Mathematics 325 Exam II Summer 2008 Name: Dr. Grow

1.(25 pts.) Complete the following table summarizing the fundamental properties of solutions to the wave and diffusion equations in the xt – plane.

	Property	Waves	Diffusions
4 pts.	Speed of propagation?	speed is finite, $\leq c$.	Infinite.
4 pts.	Singularities for $t > 0$?	Transported along characteristic lines $x \pm ct =$ constant (with speed $= c$).	Lost immediately.
4 pts.	Well-posed for $t > 0$?	Yes.	Yes (at least for bounded solutions).
3 pts.	Well-posed for $t < 0$?	Yes.	No.
4 pts	Maximum principle?	No.	Yes.
4 pts.	Behavior as $t \to \infty$?	Energy is constant so solution doesn't decay.	Solutions decay to zero (if φ is integrable).
2 pts.	Transmission of information?	Transported along characteristics without loss.	Lost gradually over time.

2.(25 pts.) Solve $u_t - u_{xx} = 0$ in the upper half-plane: $-\infty < x < \infty$, $0 < t < \infty$, subject to the initial condition $u(x,0) = x^2$ if $-\infty < x < \infty$. You may find the following facts useful:

$$\int_{-\infty}^{\infty} e^{-p^2} p\,dp = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-p^2} p^2\,dp = \frac{\sqrt{\pi}}{2}.$$

3 pts. to here. $u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$ with $k=1$ and $\varphi(y) = y^2$.

9 pts. to here.

6 pts. to here. $\therefore\ u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} y^2\,dy$.

Let $p = \frac{y-x}{\sqrt{4t}}$. Then $\sqrt{4t}\,p + x = y$ and $\sqrt{4t}\,dp = dy$.

As $\to +\infty$, $\to +\infty$.

As $y \to -\infty$, $p \to -\infty$.

12 pts. to here. $u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (\sqrt{4t}\,p + x)^2\,dp$

15 pts. to here. $= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} (4tp^2 + 2\sqrt{4t}\,px + x^2)\,dp$

18 to here. $= \frac{4t}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-p^2} p^2\,dp}_{\frac{\sqrt{\pi}}{2}} + \frac{2\sqrt{4t}\,x}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} p e^{-p^2}\,dp}_{0} + \frac{x^2}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-p^2}\,dp}_{\sqrt{\pi}}$

24 pts. to here. $\therefore\ \boxed{u(x,t) = 2t + x^2}$

Since $\varphi(y) = y^2$ is not bounded on $(-\infty,\infty)$, we "misused" the solution formula and we need to check our answer.

$u_t - u_{xx} = 2 - 2 \overset{\checkmark}{=} 0$ in the upper half-plane.

25 pts. to here. $u(x,0) = 2\cdot 0 + x^2 \overset{\checkmark}{=} x^2$ for all $-\infty < x < \infty$.

3.(25 pts.) Use Fourier transform methods to derive a formula for the solution to the following problem.

$$u_t - u_{xx} = f(x,t) \quad \text{if} \quad -\infty < x < \infty,\ 0 < t < \infty,$$
$$u(x,0) = \phi(x) \quad \text{if} \quad -\infty < x < \infty.$$

We take the Fourier transform of the PDE w.r.t. x:

1 pt. to here

$$\mathcal{F}(u_t - u_{xx})(\xi) = \mathcal{F}(f)(\xi)$$

3 pts. to here

$$\Rightarrow \frac{\partial}{\partial t}\mathcal{F}(u)(\xi) - \overset{-\xi^2}{\cancel{(i\xi)^2}}\mathcal{F}(u)(\xi) = \mathcal{F}(f)(\xi).$$

$$\mu(t) = e^{\int \xi^2 dt} = e^{\xi^2 t} \text{ is an integrating factor.}$$

5 pts. to here.

$$\therefore \quad e^{\xi^2 t}\frac{\partial}{\partial t}\mathcal{F}(u)(\xi) + \xi^2 e^{\xi^2 t}\mathcal{F}(u)(\xi) = e^{\xi^2 t}\mathcal{F}(f)(\xi)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(e^{\xi^2 t}\mathcal{F}(u)(\xi)\right) = e^{\xi^2 t}\mathcal{F}(f)(\xi)$$

7 pts. to here.

$$\Rightarrow e^{\xi^2 t}\mathcal{F}(u)(\xi) = \int_0^t e^{\xi^2 \tau}\mathcal{F}(f)(\xi)\,d\tau + c(\xi)$$

8 pts. to here.

$$\Rightarrow \mathcal{F}(u)(\xi) = e^{-\xi^2 t}\int_0^t e^{\xi^2 \tau}\mathcal{F}(f)(\xi)\,d\tau + c(\xi)e^{-\xi^2 t}.$$

10 pts. to here.

$$\mathcal{F}(\varphi)(\xi) = \mathcal{F}(u)(\xi)\Big|_{t=0} = c(\xi). \text{ Thus}$$

12 pts. to here.

$$(*) \quad \mathcal{F}(u)(\xi) = \int_0^t e^{-\xi^2(t-\tau)}\mathcal{F}(f)(\xi)\,d\tau + \mathcal{F}(\varphi)(\xi)e^{-\xi^2 t}$$

Applying formula (I) in the table of Fourier transforms with $a = \frac{1}{4t}$ gives

14 pts. to here.

$$e^{-\xi^2 t} = \mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi).$$

Also $\mathcal{F}(g)(\xi)\mathcal{F}(h)(\xi) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(g * h)(\xi)$, so $(*)$ becomes

16 pts. to here.

$$\mathcal{F}(u)(\xi) = \int_0^t \mathcal{F}\left(\frac{1}{\sqrt{2(t-\tau)}}e^{-\frac{(\cdot)^2}{4(t-\tau)}}\right)(\xi)\,\mathcal{F}(f)(\xi)\,d\tau + \mathcal{F}(\varphi)(\xi)\,\mathcal{F}\left(\frac{1}{\sqrt{2t}}e^{-\frac{(\cdot)^2}{4t}}\right)(\xi)$$

(see above right for continuation)

or equivalently,

$$\mathcal{F}(u)(\xi) = \int_0^t \mathcal{F}\left(\frac{1}{\sqrt{4\pi(t-\tau)}}e^{-\frac{(\cdot)^2}{4(t-\tau)}} * f\right)(\xi)\,d\tau + \mathcal{F}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}} * \varphi\right)(\xi).$$

18 pts. to here.

Interchange the order of integration in the first term on the right side of the above identity to obtain

$$\mathcal{F}(u)(\xi) = \mathcal{F}\left(\int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}}e^{-\frac{(\cdot)^2}{4(t-\tau)}} * f\,d\tau\right)(\xi) + \mathcal{F}\left(\frac{1}{\sqrt{4\pi t}}e^{-\frac{(\cdot)^2}{4t}} * \varphi\right)(\xi)$$

20 pts. to here.

Apply the inversion theorem to get

$$u(x,t) = \int_0^t \left(\frac{1}{\sqrt{4\pi(t-\tau)}}e^{-\frac{(\cdot)^2}{4(t-\tau)}} * f\right)(x)\,d\tau + \frac{1}{\sqrt{4\pi t}}\left(e^{-\frac{(\cdot)^2}{4t}} * \varphi\right)(x)$$

22 pts. to here.

$$= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}}e^{-\frac{(x-y)^2}{4(t-\tau)}} f(y,\tau)\,dy + \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\varphi(y)\,dy$$

25 pts. to here.

4.(25 pts.) Consider the following initial/boundary value problem.

$$① \quad u_{tt} - u_{xx} = 0 \text{ if } 0 < x < 1, \ -\infty < t < \infty,$$

$$②\text{-}③ \quad u_x(0,t) = u(1,t) = 0 \text{ if } -\infty < t < \infty,$$

$$④\text{-}⑤ \quad u(x,0) = 2\cos(\pi x/2) - \cos(5\pi x/2) \text{ and } u_t(x,0) = 0 \text{ if } 0 \le x \le 1.$$

13 (a) Show that the eigenfunctions for this problem are $\cos((2n+1)\pi x/2)$ where $n = 0,1,2,\ldots$

12 (b) Find a solution to the initial/boundary value problem above.

Bonus (10 pts.): Show that the solution to the initial/boundary value problem above is unique.

(a) We seek nontrivial solutions to ①-②-③-⑤ of the form $u(x,t) = X(x)T(t)$. Then ① $\Rightarrow X(x)T''(t) = X''(x)T(t)$

so $-\dfrac{T''(t)}{T(t)} = -\dfrac{X''(x)}{X(x)} = \text{constant} = \lambda$. Also ②-③-⑤ $\Rightarrow X'(0)T(t) = 0 = X(1)T(t)$ for $-\infty<t<\infty$ and

5 pts. to here $X(x)T'(0) = 0$ for $0 \le x \le 1$. Consequently $\begin{cases} \boxed{X''(x) + \lambda X(x) = 0, \ X'(0) = 0 = X(1)} \\ T''(t) + \lambda T(t) = 0, \ T'(0) = 0. \end{cases}$ ← The eigenvalue pr

Case $\lambda > 0$ (say $\lambda = k^2$ where $k>0$): The general solution of $X''(x) + k^2X(x) = 0$ is $X(x) = c_1\cos(kx) + c_2\sin(k$

The B.C.'s imply $0 = X'(0) = kc_2$ and $0 = X(1) = c_1\cos(k) + c_2\sin(k)$ so $c_2 = 0$ and $\cos(k) = 0$; i.e.

9 pts. to here. $k = (2n+1)\frac{\pi}{2}$ for $n = 0,1,2,\ldots$ Thus $\boxed{\lambda_n = \dfrac{(2n+1)^2\pi^2}{4} \text{ and } X_n(x) = \cos\left(\dfrac{(2n+1)\pi x}{2}\right) \text{ for } n = 0,1,2,\ldots}$

Case $\lambda = 0$: the general solution of $X''(x) = 0$ is $X(x) = c_1x + c_2$. The B.C.'s imply $0 = X'(0) = c_1$ and

11 pts. to here. $0 = X(1) = c_1 + c_2$ so $c_1 = c_2 = 0$. I.e. $\lambda = 0$ is not an eigenvalue.

Case $\lambda < 0$ (say $\lambda = -k^2$ where $k>0$): The general solution of $X''(x) - k^2X(x) = 0$ is

13 pts. to here. $X(x) = c_1\cosh(kx) + c_2\sinh(kx)$. The B.C.'s imply $0 = X'(0) = c_2k$ and $0 = X(1) = c_1\cosh(k) + c_2\sinh(k)$.

Since $k > 0$ and $\cosh(k) > 0$, it follows that $c_1 = c_2 = 0$. I.e. there are no negative eigenvalues.

3 pts. to here. (b) For $\lambda = \lambda_n = (2n+1)^2\left(\frac{\pi}{2}\right)^2$ the solution to the t-problem above is $T_n(t) = \cos\left(\frac{(2n+1)\pi t}{2}\right)$, up to a

6 pts. to here. constant factor. By the superposition principle, $u(x,t) = \sum_{n=0}^{N} a_n \cos\left(\frac{(2n+1)\pi x}{2}\right)\cos\left(\frac{(2n+1)\pi t}{2}\right)$ is a

solution to ①-②-③-⑤ for any integer $N \ge 0$ and any choice of constants $a_0, a_1, \ldots, a_N$. Thus,

to satisfy ④ we must have

$$2\cos\left(\frac{\pi x}{2}\right) - \cos\left(\frac{5\pi x}{2}\right) = u(x,0) = \sum_{n=0}^{N} a_n \cos\left(\frac{(2n+1)\pi x}{2}\right) \quad \text{for all } 0 \le x \le 1.$$

We may take $N = 2$, $a_0 = 2$, $a_1 = 0$, $a_2 = -1$ to get a solution. I.e.

12 pts. to here.

$$\boxed{u(x,t) = 2\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi t}{2}\right) - \cos\left(\frac{5\pi x}{2}\right)\cos\left(\frac{5\pi t}{2}\right)}$$

solves ①-②-③-④-⑤. (OVER FOR BONUS)

BONUS:

1 pt. to here

Let $v(x,t)$ be another solution to ①-②-③-④-⑤ and consider $w(x,t) = u(x,t) - v(x,t)$.

Then w solves

2 pts. to here.

$$\begin{cases} w_{tt} - w_{xx} = 0 & \text{in } 0<x<1,\ -\infty<t<\infty, \\ w_x(0,t) = 0 = w(1,t) & \text{if } -\infty<t<\infty, \\ w(x,0) = 0 = w_t(x,0) & \text{if } 0 \le x \le 1. \end{cases}$$

3 pts. to here. Consider the energy function $E(t) = \int_0^1 \left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx$ corresponding to w. Then

$$\frac{dE}{dt} = \int_0^1 \frac{\partial}{\partial t}\left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx = \int_0^1 w_t(x,t)w_{tt}(x,t)\,dx + \int_0^1 \underbrace{w_x(x,t)}_{v}\underbrace{w_{xt}(x,t)dx}_{dV}.$$

Integrating the second integral by parts yields

$$\frac{dE}{dt} = \int_0^1 w_t(x,t)w_{tt}(x,t)\,dx + w_x(x,t)w_t(x,t)\Big|_{x=0}^{1} - \int_0^1 w_t(x,t)w_{xx}(x,t)\,dx.$$

But the B.C.'s imply $w_x(0,t) = 0$ and $w_t(1,t) = 0$ for all real t, so $w_x(x,t)w_t(x,t)\Big|_{x=0}^{1} = 0$.

Thus

$$\frac{dE}{dt} = \int_0^1 w_t(x,t)\Big[\overbrace{w_{tt}(x,t) - w_{xx}(x,t)}^{\text{zero for all } 0<x<1 \text{ and all real } t}\Big]dx = 0.$$

7 pts. to here.

It follows that $E(t) = E(0)$ for all real t. However

$$E(0) = \int_0^1 \left[\frac{1}{2}w_t^2(x,0) + \frac{1}{2}w_x^2(x,0)\right]dx = \int_0^1 \left[\frac{1}{2}(0)^2 + \frac{1}{2}(0)^2\right]dx = 0,$$

8 pts. to here. so $\int_0^1 \left[\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t)\right]dx = E(t) = 0$ for all real t.

9 pts. to here. By the vanishing theorem $\frac{1}{2}w_t^2(x,t) + \frac{1}{2}w_x^2(x,t) = 0$ for all $0 \le x \le 1$ and each fixed real t. It follows that $w_t(x,t) = 0 = w_x(x,t)$ for $0 \le x \le 1$ and $-\infty<t<\infty$ and hence $w(x,t) = \text{constant}$ for $0 \le x \le 1$ and $-\infty<t<\infty$. But the I.C. $w(x,0) = 0$ for $0 \le x \le 1$ implies $w(x,t) = 0$ for $0 \le x \le 1$ and $-\infty<t<\infty$. That is, $0 = u(x,t) - v(x,t)$ for all

10 pts. to here. $0 \le x \le 1$ and $-\infty<t<\infty$ so the solution to ①-②-③-④-⑤ is unique.

A Brief Table of Fourier Transforms

	$f(x)$	$\hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$
A.	$\begin{cases} 1 & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(b\xi)}{\xi}$
B.	$\begin{cases} 1 & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{-ic\xi} - e^{-id\xi}}{i\xi\sqrt{2\pi}}$
C.	$\dfrac{1}{x^2 + a^2}$ $(a > 0)$	$\sqrt{\dfrac{\pi}{2}}\,\dfrac{e^{-a\|\xi\|}}{a}$
D.	$\begin{cases} x & \text{if } 0 < x \le b, \\ 2b - x & \text{if } b < x < 2b, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{-1 + 2e^{-ib\xi} - e^{-2ib\xi}}{\xi^2\sqrt{2\pi}}$
E.	$\begin{cases} e^{-ax} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$ $(a > 0)$	$\dfrac{1}{(a + i\xi)\sqrt{2\pi}}$
F.	$\begin{cases} e^{ax} & \text{if } b < x < c, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{e^{(a-i\xi)c} - e^{(a-i\xi)b}}{(a - i\xi)\sqrt{2\pi}}$
G.	$\begin{cases} e^{iax} & \text{if } -b < x < b, \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\dfrac{2}{\pi}}\,\dfrac{\sin(b(\xi-a))}{\xi - a}$
H.	$\begin{cases} e^{iax} & \text{if } c < x < d, \\ 0 & \text{otherwise.} \end{cases}$	$\dfrac{i}{\sqrt{2\pi}}\,\dfrac{e^{ic(a-\xi)} - e^{id(a-\xi)}}{a - \xi}$
I.	e^{-ax^2} $(a > 0)$	$\dfrac{1}{\sqrt{2a}}\,e^{-\xi^2/(4a)}$
J.	$\dfrac{\sin(ax)}{x}$ $(a > 0)$	$\begin{cases} 0 & \text{if } \|\xi\| \ge a, \\ \sqrt{\pi/2} & \text{if } \|\xi\| < a. \end{cases}$

Math 325

Exam II

Summer 2008

number of scores: 20

mean: 70.2

standard deviation: 22.5

Distribution of Scores:

87 – 100	5
73 – 86	6
60 – 72	4
50 – 59	1
0 – 49	4

Distribution of Letter Grades:

A	5
B	10
C	1
D	4
F	0