180^\circ
\]

\[
= \tan^{-1}(100/10000) - \tan^{-1}(100/0) - \tan^{-1}(100/1) + 180^\circ
\]

\[
= 0.5729^\circ - 90^\circ - 89.4271^\circ + 180^\circ = 0^\circ.
\]

Note that we could have easily determined the gain-crossover frequency and the phase margin pretty accurately from the Bode plots of $W [G'] (j\omega)$ as well.

Since Phase-Margin\textsubscript{desired} $\approx 45^\circ$, the phase angle of a lead compensator needed at the mid-frequency

\[
\phi_m \approx \text{Phase-Margin\textsubscript{desired}} - \text{Phase-Margin} + 0^\circ = 45^\circ - 0^\circ + 0^\circ = 45^\circ
\]

assuming that there is no need to compensate for the displaced gain-crossover frequency due to the cascaded lead compensator, since the phase plot is flat about the gain-crossover frequency.

The parameter $\alpha$ of the lead compensator is given by

\[
\alpha = \frac{1 - \sin(\phi_m)}{1 + \sin(\phi_m)} = 0.1716.
\]

The mid-frequency $\omega_m$ can be determined from

\[
|W [G'] (j\omega)|_{\text{dB}} = -20 \log(1/\sqrt{\alpha}),
\]

or

\[
|W [G'] (j\omega)| = \sqrt{\alpha}.
\]

In our case,

\[
\left| \frac{10000 + j\omega_m}{j\omega_m(1 + j\omega_m)} \right| = \sqrt{0.1716},
\]

\[
\frac{(10000^2 + \omega_m^2)}{(\omega_m^2)(1^2 + \omega_m^2)} = 0.1716,
\]

or $\omega_m = 156.0861 \text{ rad/s}$; and the lead compensator in w-transform domain

\[
W [D'] (j\omega) = \frac{j\omega/(\sqrt{\alpha}\omega_m) + 1}{j\omega/(\omega_m/\sqrt{\alpha}) + 1} = \frac{j\omega/64.65 + 1}{j\omega/376.83 + 1}.
\]

Finally, $D'(z)$ is determined from

\[
D'(z) = \left[ W [D'] (w) \right]_{w=\frac{\omega}{\sqrt{\alpha}}} = 1.01278 \left( \frac{z + 0.9695}{z + 0.9947} \right).
\]

Since $D(z) = 10D'(z)$, we have

\[
D(z) = 10.1278 \left( \frac{z + 0.9695}{z + 0.9947} \right).
\]

Indeed, the compensated system has a phase margin of 46.26°.
Lag compensator design

The desired gain-crossover frequency $\omega_{\text{desired}}$ is when the phase angle is

$$\angle \left( W \left[ G' \right] (j\omega_{\text{desired}}) \right) = -180^\circ + \text{Phase-Margin}_{\text{desired}} + 5^\circ = -180^\circ + 45^\circ + 5^\circ = -130^\circ.$$  

Since

$$\angle \left( W \left[ G' \right] (j\omega_{\text{desired}}) \right) = \angle (10000 + j\omega_{\text{desired}}) - \angle (j\omega_{\text{desired}}) - \angle (1 + j\omega_{\text{desired}}) = \tan^{-1}(\omega_{\text{desired}}/10000) - \tan^{-1}(\omega_{\text{desired}}/0) - \tan^{-1}(\omega_{\text{desired}}/1) = \tan^{-1}(\omega_{\text{desired}}/10000) - 90^\circ - \tan^{-1}(\omega_{\text{desired}}) = -130^\circ,$$

or

$$\tan^{-1}(\omega_{\text{desired}}/10000) - \tan^{-1}(\omega_{\text{desired}}) = -40^\circ.$$  

After taking the tangent of the above equation, we get

$$\frac{\tan(\tan^{-1}(\omega_{\text{desired}}/10000)) - \tan(\tan^{-1}(\omega_{\text{desired}}))}{1 + \tan(\tan^{-1}(\omega_{\text{desired}}/10000))\tan(\tan^{-1}(\omega_{\text{desired}}))} = \tan(-40^\circ),$$

$$\frac{\omega_{\text{desired}}/10000 + \omega_{\text{desired}}}{1 - \omega_{\text{desired}}/10000} = -0.8391,$$

we get $\omega_{\text{desired}} \approx 0.8390 \text{ rad/s}$ or $\omega_{\text{desired}} \approx 11920 \text{ rad/s}$. The second frequency is when the zero at $\omega = 10000 \text{ rad/s}$ starts to increase the phase angle. The frequency that will determine the phase margin is the first frequency. The gain at this frequency

$$\left| W \left[ G' \right] (j\omega) \right|_{\omega=0.8390} = 20 \log \left| \frac{10000 + j\omega}{j\omega(1 + j\omega)} \right|_{\omega=0.8390} = 20 \log(9130.89) = 79.2 \text{ dB}.$$  

Note here that the asymptotic sketch of the bode plot gives the gain of 80 dB close to the frequency $\omega = 1 \text{ rad/s}$, but the 3 dB drop due to pole with the corner frequency at $\omega = 1 \text{ rad/s}$ would bring it down to 77 dB. Since this gain is to be reduced to 0 dB by the lag compensator, the compensator gain $\beta$ is such that

$$|\beta|_{\text{dB}} = 20 \log(\beta) = 20 \log(9130.89),$$

or $\beta \approx 9131$. However, we need to remember that this value of $\beta$ is too large to be practically feasible.

Choosing the cut-off frequency of the compensator 10 times slower than the desired gain-crossover frequency, so that the negative phase angle of the compensator doesn't affect the phase angle directly, we get the lag compensator in the $\omega$-transform domain as

$$W \left[ D' \right] (j\omega) = \frac{j\omega/(\omega_{\text{desired}}/10) + 1}{j\omega/(\omega_{\text{desired}}/(10\beta)) + 1} = \frac{j\omega/0.0839 + 1}{j\omega/0.00000918848 + 1}.$$
Finally, $D'(z)$ is determined from

$$D'(z) = \left[ W \left[ D' \right] (w) \right]_{w=\frac{2}{4+1}} = 0.000118704 \left( \frac{z - 0.845189}{z - 0.999982} \right).$$

Since $D(z) = 10D'(z)$, we have

$$D(z) = 0.00118704 \left( \frac{z - 0.845189}{z - 0.999982} \right).$$

In this case, the compensated system has a phase margin of 44.3°.

Note that even though the lag-compensator design is considerably simpler, the bandwidth and the settling time of the system with the lag compensator are also considerably narrower and longer, respectively. Moreover, the lag compensator requires more accuracy during its implementation.