Mathematics 204

Spring 2012

Exam I

[1] Your Printed Name: Dr. Grow

[1] Your Instructor's Name: ____________________

Your Section (or Class Meeting Days and Time): ____________________

1. **Do not open this exam until you are instructed to begin.**
2. All cell phones and other electronic devices must be **turned off or completely silenced** (i.e. not on vibrate) for the duration of the exam.
3. You are **not allowed to use a calculator** on this exam.
4. Exam I consists of this cover page and 4 pages of problems containing 7 numbered problems.
5. Once the exam begins, you will have 60 minutes to complete your solutions.
6. **Show all relevant work. No credit** will be awarded for unsupported answers and partial credit depends upon the work you show.
7. You may use the back of any page for extra scratch paper, but if you would like it to be graded, clearly indicate in the space of the original problem where the work is to be found.
8. The symbol [14] at the beginning of a problem indicates the point value of that problem is 14. The maximum possible score on this exam is 100.

problem	0	1	2	3	4	5	6	7	**Sum**
points earned									
maximum points	2	12	14	14	14	14	14	16	100

1.[12] Classify each differential equation by completing the columns in the following table. For each nonlinear differential equation, **CIRCLE** a term that makes it nonlinear.

Differential Equation	Order?	Linear? (Y/N)	Homogeneous? (Y/N)
$tu''+t^2-\sin(t)u'=0$	2	Y	N
$y'+y(1-y)=0$	1	N	NA
$\sqrt{\tan(x^4)}\,y'+x-y=0$	1	Y	N
$u\dfrac{d^3u}{dx^3}=e^x\dfrac{du}{dx}$	3	N	NA

First order, nonlinear

2.[14] Find the explicit solution to $y'-ty^2=t$ satisfying the initial condition $y(0)=1$.

$$\frac{dy}{dt} = ty^2+t$$

$$\frac{dy}{dt} = t(y^2+1) \quad (\text{Separable})$$

$$\int \frac{dy}{y^2+1} = \int t\,dt$$

$$\text{Arctan}(y) = \frac{t^2}{2}+c$$

Apply the initial condition: $y=1$ when $t=0$.

$$\text{Arctan}(1) = \frac{0^2}{2}+c$$

$$\frac{\pi}{4} = c.$$

$$\therefore \ \text{Arctan}(y) = \frac{t^2}{2}+\frac{\pi}{4}$$

$$\boxed{y(t) = \tan\left(\frac{t^2}{2}+\frac{\pi}{4}\right)}$$

First order, linear, nonhomogeneous.

3.[14] Find the general solution of $(20+t)y' + 2y = \frac{3}{2}(20+t)$ on the interval $t > 0$.

$$(*) \quad y' + \frac{2}{20+t} y = \frac{3}{2}$$

Integrating factor: $\mu(t) = e^{\int p(t)dt}$

$$\mu(t) = e^{\int \frac{2}{20+t}dt} = e^{2\ln(20+t) + \cancel{c}^{\,0}}$$

$$\mu(t) = e^{\ln(20+t)^2} = (20+t)^2.$$

Multiply through $(*)$ with the integrating factor:

$$\underbrace{(20+t)^2 y' + 2(20+t)y}_{\text{Exact ?}} = \frac{3}{2}(20+t)^2$$

$$\frac{d}{dt}\left((20+t)^2 y \right) \overset{\checkmark}{=} \frac{3}{2}(20+t)^2$$

Integrating both sides of the DE yields

$$(20+t)^2 y = \int \frac{3}{2}(20+t)^2 dt$$

$$(20+t)^2 y = \frac{1}{2}(20+t)^3 + c$$

$$\boxed{y(t) = \frac{20+t}{2} + \frac{c}{(20+t)^2}}$$

where c is an arbitrary constant.

4.[14] A population of insects in a certain region has a daily birth rate that equals the square of the current population. Assume that the population's daily death rate is triple the current insect population. On any given day, there is a net migration into the region of 2 million insects. If there are half a million insects initially, write, **BUT DO NOT SOLVE,** an initial value problem which models the population of insects in the region at any time $t > 0$.

Let $p(t)$ denote the insect population (in millions) at time t (in days).

Net Rate = Rate In − Rate Out ← - If no IVP then 1 pt and

= (Birth Rate + Immigration Rate) − Death Rate ← 1 pt

(−1 pt. for wrong sign)

$$\therefore \boxed{\frac{dp}{dt} = p^2 + 2 - 3p \,, \quad p(0) = \frac{1}{2}.}$$

4 pts. 3pts. 3pts. 3pts. 1pt.

(−3 if they add the initial population to the diferential equatio

Also acceptable: $\frac{dA}{dt} = A^2 + 2{,}000{,}000 - 3A \,, \quad A(0) = 500{,}000$

5.[14] For the autonomous differential equation $y' = y^4 + 4y^3 + 3y^2$,
(a) find the critical (or equilibrium) points;
(b) draw the phase line (or phase portrait);
(c) classify the critical points as asymptotically stable, unstable, or semistable.
(d) If $y(0) = -2$, determine $\lim_{t\to\infty} y(t)$.

(a) $y' = y^2(y^2 + 4y + 3)$

$y' = y^2(y+1)(y+3)$

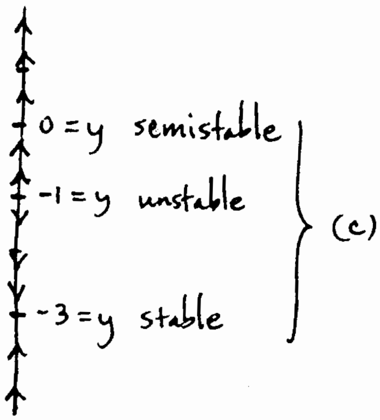
Critical points: $\boxed{y=0,\ y=-1,\ y=-3}$

(d) Since $y=-3$ is a stable equilibrium point, $\lim_{t\to\infty} y(t) = \boxed{-3}$.

(b)

interval	sign of $y' = y^2(y+1)(y+3)$
$0<y<\infty$	$+$
$-1<y<0$	$+$
$-3<y<-1$	$-$
$-\infty<y<-3$	$+$

Phase Line



6.[14] Find the general solution of each differential equation.
(a) $6y'' + 17y' + 5y = 0$
(b) $y'' - 6y' + 25y = 0$

(a) $y = e^{rt}$ in the DE leads to $6r^2 + 17r + 5 = 0 \Rightarrow (3r+1)(2r+5) = 0$

$\Rightarrow r = -\frac{1}{3},\ r = -\frac{5}{2}$. Therefore the general solution is

$$\boxed{y(t) = c_1 e^{-\frac{1}{3}t} + c_2 e^{-\frac{5}{2}t}}$$

where c_1 and c_2 are arbitrary constants.

(b) $y = e^{rt}$ in the DE leads to $r^2 - 6r + 25 = 0 \Rightarrow r = \dfrac{6 \pm \sqrt{36-100}}{2}$

$\Rightarrow r = \dfrac{6 \pm 8i}{2} = 3 \pm 4i$. Therefore the general solution is

$$\boxed{y(t) = e^{3t}\left(c_1 \cos(4t) + c_2 \sin(4t)\right)}$$

where c_1 and c_2 are arbitrary constants.

7.[16] Given that $y_1(t) = t^2$ is a solution of the differential equation $t^2 y'' + 2ty' - 6y = 0$, use reduction of order to find a second linearly independent solution y_2 on the interval $t > 0$. Verify the linear independence of y_1 and y_2 on $t > 0$ using the Wronskian.

$y_2(t) = u(t)y_1(t) = t^2 u$ so $y_2' = 2tu + t^2 u'$ and $y_2'' = 2u + 2tu' + 2tu' + t^2 u''$
$= 2u + 4tu' + t^2 u''$.

We want $t^2 y_2'' + 2ty_2' - 6y_2 = 0$ so $t^2(\cancel{2u} + 4tu' + t^2 u'') + 2t(\cancel{2tu} + t^2 u') - \cancel{6t^2 u} = 0$

$\Rightarrow t^4 u'' + (4t^3 + 2t^3)u' = 0 \Rightarrow tu'' + 6u' = 0$. Let $v = u'$.

Then $v' = u''$ so the DE becomes $tv' + 6v = 0 \Rightarrow t\frac{dv}{dt} = -6v$

$\Rightarrow \int \frac{dv}{v} = \int -\frac{6}{t}\,dt \Rightarrow \ln|v| = -6\ln(t) + c \Rightarrow v = Kt^{-6}$

where $\pm e^c = K$. Then $u' = Kt^{-6} \Rightarrow u = \int Kt^{-6}\,dt = \frac{Kt^{-5}}{-5} + c_2$

Thus, $u(t) = c_1 t^{-5} + c_2$ where c_1 and c_2 are arbitrary constants. Therefore $y_2(t) = t^2(c_1 t^{-5} + c_2) = c_1 t^{-3} + c_2 t^2$. To get a solution linearly independent from $y_1(t) = t^2$, we take $c_1 = 1$ and $c_2 = 0$. Consequently

$$\boxed{y_2(t) = t^{-3}}.$$

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} t^2 & t^{-3} \\ 2t & -3t^{-4} \end{vmatrix} = -\frac{3}{t^2} - \frac{2}{t^2}$$

$= \boxed{-\frac{5}{t^2} \neq 0}$ on $t > 0$. Therefore $\{t^2, t^{-3}\}$ is a fundamental set of solutions for the DE on $t > 0$.

2012 Spring Semester, Math 204 Hour Exam I, Master List

100	卌 卌 ||||		59	||||		19
99	卌 |		58			18
98	卌		57	|		17
97	卌 卌		56	|		16
96	卌 |||	94 A's	55	||||		15
95	卌 |||		54	卌 |		14
94	卌 ||||		53			13
93	卌 ||||		52	|		12
92	卌 |		51	||	37 F's	11
91	卌 ||||		50	|		10
90	卌 卌 |		49	|||		9
89	卌 卌 |		48			8
88	卌 |		47	|		7
87	||||		46	||		6
86	卌 卌 |||	90 B's	45			5
85	卌 |		44	|		4
84	卌 |		43			3
83	卌 卌 ||		42	|		2
82	卌 卌 卌 ||		41	||		1
81	卌 卌		40			0
80	卌		39			
79	卌 卌 ||		38	|		
78	卌 |||		37	|		
77	卌 |||	75 C's	36	|		
76	卌 ||		35	||		
75	卌 ||		34	|		
74	卌 卌		33	|		
73	卌 ||		32			
72	卌 ||		31			
71	||||		30			
70	卌		29			
69	卌 ||		28			
68	|		27			
67	卌		26			
66	||		25			
65	||	33 D's	24			
64	||||		23			
63	||||		22			
62	||		21			
61	||||		20			
60	||					

Number taking exam: 329
Median: 82
Mean: 79.0
Standard Deviation: 15.5

Number receiving A's: 94 — 28.6 %
Number receiving B's: 90 — 27.4
Number receiving C's: 75 — 22.8
Number receiving D's: 33 — 10.0
Number receiving F's: 37 — 11.2