Mathematics 325
Winter 2002
Two Take-Home Problems for Exam III

1. (10 pts.) (a) Let n be a nonnegative integer. Show that the operator T given by

$$Tf(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{n^2}{r^2} f(r) \qquad (0 < r \le 1)$$

is hermitian on the vector space

$$V_B = \{\ f \in C^2(0,1] \ :\ f(1) = 0,\ f \text{ and } f' \text{ bounded on } (0,1]\ \}$$

equipped with the inner product

(*) $$\langle f,g\rangle = \int_0^1 f(r)\overline{g(r)}\,r\,dr.$$

(b) Are the eigenvalues of T on V_B real numbers?

(c) Are the eigenvalues of T on V_B positive?

(d) Are the eigenfunctions of T on V_B, corresponding to distinct eigenvalues, orthogonal on (0,1) relative to the inner product (*)?

(Please give reasons for your answers to (b)-(d).)

.(10 pts.) Use separation of variables to solve the variable density vibrating string problem:

$$\frac{1}{(1+x)^2} u_{tt} - u_{xx} = 0 \qquad \text{for } 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0 = u(1,t) \qquad \text{for } 0 \le t < \infty,$$

$$u(x,0) = x(1-x)\sqrt{1+x} \quad \text{and} \quad u_t(x,0) = 0 \qquad \text{for } 0 \le x \le 1.$$

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|\xi\|}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (a > 0)$	$\frac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin(b(\xi-a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\frac{i}{\sqrt{2\pi}} \frac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	$e^{-ax^2} \quad (a > 0)$	$\frac{1}{\sqrt{2a}} e^{-\xi^2/(4a)}$
J.	$\frac{\sin(ax)}{x} \quad (a > 0)$	$\begin{cases} 0 & \text{if } \|\xi\| \ge a, \\ \sqrt{\pi/2} & \text{if } \|\xi\| < a. \end{cases}$